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A NOTE ON THE VALUE DISTRIBUTION OF DIFFERENTIAL POLYNOMIALS

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Abstract. Let f be a transcendental meromorphic function, defined in the complex plane $\mathbb{C}$. In this paper, we give a quantitative estimations of the characteristic function $T(r,f)$ in terms of the counting function of a homogeneous differential polynomial generated by f. Our result improves and generalizes some recent results.

1. Introduction

Throughout of this article, we adopt the standard notations and results of classical value distribution theory [See, Hayman's Monograph ([2])]. A meromorphic function g is said to be rational if and only if $T(r,g) = O(\log r)$, otherwise, g is called a transcendental meromorphic function. Let f be a transcendental meromorphic function, defined in the complex plane $\mathbb{C}$.

We denote by $S(r,f)$, the quantity satisfying

$$S(r,f) = o(T(r,f)) \text{ as } r \to \infty,\ r \notin E,$$

where E is a subset of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

In addition, in this paper, we also use another type of notation $S^*(r,f)$ which is defined as

$$S^*(r,f) = o(T(r,f)) \text{ as } r \to \infty,\ r \notin E^*,$$

where E^* is a subset of positive real numbers of logarithmic density 0.

Definition 1.1. A meromorphic function $b(z)(\not\equiv 0,\infty)$ defined in $\mathbb{C}$ is called a "small function" with respect to f if $T(r,b(z)) = S(r,f)$.

Definition 1.2. Let k be a positive integer, for any constant a in the complex plane. We denote

i) by $N_{k)}(r,\frac{1}{(f-a)})$ the counting function of a-points of f with multiplicity $\leq k$,

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ii) by $N_{(k}(r, \frac{1}{(f-a)})$ the counting function of a-points of f with multiplicity $\geq k$.

Similarly, the reduced counting functions $\overline{N}_{k)}(r, \frac{1}{(f-a)})$ and $\overline{N}_{(k}(r, \frac{1}{(f-a)})$ are defined.

Definition 1.3 ([5])**.** For a positive integer k, we denote $N_k(r, 0; f)$ the counting function of zeros of f, where a zero of f with multiplicity q is counted q times if $q \leq k$, and is counted k times if $q > k$.

In 1979, E. Mues ([7]) proved that for a transcendental meromorphic function $f(z)$ in $\mathbb{C}$, $f^2 f' - 1$ has infinitely many zeros.

Later, in 1992, Q. Zhang ([9]) proved a quantitative version of Mues's result as follows:

Theorem A. *For a transcendental meromorphic function f, the following inequality holds*:

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2 f' - 1}\right) + S(r, f).$$

In this direction, X. Huang and Y. Gu ([3]) further generalised Theorem A. They proved the following result:

Theorem B. *Let f be a transcendental meromorphic function and k be a positive integer. Then*

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r, f).$$

Next we recall the following definition:

Definition 1.4. Let $q_{0j}, q_{1j}, \ldots, q_{kj}$ be non negative integers. Then the expression

$$M_j[f] = (f)^{q_{0j}}(f^{(1)})^{q_{1j}} \cdots (f^{(k)})^{q_{kj}}$$

is called a differential monomial generated by f of degree $d(M_j) = \sum_{i=0}^{k} q_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)q_{ij}$. The sum

$$P[f] = \sum_{j=1}^{t} b_j M_j[f]$$

is called a differential polynomial generated by f of degree $\overline{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

The differential polynomial $P[f]$ is said to be homogeneous if $\overline{d}(P)$=$\underline{d}(P)$, otherwise, $P[f]$ is called non-homogeneous differential polynomial. The term

$d(P) = \overline{d}(P) = \underline{d}(P)$ is called the degree of the homogeneous differential polynomial $P[f]$.

We also denote by $\nu = \max\{\Gamma_{M_j} - d(M_j) : 1 \le j \le t\} = \max\{q_{1j} + 2q_{2j} + \cdots + kq_{kj} : 1 \le j \le t\}$.

In 2003, I. Lahiri and S. Dewan ([5]) considered the value distribution of a differential polynomial in more general settings. They proved the following theorem.

Theorem C. *Let f be a transcendental meromorphic function and $\alpha = \alpha(z)(\not\equiv 0, \infty)$ be a small function of f. If $\psi = \alpha(f)^n(f^{(k)})^p$, where $n(\ge 0)$ $p(\ge 1)$, $k(\ge 1)$ are integers, then for any small function $a = a(z)(\not\equiv 0, \infty)$ of ψ,*

$$(p+n)T(r,f) \le \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + pN_k(r,0;f) + \overline{N}(r,a;\psi) + S(r,f).$$

The following result is an immediate corollary of the above theorem.

Theorem D. *Let f be a transcendental meromorphic function. Let $l(\ge 3)$, $n(\ge 1)$, $k(\ge 1)$ be positive integers. Then*

$$T(r,f) \le \frac{1}{l-2}\overline{N}\left(r, \frac{1}{f^l(f^{(k)})^n - 1}\right) + S(r,f).$$

Since $f^l(f^{(k)})$ is a specific form of a differential monomial, so the following questions are natural:

Question 1.1 ([1]). Does there exist positive constants $B_1(>0)$, $B_2(>0)$ such that the following two inequalities hold?

i) $T(r,f) \le B_1\ N\left(r, \frac{1}{M[f]-c}\right) + S(r,f)$,

ii) $T(r,f) \le B_2\ \overline{N}\left(r, \frac{1}{M[f]-c}\right) + S(r,f)$,

where $M[f]$ is any differential monomial as defined above generated by a transcendental meromorphic function f and c is any non zero constant.

In connection to the above questions, the following theorems are given in ([1]).

Theorem E ([1])**.** *Let f be a transcendental meromorphic function defined in $\mathbb{C}$, and $M[f] = a(f)^{q_0}(f')^{q_1} \cdots (f^{(k)})^{q_k}$ be a differential monomial generated by f, where a is a non zero complex constant; and $k(\ge 2)$, $q_0(\ge 2)$, $q_i(\ge 0)$ $(i = 1, 2, \ldots, k-1)$, $q_k(\ge 2)$ are integers. Let $\mu = q_0 + q_1 + \cdots + q_k$ and $\mu_* = q_1 + 2q_2 + \cdots + kq_k$. Then*

$$T(r,f) \le \frac{1}{q_0 - 1}N\left(r, \frac{1}{M[f]-1}\right) + S^*(r,f), \tag{1}$$

where $S^*(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$, E is a set of logarithmic density 0.

Thus Theorem E improves, extends and generalizes Theorem D.

Theorem F ([1]). *Let f be a transcendental meromorphic function defined in $\mathbb{C}$, and $M[f] = a(f)^{q_0}(f')^{q_1}\cdots(f^{(k)})^{q_k}$ be a differential monomial generated by f, where a is a non zero complex constant; and $k(\geq 1)$, $q_0(\geq 1)$, $q_i(\geq 0)$ $(i = 1, 2, \ldots, k-1)$, $q_k(\geq 1)$ are integers. Let $\mu = q_0 + q_1 + \cdots + q_k$ and $\mu_* = q_1 + 2q_2 + \cdots + kq_k$. If $\mu - \mu_* \geq 3$, then*

$$T(r,f) \leq \frac{1}{\mu - \mu^* - 2}\overline{N}\left(r, \frac{1}{M[f]-1}\right) + S(r,f), \tag{2}$$

where $S(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$, E is a set of finite linear measure.

Theorem G ([1]). *Let f be a transcendental meromorphic function defined in $\mathbb{C}$, and $M[f] = a(f)^{q_0}(f')^{q_1}\cdots(f^{(k)})^{q_k}$ be a differential monomial generated by f, where a is a non zero complex constant; and $k(\geq 1)$, $q_0(\geq 1)$, $q_i(\geq 0)$ $(i = 1, 2, \ldots, k-1)$, $q_k(\geq 1)$ are integers. Let $\mu = q_0 + q_1 + \cdots + q_k$ and $\mu_* = q_1 + 2q_2 + \cdots + kq_k$. If $\mu - \mu_* \geq 5 - q_0$, then*

$$T(r,f) \leq \frac{1}{\mu - \mu^* - 4 + q_0}\overline{N}\left(r, \frac{1}{M[f]-1}\right) + S(r,f). \tag{3}$$

The aim of this paper is to extend "Theorem D–Theorem G" for homogeneous differential polynomials.

2. Main results

Theorem 2.1. *Let f be a transcendental meromorphic function and $P[f]$ be a homogeneous differential polynomial generated by f. Let $k(\geq 2)$ be the highest order of the derivative of f in $P[f]$. If $q_{0j}(\geq 2)$, $q_{ij}(\geq 0)$ $(i = 1, 2, \ldots, k-1)$, $q_{kj}(\geq 2)$ are integers for $1 \leq j \leq t$, then either*

i) $P[f] \equiv 0$, *or*

ii) $T(r,f) \leq \frac{1}{q_0-1}N\left(r, \frac{1}{P[f]-1}\right) + S^*(r,f)$,

where $S^*(r,f) = o(T(r,f))$ as $r \to \infty$ and $r \notin E$, E is a set of logarithmic density 0.

Example 2.1. Let us take $f(z) = e^z$ and

$$P[f] = f^2\left(f'(f'')^2(f^{(3)})^2 - (f')^2 f''(f^{(3)})^2\right).$$

Then $P[f] \equiv 0$. Thus the condition (i) in Theorem 2.1 is necessary.

Remark 2.1. If $M[f]$ is differential monomial and f is a transcendental meromorphic function, then $M[f] \not\equiv 0$. Thus for the case of a differential monomial, the condition (i) in Theorem 2.1 is redundant.

Remark 2.2. Under the suppositions of Theorem 2.1, it is clear that either $P[f]$ is identically zero or the equation $P[f] = 1$ has infinitely many solutions.

Theorem 2.2. *Let f be a transcendental meromorphic function. Also, let $P[f]$ be a homogeneous differential polynomial generated by f, and of degree $d(P)$. Let $k(\geq 1)$ be the highest order of the derivative of f in $P[f]$ and $q_{0j}(\geq 1)$,*

$q_{ij}(\geq 0)$ $(i = 1, 2, \ldots, k-1)$, $q_{kj}(\geq 1)$ *be integers for* $1 \leq j \leq t$. *If* $d(P) - \nu > 2$, *then either*

i) $P[f] \equiv 0$ *or,*

ii) $T(r, f) \leq \frac{1}{d(P)-\nu-2}\overline{N}(r, \frac{1}{P[f]-1}) + S(r, f)$,

where $S(r, f) = o(T(r, f))$ *as* $r \to \infty$ *and* $r \notin E$, E *is a set of finite linear measure.*

Example 2.2. Let us take $f(z) = e^{-z}$ and

$$P[f] = f^6\left(f'f^{(3)} + f''f^{(3)}\right).$$

Then $P[f] \equiv 0$. Thus the condition (i) in Theorem 2.2 is necessary.

Remark 2.3. If $M[f]$ is differential monomial and f is a transcendental meromorphic function, then $M[f] \not\equiv 0$. Thus for the case of a differential monomial, the condition (i) in Theorem 2.2 is redundant.

Remark 2.4. Under the suppositions of Theorem 2.2, it is clear that either $P[f]$ is identically zero or the equation $P[f] = 1$ has infinitely many solutions.

Theorem 2.3. *Let* f *be a transcendental meromorphic function. Also, let* $P[f]$ *be a homogeneous differential polynomial generated by* f *and of degree* $d(P)$. *Let* $k(\geq 1)$ *be the highest order of the derivative of* f *in* $P[f]$ *and* $q_{0j}(\geq 1)$, $q_{ij}(\geq 0)$ $(i = 1, 2, \ldots, k-1)$, $q_{kj}(\geq 1)$ *be integers for* $1 \leq j \leq t$. *If* $d(P) + kq_* > 2(k+1) + \nu$, *then either*

i) $P[f] \equiv 0$ *or,*

ii) $T(r, f) \leq \frac{(k+1)}{d(P)+kq_*-\nu-2(k+1)}\overline{N}(r, \frac{1}{P[f]-1}) + S(r, f)$,

where $S(r, f) = o(T(r, f))$ *as* $r \to \infty$ *and* $r \notin E$, E *is a set of finite linear measure and* $q_* = \min\limits_{j=1}^{t}\{q_{0j}\}$.

Example 2.3. Let us take $f(z) = e^{z}$ and

$$P[f] = f^5(f')^3 - f^3(f')^5.$$

Then $P[f] \equiv 0$. Thus the condition (i) in Theorem 2.3 is necessary.

Remark 2.5. If $M[f]$ is differential monomial and f is a transcendental meromorphic function, then $M[f] \not\equiv 0$. Thus for the case of a differential monomial, the condition (i) in Theorem 2.3 is redundant.

Remark 2.6. Under the suppositions of Theorem 2.3, it is clear that either $P[f]$ is identically zero or the equation $P[f] = 1$ has infinitely many solutions.

3. Lemmas

Lemma 3.1 ([4])**.** *For a non constant meromorphic function* g, *the following equality holds*:

$$N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) = \overline{N}(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right).$$

The next lemma plays the major role to prove Theorem 2.1, which is an immediate corollary of Yamanoi's Celebrated Theorem ([8]). Yamanoi's Theorem is a correspondent result to the famous Gol'dberg Conjecture.

Lemma 3.2 ([8])**.** *Let f be a transcendental meromorphic function in $\mathbb{C}$ and let $k(\geq 2)$ be an integer. Then*

$$(k-1)\overline{N}(r,f) \leq N\left(r, \frac{1}{f^{(k)}}\right) + S^*(r,f),$$

where $S^*(r,f) = o(T(r,f))$ as $r \to \infty$ and $r \notin E$, E is a set of logarithmic density 0*.*

Lemma 3.3. *Let f be a transcendental meromorphic function and $b(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If $P[f]$ is a homogeneous differential polynomial of degree $d(P)$ and $q_{0j}(\geq 1)$, then either $P[f] \equiv 0$ or, $b(z)P[f]$ can not be a non zero constant.*

Proof. Let us assume that

$$b(z)P[f] \equiv C \tag{4}$$

for some constant C.

If $C = 0$, then there is nothing to prove. Thus we assume that $C \neq 0$. Then from equation (4) and Lemma of logarithmic derivative, we have

$$d(P)m(r, \frac{1}{f}) = m\left(r, \frac{b(z)P[f]}{Cf^{d(P)}}\right) = S(r,f). \tag{5}$$

Since $q_{0j} \geq 1$, it is clear from equation (4) that

$$N(r,0;f) \leq N(r,0;P[f]) + S(r,f) = S(r,f). \tag{6}$$

Thus $T(r,f) = S(r,f)$, which is absurd since f is a transcendental meromorphic function. □

Lemma 3.4. *Let f be a transcendental meromorphic function. Then*

$$T\biggl(r, b(z)P[f]\biggr) = O(T(r,f)) \text{ and } S\biggl(r, b(z)P[f]\biggr) = S(r,f).$$

Proof. The proof is similar to the proof of Lemma 2.4 of ([6]). □

Lemma 3.5. *Let f be a transcendental meromorphic function and $b(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If $P[f]$ is a homogeneous differential polynomial of degree $d(P)$ and $q_{0j}(\geq 1)$, then either $P[f] \equiv 0$, or*

$$d(P)T(r,f) \leq d(P)N\left(r, \frac{1}{f}\right) + \overline{N}(r,\infty;f) + N\left(r, \frac{1}{b(z)P[f]-1}\right) - N\left(r, \frac{1}{(b(z)P[f])'}\right) + S(r,f).$$

Proof. Assume that $P[f] \not\equiv 0$. Then by Lemma 3.3, we have $b(z)P[f]$ can not be a non zero constant. Thus we can write

$$\frac{1}{f^{d(P)}} = \frac{b(z)P[f]}{f^{d(P)}} - \frac{(b(z)P[f])'}{b(z)P[f]}\frac{b(z)P[f]}{f^{d(P)}}\frac{(b(z)P[f]-1)}{(b(z)P[f])'}.$$

Thus in view of first fundamental theorem, Lemma 3.4 and Lemma 3.1, we have

$$\begin{aligned}(7)\qquad & d(P)m(r,\frac{1}{f}) \\ & \leq m\left(r,\frac{b(z)P[f]-1}{(b(z)P[f])'}\right) + S(r,f) \\ & \leq T\left(r,\frac{b(z)P[f]-1}{(b(z)P[f])'}\right) - N\left(r,\frac{b(z)P[f]-1}{(b(z)P[f])'}\right) + S(r,f) \\ & \leq N\left(r,\frac{(b(z)P[f])'}{b(z)P[f]-1}\right) - N\left(r,\frac{b(z)P[f]-1}{(b(z)P[f])'}\right) + S(r,f) \\ & \leq \overline{N}(r,\infty;f) + N\left(r,\frac{1}{b(z)P[f]-1}\right) - N\left(r,\frac{1}{(b(z)P[f])'}\right) + S(r,f).\end{aligned}$$

Thus

$$\begin{aligned} d(P)T(r,f) \leq d(P)N\left(r,\frac{1}{f}\right) + \overline{N}(r,\infty;f) + N\left(r,\frac{1}{b(z)P[f]-1}\right) \\ - N\left(r,\frac{1}{(b(z)P[f])'}\right) + S(r,f).\end{aligned}$$

□

Lemma 3.6. *Let f be a transcendental meromorphic function and $b(z)(\not\equiv 0,\infty)$ be a small function with respect to f. If $P[f]$ is a homogeneous differential polynomial of degree $d(P)$ and $q_{0j}(\geq 1)$, $q_{kj}(\geq 1)$, then either $P[f]\equiv 0$, or*

$$\begin{aligned}(8)\qquad & d(P)T(r,f) \\ & \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \nu\overline{N}_{(k+1}\left(r,\frac{1}{f}\right) + \left(d(P)-q_*\right)N_{k)}\left(r,\frac{1}{f}\right) \\ & \quad + \overline{N}\left(r,\frac{1}{P[f]-1}\right) - N_0\left(r,\frac{1}{(P[f])'}\right) + S(r,f),\end{aligned}$$

where $N_0\left(r,\frac{1}{(P[f])'}\right)$ is the counting function of the zeros of $(P[f])'$ but not the zeros of $f(P[f]-1)$ and $q_* = \min\limits_{j=1}^{t}\{q_{0j}\}$.

Proof. Clearly

$$\begin{aligned}(9)\qquad & d(P)N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P[f]-1}\right) - N\left(r,\frac{1}{(P[f])'}\right) \\ & = d(P)N\left(r,\frac{1}{f}\right) + \overline{N}(r,1;P[f]) + \sum_{t=2}^{\infty}\overline{N}_{(t}(r,1;P[f]) - N(r,0;(P[f])')\end{aligned}$$

$$\leq d(P)N\left(r,\frac{1}{f}\right)-N_\star\left(r,\frac{1}{(P[f])'}\right)+\overline{N}\left(r,\frac{1}{P[f]-1}\right)-N_0\left(r,\frac{1}{(P[f])'}\right),$$

where $N_\star\left(r,\frac{1}{(P[f])'}\right)$ is the counting function of the zeros of $(P[f])'$ which comes from the zeros of f.

Let z_0 be a zero of f with multiplicity q such that $b_j(z_0)\neq 0,\infty$.

Case-1 If $q\leq k$, then z_0 is the zero of $(P[f])'$ of order at least qq_*-1.

Case-2 If $q\geq k+1$, then z_0 is the zero of $(P[f])'$ of order at least

$$\begin{aligned}&\min_{1\leq j\leq t}\{q_{0j}q+q_{1j}(q-1)+\cdots+q_{kj}(q-k)\}-1\\&=qd(P)-\max_{1\leq j\leq t}\{q_{1j}+2q_{2j}+\cdots+kq_{kj}\}-1\\&=qd(P)-\nu-1,\end{aligned}$$

where

$$qd(P)-\nu-1\geq(kd(P)-\nu)+(d(P)-1)>0.$$

Thus

$$\begin{aligned}d(P)N\left(r,\frac{1}{f}\right)\leq{}&N_\star\left(r,\frac{1}{(P[f])'}\right)+(\nu+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)\\&+(d(P)-q_*)N_{k)}\left(r,\frac{1}{f}\right)+\overline{N}_{k)}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}\tag{10}$$

Assume that $P[f]\not\equiv 0$. Now using Lemma 3.5 and the inequalities (9), (10), we have

$$\begin{aligned}&d(P)T(r,f)\\&\leq d(P)N\left(r,\frac{1}{f}\right)+\overline{N}(r,\infty;f)+N\left(r,\frac{1}{b(z)P[f]-1}\right)\\&\quad-N\left(r,\frac{1}{(b(z)P[f])'}\right)+S(r,f)\\&\leq N_\star\left(r,\frac{1}{(P[f])'}\right)+(\nu+1)\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)+(d(P)-q_*)N_{k)}\left(r,\frac{1}{f}\right)\\&\quad+\overline{N}_{k)}\left(r,\frac{1}{f}\right)+\overline{N}(r,\infty;f)+N\left(r,\frac{1}{b(z)P[f]-1}\right)\\&\quad-N\left(r,\frac{1}{(b(z)P[f])'}\right)+S(r,f)\\&\leq\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\nu\overline{N}_{(k+1}\left(r,\frac{1}{f}\right)+(d(P)-q_*)N_{k)}\left(r,\frac{1}{f}\right)\\&\quad+\overline{N}\left(r,\frac{1}{P[f]-1}\right)-N_0\left(r,\frac{1}{(P[f])'}\right)+S(r,f).\end{aligned}$$

□

4. Proof of the Theorems

Proof of Theorem 2.1. Given that f is a transcendental meromorphic function and $k \geq 2$, $q_{0j} \geq 2$, $q_{kj} \geq 2$ for $1 \leq j \leq t$. Let $P[f] \not\equiv 0$. Now

$$(q_0 - 1)N(r, 0; f) + (q_k - 1)N(r, 0; f^{(k)}) \leq N(r, 0; (P[f])'). \tag{11}$$

Now in view of equation (11) and Lemmas 3.2 and 3.5, we have

$$\begin{aligned}
&d(P)T(r, f) \\
&\leq d(P)N\left(r, \frac{1}{f}\right) + \overline{N}(r, \infty; f) + N\left(r, \frac{1}{b(z)P[f] - 1}\right) \\
&\quad - N\left(r, \frac{1}{(b(z)P[f])'}\right) + S(r, f) \\
&\leq \left(d(P) - (q_0 - 1)\right) N\left(r, \frac{1}{f}\right) + \overline{N}(r, \infty; f) + N\left(r, \frac{1}{b(z)P[f] - 1}\right) \\
&\quad - (q_k - 1)N(r, 0; f^{(k)}) + S(r, f) \\
&\leq \left(d(P) - (q_0 - 1)\right) N\left(r, \frac{1}{f}\right) + \left(1 - (k-1)(q_k - 1)\right) \overline{N}(r, \infty; f) \\
&\quad + N\left(r, \frac{1}{b(z)P[f] - 1}\right) + S^*(r, f) + S(r, f) \\
&\leq (d(P) - q_0 + 1)T(r, f) + N\left(r, \frac{1}{b(z)P[f] - 1}\right) + S^*(r, f),
\end{aligned}$$

i.e.,

$$T(r, f) \leq \frac{1}{q_0 - 1} N\left(r, \frac{1}{P[f] - 1}\right) + S^*(r, f).$$

This completes the proof. □

Proof of Theorem 2.2. If $P[f] \not\equiv 0$, then using Lemma 3.6, we have

$$\begin{aligned}
&d(P)T(r, f) \\
&\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \nu \overline{N}_{(k+1}\left(r, \frac{1}{f}\right) + (d(P) - q_*)N_{k)}\left(r, \frac{1}{f}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{P[f] - 1}\right) - N_0\left(r, \frac{1}{(P[f])'}\right) + S(r, f).
\end{aligned}$$

That is,

$$\begin{aligned}
d(P)T(r, f) &\leq 2T(r, f) + (\nu - d(P) + q_*)\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) \\
&\quad + (d(P) - q_*)\left(\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) + N_{k)}\left(r, \frac{1}{f}\right)\right) \\
&\quad + \overline{N}\left(r, \frac{1}{P[f] - 1}\right) - N_0\left(r, \frac{1}{(P[f])'}\right) + S(r, f)
\end{aligned} \tag{12}$$

$$\leq (\nu+2)T(r,f) + \overline{N}\left(r, \frac{1}{P[f]-1}\right) + S(r,f),$$

i.e.,

$$T(r,f) \leq \frac{1}{d(P)-\nu-2}\overline{N}\left(r, \frac{1}{P[f]-1}\right) + S(r,f).$$

This completes the proof. □

Proof of Theorem 2.3 . If $P[f] \not\equiv 0$, then from the inequality (12), we have

$$\begin{aligned}
&(q_*-2)T(r,f) \\
\leq\ &(\nu - d(P) + q_*)\overline{N}_{(k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P[f]-1}\right) \\
&- N_0\left(r, \frac{1}{(P[f])'}\right) + S(r,f) \\
\leq\ &\frac{\nu - d(P) + q_*}{k+1}T(r,f) + \overline{N}\left(r, \frac{1}{P[f]-1}\right) + S(r,f),
\end{aligned}$$

i.e.,

$$T(r,f) \leq \frac{(k+1)}{d(P)+kq_* - \nu - 2(k+1)}\overline{N}\left(r, \frac{1}{P[f]-1}\right) + S(r,f).$$

This completes the proof. □

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