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COMPUTATION OF THE MATRIX OF THE TOEPLITZ OPERATOR ON THE HARDY SPACE

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Abstract. The matrix representation of the Toeplitz operator on the Hardy space with respect to a generalized orthonormal basis for the space of square integrable functions associated to a bounded simply connected region in the complex plane is completely computed in terms of only the Szegő kernel and the Garabedian kernels.

1. Introduction

Some progresses for Toeplitz operators on the Hardy spaces associated to general bounded regions have been recently made by virtue of the author (see [4–6]). The author constructed an orthonormal basis for the space of square integrable functions on bounded domains in the complex plane for which the Laurent operators and the Toeplitz operators were classified in terms of their corresponding matrices. Moreover he computed in an abstract form the matrices of the Laurent operators and the Toeplitz operators relative to the Fourier series expansion of the symbol involving the inner products of functions in the basis. As the special case of the unit disc, an orthonormal basis has been generalized with only rational functions and by using this basis, several algebraic properties such as computing problems of the Toeplitz operators on the Hardy space have been proved in this general setting.

On the other hand, the matrix of the Toeplitz operator is very hard to compute in general even in simply connected regions. So I would like to find a compact form of the matrix of the operator having no such inner products of elements of functions in the orthonormal basis. More specifically I want to write the matrix in terms of functions which consist of only the Szegő kernels and analytic parts of the Garabedian kernels.

In Section 2, we give several notations to be used in this paper and introduce previous results and history. And we in Section 3 work on the main results expressing the matrix of the Toeplitz operator.

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2. Preliminaries and notes

Throughout the paper, we assume that Ω is a simply connected bounded region with C^∞ smooth boundary and the base point a in Ω is fixed unless otherwise specified.

Let $L^2(b\Omega)$ be the space of square integrable functions on the boundary $b\Omega$ and let $H^2(b\Omega)$ be the classical Hardy space which consist of holomorphic functions on Ω with boundary values in $L^2(b\Omega)$. Then there exists a well-known orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$ which is called the Szegő projection denoted by P.

The Toeplitz operator T_φ with symbol φ belong to the space $L^\infty(b\Omega)$ of the essentially bounded Lebesgue-measurable functions is the bounded operator on the Hardy space $H^2(b\Omega)$ defined by

$$T_\varphi(h) = P(\varphi h),\ h \in H^2(b\Omega),$$

namely, the composite operator of the Szegő projection with the Laurent operator restricted to the Hardy space. What is then the matrix representation of the Toeplitz operator with respect to an orthonormal basis on $H^2(b\Omega)$?

Suppose that U is the unit disc. It is very easy to see that the class $\mathcal{L}_0(bU)$ of monomials and their reciprocals $\frac{1}{\sqrt{2\pi}}z^p$ for $p \in \mathbb{Z}$ is an orthonormal basis for $L^2(b\Omega)$ with respect to the inner product

$$\langle u, v\rangle = \int_{bU} u\overline{v}\, ds,$$

where ds is the differential element of arc length on the boundary bU of U and in particular, the monomials $\frac{1}{\sqrt{2\pi}}z^p$ with nonnegative integers p form an orthonormal basis for the Hardy space $H^2(b\Omega)$ which is denoted by $\mathcal{H}_0(bU)$. Moreover for the Fourier series representation $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p \frac{1}{\sqrt{2\pi}} z^p$ of φ, the (m,l)-th entry of the matrix $\mathcal{M}_0(bU)$ of the Toeplitz operator T_φ with respect to the basis $\mathcal{H}_0(bU)$ is given by

$$\mathcal{M}_0(bU)_{ml} = \alpha_{m-l}, \quad m, l \geq 0$$

which becomes a Toeplitz matrix of order 1 in the sense that each entry is the same as the one with indices increased by one step row-wise and column-wise.

As a generalization of this result, the author proved in [6] that given any point $a \in U$, the class

$$\mathcal{L}_a(bU) := \left\{ \sqrt{\frac{1-|a|^2}{2\pi}} \frac{(z-a)^p}{(1-\overline{a}z)^{p+1}} \mid p \in \mathbb{Z} \right\}$$

of rational functions and the subclass

$$\mathcal{H}_a(bU) := \left\{ \sqrt{\frac{1-|a|^2}{2\pi}} \frac{(z-a)^p}{(1-\overline{a}z)^{p+1}} \mid p \geq 0 \right\}$$

are orthonormal bases for $L^2(bU)$ and $H^2(bU)$, respectively. Furthermore I showed that the matrix $\mathcal{M}_a(bU)$ of the operator T_φ with respect to $\mathcal{H}_a(bU)$ satisfies the identity

$$\mathcal{M}_a(bU)_{ml} = \frac{1}{\sqrt{2\pi(1-|a|^2)}}\alpha_{m-l} + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}}\alpha_{m-l-1}$$

which implies

$$\mathcal{M}_a(bU) = \frac{1}{\sqrt{2\pi(1-|a|^2)}}\mathcal{M}_0(bU) + \frac{\overline{a}}{\sqrt{2\pi(1-|a|^2)}}\mathcal{M}_0(bU)L,$$

where L is the one-way infinite lower shift matrix given by

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Now we want to generalize the case of the unit disc to more general simply connected regions as mentioned early. Suppose that Ω is a simply connected bounded region with C^∞ smooth boundary and let $S(z,w)$ be the reproducing kernel for P which is called the Szegő kernel. And suppose that T is the unit tangent vector function on the boundary $b\Omega$ with positive orientation. Then the function

$$C_a(z) := \overline{\left(\frac{1}{2\pi i}\frac{T(z)}{z-a}\right)}$$

is the Cauchy kernel which reproduces holomorphic functions on Ω in $C^\infty(\overline{\Omega})$ by means of the Cauchy integral formula. The function

$$G(z,a) := \frac{1}{2\pi(z-a)} + P(\overline{iC_aT})(z)$$

is called the Garabedian kernel which is a reproducing kernel for the orthogonal complement $P^\perp$ in some sense. An important relation between the Szegő kernel and the Garabedian kernel which is often used in the paper as a key element is given by

$$G(z,a) = i\,\overline{S(z,a)}\,\overline{T(z)}, \quad (z,a)\in \mathrm{b}\Omega\times\Omega \tag{2.1}$$

(see [2] for details). Finally the quotient of two kernels above

$$f_a(z) := \frac{S(z,a)}{G(z,a)}$$

is called the Riemann mapping function associated to a which is a conformal mapping of Ω onto U, satisfying the extremal problem

$$f_a'(a) = \max\{h : \Omega \to U \mid h \text{ is holomorphic in } \Omega, h(a)=0, h'(a)\in\mathbb{R}\}.$$

3. Main results

In this section, we compute the matrix representation of the Toeplitz operator on the Hardy space associated to the given simply connected region Ω with respect to an orthonormal basis of the Hardy space and the given symbol. For simplicity, we use the shorthand notations $S_a = S(\cdot, a)$ and $G_a = G(\cdot, a)$ as functions of the first variables.

First we need the following proposition proved by the author [4].

Proposition 3.1. *Let Ω be a simply connected bounded region with C^∞ smooth boundary and let a be fixed in Ω. Then the class*

$$\mathcal{L}_a(b\Omega) := \{E_k \,|\, k \geq 0\} \cup \{E_{-k} \,|\, k \geq 1\}$$

is an orthonormal basis for $L^2(b\Omega)$ and the subclass $\mathcal{H}_a(b\Omega) := \{E_k \,|\, k \geq 0\}$ is an orthonormal basis for $H^2(b\Omega)$, where $E_k = \sigma S_a {f_a}^k$ for $k \geq 0$, $E_{-k} = \sigma G_a \overline{f_a}^{-k+1}$ for $k \geq 1$, $\sigma = 1/\sqrt{S(a,a)}$.

In order to compute entries of the matrix of the Toeplitz operator T_φ, we have to work on the inner product of the form $\langle E_k E_m, E_l \rangle$ for arbitrary integer k and nonnegative integers m and l.

Lemma 3.2. *Let k be a positive integer. Then*

$$\langle E_k E_0, E_0 \rangle = 0.$$

Proof. Note that for a holomorphic function $h \in H^2(b\Omega)$,

$$h(a) = \langle h, S_a \rangle.$$

It is then obvious because

$$\langle E_k E_0, E_0 \rangle = \sigma^3 \langle S_a {f_a}^k S_a, S_a \rangle = \sigma^3 {S_a(a)}^2 {f_a(a)}^k = 0. \qquad \square$$

Lemma 3.3.

$$\langle E_0 E_0, E_0 \rangle = \sigma^3 S_a(a)^2.$$

Proof. It is also obvious because

$$\langle E_0 E_0, E_0 \rangle = \sigma^3 \langle {S_a}^2, S_a \rangle = \sigma^3 {S_a(a)}^2. \qquad \square$$

Observe that the Garabedian kernel G_a is a meromorphic function on Ω with a simple single pole at a with residue $1/(2\pi)$. Hence the function $\widetilde{G_a}$ defined by

$$\widetilde{G_a}(z) := G(z,a) - \frac{1}{2\pi(z-a)} \tag{3.1}$$

is the analytic part of the kernel $G(z,a)$ at $z = a$.

Lemma 3.4.

$$\langle E_{-1} E_0, E_0 \rangle = \frac{\sigma^3}{2\pi} S_a'(a) + 2\sigma^3 S_a(a) \widetilde{G_a}(a). \tag{3.2}$$

Proof.

$$\langle E_{-1}E_0, E_0\rangle = \sigma^3\langle G_aS_a, S_a\rangle = \sigma^3\int_{b\Omega} G_aS_a\overline{S_a}\;ds.$$

It is then from the identities (2.1) and $ds = \overline{T}dz$ that the identity above equals

$$-\,i\,\sigma^3\int_{b\Omega} {G_a}^2 S_a\;dz. \tag{3.3}$$

It is easy to see from (3.1) that the residue of the integrand ${G_a}^2S_a$ at a is equal to

$$\frac{1}{4\pi^2}{S_a}'(a) + \frac{1}{\pi}S_a(a)\widetilde{G_a}(a)$$

and it hence follows from the Residue theorem that the above identity (3.3) is the same as the one (3.2) which proves Lemma 3.4. □

Lemma 3.5.

$$\langle E_{-2}E_0, E_0\rangle = 3\,\sigma^3\,\widetilde{G_a}^2(a) + \frac{3}{2\pi}\sigma^3\widetilde{G_a}'(a). \tag{3.4}$$

Proof. The proof is very similar to the one of the previous lemma. In fact, it follows from the formula $f_a = S_a/G_a$ that

$$\langle E_{-2}E_0, E_0\rangle = \sigma^3\langle G_a{f_a}^{-1}S_a, S_a\rangle = -i\,\sigma^3\int_{b\Omega} {G_a}^3\;dz. \tag{3.5}$$

Note from (3.1) that the residue of ${G_a}^3$ at a is equal to

$$\frac{3}{4\pi^2}\widetilde{G_a}'(a) + \frac{3}{2\pi}\widetilde{G_a}(a)^2.$$

Thus by the Residue theorem we can rewrite the above identity (3.5) as the one (3.4) in Lemma 3.5. □

Now we work on general k's with $k \geq 3$. We first introduce the higher order derivative rule of the reciprocal which is well-known.

Lemma 3.6 ([7])**.** *Let n be a nonnegative integer and let h be a holomorphic function at a point z with $h(z) \neq 0$. Then the nth derivative of the reciprocal $1/h$ at z is given by*

$$\left(\frac{1}{h}\right)^{(n)}(z) = \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\frac{1}{h(z)^{k+1}}\left(h^k\right)^{(n)}(z).$$

Using Lemma 3.6 and the Leibnitz rule, we can obtain the rule of the higher order derivative for the quotient of two holomorphic functions as follows.

Lemma 3.7. *Let n be a nonnegative integer and let g, h be holomorphic functions at a point z with $h(z) \neq 0$. Then the n-th derivative of the quotient g/h at z is given by*

$$\left(\frac{g}{h}\right)^{(n)}(z) = \sum_{k=0}^{n}\binom{n}{k}g^{(k)}(z)\sum_{r=0}^{n-k}(-1)^r\binom{n-k+1}{r+1}\frac{1}{h(z)^{r+1}}\left(h^r\right)^{(n-k)}(z).$$

Going back to our work, we compute the inner product of the triple (E_k, E_0, E_0) for $k \geq 3$. Similarly as in the previous lemmas, we get the identities

$$\begin{aligned}\langle E_{-k}E_0, E_0\rangle &= \sigma^3 \langle G_a {f_a}^{-k+1} S_a, S_a\rangle \\ &= \sigma^3 \int_{b\Omega} G_a \frac{{G_a}^{k-1}}{{S_a}^{k-1}} S_a \overline{S_a}\ ds \\ &= \sigma^3 \int_{b\Omega} \frac{{G_a}^{k}}{{S_a}^{k-2}} (-iG_aT)\overline{T}\ dz \\ &= -i\,\sigma^3 \int_{b\Omega} \frac{{G_a}^{k+1}}{{S_a}^{k-2}}\ dz \\ &= 2\pi\sigma^3 \operatorname{Res}\left(\frac{{G_a}^{k+1}}{{S_a}^{k-2}}; a\right).\end{aligned}$$

Note from the binomial formula that

$$\frac{{G_a}^{k+1}}{{S_a}^{k-2}} = \sum_{j=0}^{k+1} \binom{k+1}{j} \left(\frac{1}{2\pi(z-a)}\right)^{k+1-j} \frac{\widetilde{G_a}^{j}}{{S_a}^{k-2}}.$$

It turns out from the definition of residue that

$$\operatorname{Res}\left(\frac{{G_a}^{k+1}}{{S_a}^{k-2}}; a\right) = \sum_{j=0}^{k} \binom{k+1}{j} \frac{1}{(k-j)!} \frac{1}{(2\pi)^{k+1-j}} \left(\frac{\widetilde{G_a}^{j}}{{S_a}^{k-2}}\right)^{(k-j)} (a). \tag{3.6}$$

Notice that the index $k+1$ for j in the summation is removed because S_a does not have a zero at a.

Hence we have proved the following formula from Lemma 3.7.

Lemma 3.8. *Let k be an integer with $k \geq 3$. Then*

$$\begin{aligned}&\langle E_{-k}E_0, E_0\rangle \\ &= \sigma^3 \sum_{j=0}^{k} \binom{k+1}{j} \frac{1}{(k-j)!} \frac{1}{(2\pi)^{k-j}} \sum_{r=0}^{k-j} \binom{k-j}{r} \left(\widetilde{G_a}^{j}\right)^{(r)}(a) \\ &\quad \cdot \sum_{l=0}^{k-j-r} (-1)^l \binom{k-j-r+1}{l+1} \frac{1}{S_a(a)^{(k-2)(l+1)}} \left({S_a}^{l(k-2)}\right)^{(k-j-r)}(a).\end{aligned}$$

We are now ready to compute the matrix of the Toeplitz operator. Let

$$\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$$

be the Fourier series representation with respect to the orthonormal basis $\mathcal{L}_a(b\Omega)$ in $L^\infty(b\Omega)$. It is proved that the matrix representation $\mathcal{M}_a(b\Omega)$ of the operator T_φ with respect to the basis $\mathcal{H}_a(b\Omega)$ is a Toeplitz matrix of order 1 (see [4]). In fact, for $m, l \geq 0$, it follows from analyticity of E_p for $p \geq 0$ that

$$\begin{aligned}\langle T_\varphi(E_{m+1}), E_{l+1}\rangle &= \langle P(\varphi E_{m+1}), E_{l+1}\rangle \\ &= \langle \varphi E_{m+1}, E_{l+1}\rangle \\ &= \sigma^2 \langle \varphi S_a {f_a}^{m+1}, S_a {f_a}^{l+1}\rangle\end{aligned}$$

which from the identity $|f_a(z)|= 1$ for $z \in b\Omega$ yields

$$\sigma^2 \langle \varphi S_a {f_a}^{m}, S_a {f_a}^{l}\rangle,$$

which is reversely exactly the same as

$$\langle T_\varphi(E_m), E_l\rangle.$$

It then follows that for $m \geq l \geq 0$,

$$\mathcal{M}_a(b\Omega)_{ml} = \mathcal{M}_a(b\Omega)_{m-l,0} = \sum_{p=-\infty}^{\infty} \alpha_p \langle E_p E_0, E_{m-l}\rangle. \tag{3.7}$$

On the other hand, as in the above remark, one can show that for any integers $p_i, q_i, r_i,\ i = 1, 2,$

$$\langle E_{p_1} E_{q_1}, E_{r_1}\rangle = \langle E_{p_2} E_{q_2}, E_{r_2}\rangle$$

provided $p_1 + q_1 - r_1 = p_2 + q_2 - r_2$. (See Theorem 4.3 of [5] for more general results.) Thus the identity (3.7) is equal to

$$\sum_{p=-\infty}^{\infty} \alpha_p \langle E_{p-m+l} E_0, E_0\rangle,$$

which from Lemma 3.2 is identical to the negative one-way summation

$$\sum_{p=-\infty}^{m-l} \alpha_p \langle E_{p-m+l} E_0, E_0\rangle. \tag{3.8}$$

Finally by applying explicit formulas obtained by Lemmas 3.3, 3.4, 3.5 and Lemma 3.8 to the above identity (3.8), we have the following main result.

Theorem 3.9. *Suppose that Ω is a bounded simply connected region with C^∞ smooth boundary. Let a be fixed in Ω and let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be the Fourier series representation with respect to the orthonormal basis $\mathcal{L}_a(b\Omega)$ in $L^\infty(b\Omega)$. For nonnegative integers m, l, the m-th and l-th entry of the matrix $\mathcal{M}_a(b\Omega)$ of the Toeplitz operator T_φ on the Hardy space $H^2(b\Omega)$ with respect to the orthonormal basis $\mathcal{H}_a(b\Omega)$ is given by*

$$\begin{aligned}\frac{1}{\sigma^3}\mathcal{M}_a(b\Omega)_{ml} = \alpha_{m-l} S_a(a)^2 &+ \alpha_{m-l-1}\left(\frac{1}{2\pi} S_a'(a) + 2 S_a(a)\widetilde{G_a}(a)\right) \\ &+ \alpha_{m-l-2}\left(3\widetilde{G_a}^2(a) + \frac{3}{2\pi}\widetilde{G_a}'(a)\right)\end{aligned}$$

$$+\sum_{p=-\infty}^{m-l-3}\alpha_p\Bigg[\sum_{j=0}^{m-l-p}\binom{m-l-p+1}{j}\frac{1}{(m-l-p-j)!}\frac{1}{(2\pi)^{m-l-p-j}}$$
$$\cdot\sum_{r=0}^{m-l-p-j}\binom{m-l-p-j}{r}\left(\widetilde{G_a}^{j}\right)^{(r)}(a)$$
$$\cdot\sum_{l=0}^{m-l-p-j-r}(-1)^l\binom{m-l-p-j-r+1}{l+1}\frac{1}{S_a(a)^{(m-l-p-2)(l+1)}}$$
$$\cdot\left({S_a}^{l(m-l-p-2)}\right)^{(m-l-p-j-r)}(a)\Bigg].$$

Remark 3.10. As a final remark, we show that the induced matrix by the Toeplitz operator can be expressed in terms of the Bell polynomials.

Suppose that n and k be nonnegative integers with $n\geq k$ and let $f_1, f_2, \ldots, f_{n-k+1}$ be indeterminates. The partial exponential Bell polynomial $B_{n,k}$ (see [1,3] for reference) is a sum of polynomials of f_j's defined by

$$B_{n,k}(f_1,f_2,\ldots,f_{n-k+1})$$
$$=\sum_{j_1,j_2,\ldots,j_{n-k+1}}\frac{n!}{j_1!\,j_2!\cdots j_{n-k+1}!}\left(\frac{f_1}{1!}\right)^{j_1}\left(\frac{f_2}{2!}\right)^{j_2}\cdots\left(\frac{f_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum is taken over all $j_i\geq 0$ satisfying $j_1+\cdots+j_{n-k+1}=k,\ j_1+2j_2+3j_3+\cdots+(n-k+1)j_{n-k+1}=n$. It is known that the corresponding formula to the identity in Lemma 3.7 of higher order derivatives of fractions is given by

$$\left(\frac{g}{h}\right)^{(n)}(z)=\sum_{k=0}^{n}\binom{n}{k}g^{(n-k)}(z)\sum_{r=0}^{k}(-1)^r\frac{r!}{h(z)^{r+1}}$$
$$\cdot\ B_{k,r}\left(h(z)^{(1)},h(z)^{(2)},\ldots,h(z)^{(k-r+1)}\right).$$

Similarly as in the Lemma 3.8 and thereafter, we can thus obtain the following result.

Theorem 3.11. *Suppose that Ω is a bounded simply connected region with C^∞ smooth boundary. Let a be fixed in Ω and let $\varphi=\sum_{p=-\infty}^{\infty}\alpha_pE_p$ be the Fourier series representation with respect to the orthonormal basis $\mathcal{L}_a(b\Omega)$ in $L^\infty(b\Omega)$. For nonnegative integers m,l, the m-th and l-th entry of the matrix $\mathcal{M}_a(b\Omega)$ of the Toeplitz operator T_φ on the Hardy space $H^2(b\Omega)$ with respect to the orthonormal basis $\mathcal{H}_a(\Omega)$ is given by*

$$\frac{1}{\sigma^3}\mathcal{M}_a(b\Omega)_{ml}=\alpha_{m-l}S_a(a)^2+\alpha_{m-l-1}\left(\frac{1}{2\pi}S_a'(a)+2S_a(a)\widetilde{G_a}(a)\right)$$
$$+\alpha_{m-l-2}\left(3\widetilde{G_a}^2(a)+\frac{3}{2\pi}\widetilde{G_a}'(a)\right)$$

$$+\sum_{p=-\infty}^{m-l-3}\alpha_p\Bigg[\sum_{j=0}^{m-l-p}\binom{m-l-p+1}{j}\frac{1}{(m-l-p-j)!}\frac{1}{(2\pi)^{m-l-p-j}}$$

$$\cdot\sum_{r=0}^{m-l-p-j}\binom{m-l-p-j}{r}\left(\widetilde{G_a}^{j}\right)^{(m-l-p-j-r)}(a)$$

$$\cdot\sum_{t=0}^{r}(-1)^t\frac{t!}{S_a(a)^{(m-l-p-2)(t+1)}}$$

$$\cdot\ B_{r,t}\Bigg(\left({S_a}^{m-l-p-2}\right)^{(1)}(a),\left({S_a}^{m-l-p-2}\right)^{(2)}(a),\ldots,$$

$$\left({S_a}^{m-l-p-2}\right)^{(r-t+1)}(a)\Bigg)\Bigg].$$

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