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A REMARK ON CONVERGENCE THEORY FOR ITERATIVE PROCESSES OF PROINOV CONTRACTION

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Abstract. In this paper, we extend the study of general convergence theorems for the Picard iteration of Proinov contraction from the class of continuous mappings to the class of discontinuous mappings. As a by product we provide a new affirmative answer to the open problem posed in [20].

1. Introduction

Let Φ_1 [19] denote the class of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying; for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < t < \delta$ implies $\varphi(t) \leq \epsilon$. Let T be a self-mapping on a metric space (X, d). A fixed point of T is said to be contractive [10] if all of the Picard iterates of T converge to this fixed point. A mapping T is called asymptotically regular [6] if $\lim_{n\to\infty} d(T^{n-1}x, T^n x) = 0$ for each $x \in X$ and $n \in \mathbb{N}$.

In 2006, Proinov [19] proved the following fixed point theorem which subsumes a vast class of fixed point theorems in the existing literature (see, [2, 5, 8, 9, 11–13, 17, 18, 21]).

Theorem 1.1. *Let (X, d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that*

(i) *there exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(D(x, y))$ for all $x, y \in X$, where $D(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]$, where $\gamma \geq 0$;*
(ii) *$d(Tx, Ty) < D(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Then T has a contractive fixed point. Moreover, if $D(x, y) = d(x, y) + d(x, Tx) + d(y, Ty)$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then the continuity of T can be dropped.

It may be observed that except $\gamma = 1$, the above theorem demands continuity of the mapping T. The equivalent version of Theorem 1.1 is the following.

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Theorem 1.2 ([19])**.** *Let (X, d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that*

(i) *for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < D(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;*
(ii) *$d(Tx, Ty) < D(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

Definition 1.1 ([17])**.** A self-mapping T of a metric space (X, d) is called k-continuous, $k = 1, 2, 3, \ldots$, if $T^k x_n \to Tz$, whenever $\{x_n\}$ is a sequence in X such that $T^{k-1} x_n \to z$.

Remark 1. It is important to note that for a self-mapping T of a metric space (X, d), the notion of 1-continuity coincides with continuity. However,

$$\text{1-continuity} \Rightarrow \text{2-continuity} \Rightarrow \text{3-continuity} \Rightarrow \cdots,$$

but not conversely. The following example illustrates this very fact.

Example 1.2 ([17])**.** Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = 1 \text{ if } x \in [0, 1], \quad T(x) = 0 \text{ if } x \in (1, 2].$$

Then $Tx_n \to t \Rightarrow T^2 x_n \to t$, since $Tx_n \to t$ implies $t = 0$ or $t = 1$ and $T^2 x_n \to 1 = T1$ for all n. Hence T is 2-continuous. However, T is not 1-continuous at $x = 1$.

The question whether there exists a contractive definition which possesses a fixed point but does not force the map to be continuous at the fixed point was ingeminated by Rhoades in [20] as an existing open problem. The question of the existence of contractive mappings which are discontinuous at their fixed points was settled in the affirmative by Pant [15]. Several other distinct answers of the open problem are addressed in [2–4, 14, 16, 17]. In this paper we show that contractive definition given by Proinov [19] need not be continuous at the fixed point. We not only relax the continuity requirement in the results proved by Proinov but also provide more answers to the open question posed in [20].

Within the paper we will use the following notations: $\mathbb{N}$ denotes the set of natural numbers, $d(X)$ and $d(T(X))$ the diameter of X and the diameter of the range of T, respectively.

2. Main results

Our first main result is the following.

Theorem 2.1. *Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that*

(i) *there exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(D(x, y))$ for all $x, y \in X$;*
(ii) *$d(Tx, Ty) < D(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Suppose T is k-continuous for some $k \in \mathbb{N}$. Then T has a contractive fixed point z, and $T^n x \to z$ for each $x \in X$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$ for all $n \in \mathbb{N}\bigcup\{0\}$. We shall show that $\{x_n\}$ is a Cauchy sequence. Fix an $\epsilon > 0$. Since $\varphi \in \Phi_1$ there exists $\delta > \epsilon$ such that $t \in \mathbb{R}_+$,

$$\epsilon < t < \delta \Longrightarrow \varphi(t) \leq \epsilon. \tag{2.1}$$

Without loss of generality, we may assume that $\delta < 2\epsilon$. Since T is asymptotically regular, i.e., $c_n = d(x_n, x_{n+1}) = d(T^{n-1}x_0, T^n x_0) \to 0$, there exists $k \in \mathbb{N}$ such that for $n \geq k$ (take $k = 1$),

$$d(x_n, x_{n+1}) < \frac{\delta - \epsilon}{1 + 2\gamma}. \tag{2.2}$$

Following Proinov [19] we shall use induction to show that, for any $n, m \in \mathbb{N}$ and $m \geq n \geq k$,

$$d(x_n, x_m) < \frac{\delta + 2\gamma\epsilon}{1 + 2\gamma}. \tag{2.3}$$

Inequality (2.3) holds for $n = m$. Assuming (2.3) is true for some m we shall prove it for $m + 1$. By the triangle inequality, we have

$$\begin{aligned} d(x_n, x_{m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) \\ &= d(x_n, x_{n+1}) + d(Tx_n, Tx_m). \end{aligned} \tag{2.4}$$

Observe that it suffices to show that

$$d(Tx_n, Tx_m) \leq \epsilon. \tag{2.5}$$

To show it we shall prove the following two cases:

Case 1. $D(x_n, x_m) \leq \epsilon$. By (i) and (ii) it follows that $d(Tx_n, Tx_m) \leq D(x_n, x_m) \leq \epsilon$ and so (2.5) holds.

Case 2. Let $D(x_n, x_m) > \epsilon$. Then by (i) we get

$$d(Tx_n, Tx_m) \leq \varphi(D(x_n, x_m)). \tag{2.6}$$

Using definition of $D(x, y)$ we get

$$\begin{aligned} D(x_n, x_m) &= d(x_n, x_m) + \gamma[d(x_n, Tx_n) + d(x_m, Tx_m)] \\ &= d(x_n, x_m) + \gamma[d(x_n, x_{n+1}) + d(x_m, x_{m+1})] \\ &= d(x_n, x_m) + \gamma[c_n + c_m]. \end{aligned}$$

Hence, from (2.2) and (2.3), we get

$$D(x_n, x_m) < \frac{\delta + 2\gamma\epsilon}{1 + 2\gamma} + 2\gamma\frac{\delta - \epsilon}{1 + 2\gamma} = \delta.$$

Therefore, $\delta < D(x_n, x_m) < \epsilon$. Hence, from (2.1) we have $\varphi(D(x_n, x_m)) \leq \epsilon$. Then (2.6) implies (2.5).

Now, from (2.2), (2.4) and (2.5), we get

$$d(x_n, x_{m+1}) \leq c_n + \epsilon < \frac{\delta - \epsilon}{1 + 2\gamma} + \epsilon = \frac{\delta + 2\gamma\epsilon}{1 + 2\gamma}.$$

Thus (2.3) is proved, which completes the induction. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Also for each $k \geq 1$ we have $T^k x_n \to z$.

Suppose that T is k-continuous. Since $T^{k-1}x_n \to z$, k-continuity of T implies that $\lim_{n\to\infty} T^k x_n = Tz$. This yields $z = Tz$, that is, z is a fixed point of T. Uniqueness of the fixed point follows from (ii). □

Equivalent version of Theorem 2.1 is the following result.

Theorem 2.2. *Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that*

(i) *for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < D(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;*
(ii) *$d(Tx, Ty) < D(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Suppose T is k-continuous for some $k \in \mathbb{N}$. Then T has a contractive fixed point z, and $T^n x \to z$ for each $x \in X$.

The following corollaries are easy consequences of Theorem 2.1.

Corollary 2.3. *Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that*

(i) *there exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leq \varphi(m(x, y))$ for all $x, y \in X$, where*

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\};$$

(ii) *$d(Tx, Ty) < m(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Suppose T is k-continuous for some $k \in \mathbb{N}$. Then T has a contractive fixed point z, and $T^n x \to z$ for each $x \in X$.

Corollary 2.4. *Let (X, d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that*

(i) *for any $\epsilon > 0$ there exists $\delta > \epsilon$ such that $\epsilon < m(x, y) < \delta$ implies $d(Tx, Ty) \leq \epsilon$;*
(ii) *$d(Tx, Ty) < m(x, y)$ for all $x, y \in X$ with $x \neq y$.*

Suppose T is k-continuous for some $k \in \mathbb{N}$. Then T has a contractive fixed point z, and $T^n x \to z$ for each $x \in X$.

The following example illustrates the above theorems:

Example 2.1 ([15])**.** Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{if } 1 < x \leq 2. \end{cases}$$

Then T satisfies all the conditions of Theorems 2.1 and 2.2 and has a unique fixed point $x = 1$ at which T is discontinuous. The mapping T satisfies the contractive condition (i) with

$$\varphi(t) = \begin{cases} (1+t)/2, & \text{if } t > 1; \\ t/2, & \text{if } t \leq 1. \end{cases}$$

It can also be easily seen that $\lim_{x\to 1} D(x,1) \neq 0$. The mapping T also satisfies condition (i) of Theorem 2.2 with

$$\delta(\epsilon) = \begin{cases} 1, & \text{if } \epsilon \geq 1; \\ 1-\epsilon, & \text{if } \epsilon < 1. \end{cases}$$

In the above example, it can be easily verified that $d(T(X)) = 1$ and φ is continuous only in the open interval $(0,1)$. In view of this very fact we can consider $\varphi(t)$ to be continuous for each $t > 0$ only in the open interval $(0, d(T(X)))$ instead of in whole of $\mathbb{R}_+$ (as assumed in Theorem 1.1 above).

Theorem 2.5. *Let (X,d) be a complete metric space. Let T be an asymptotically regular self-mapping on X such that*

(i) *there exists $\varphi \in \Phi_1$ such that $d(Tx,Ty) \leq \varphi(d(x,y) + d(x,Tx) + d(y,Ty))$ for all $x,y \in X$;*

(ii) *$d(Tx,Ty) < d(x,y) + d(x,Tx) + d(y,Ty)$ for all $x,y \in X$ with $x \neq y$.*

If φ is continuous for each t in the open interval $(0, d(T(X)))$ and satisfies $\varphi(t) < t$ for all $t > 0$, then T has a contractive fixed point.

Remark 2. The above theorems generalize and improve the results due to Abtahi [1], Proinov [19], Boyd and Wang [5], Bisht and Pant [3], Ciric [7], Jachymski [8], Kuczma et al. [9], Maiti and Pal [11], Matkowski [12], Meir and Keeler [13], and Pant and Pant [17].

Remark 3. Theorem 2.1 provides a new answer to the once open question (see Rhoades [20], p. 242) on the existence of contractive mappings which admit discontinuity at the fixed point.

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