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A COMBINATORIAL APPROACH TO ASYMPTOTIC BEHAVIOR OF KIRILLOV MODEL FOR GL_2

Yusuf Danisman

Abstract. We find the asymptotic behavior of Kirillov model for irreducible induced representations of GL_2 by using combinatorial methods.

1. Introduction

Let F be a non-archimedean local field. L-factors with ϵ-factors are crucial to determine the surjective, finite to one map in local Langlands Conjecture uniquely. Langlands has conjectured that there is a close connection between the irreducible admissible representations of a reductive group $G(F)$ and the representations of the Weil-Deligne group in the dual group ${}^{L}G(\mathbb{C})$. The Weil-Deligne group is roughly product of a subgroup of $Gal(\bar{F}/F)$ and $SL_2(\mathbb{C})$ and ${}^{L}G(\mathbb{C})$ is derived from the root system of $G(F)$.

The local Langlands conjecture assigns an L and ϵ-factor to each irreducible representation of $G(F)$ namely L and ϵ-factor of the corresponding representation of Weil-Delign group defined by Deligne and Langlands [7, 10]. Instead of this assignment, to get more information on this correspondence it is better to define these factors of $G(F)$ by attaching an integral representation which would compute these local factors. One way to do this is, as in $GL_n(F)$ case as follows: for generic representations induced from a parabolic subgroup an integral representation is defined by using Whittaker model; then the definition is extended to quotients, i.e., nongeneric representations by Langlands classification [9].

Let us first briefly explain the integral representation for $GL_2(F)$. Let (π, V_π) be an infinite dimensional irreducible representation of $GL_2(F)$ and ψ be a nontrivial additive character of F. A Whittaker functional on V is a linear functional L such that

$$L : V \to \mathbb{C} \text{ and } L(\pi \left[\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right] v) = \psi(n)L(v)$$

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for all $n \in F$. The space $\mathbf{W} = \{W_v(g) = L(\pi(g)v) : g \in GL_2(F), v \in V\}$ is called the Whittaker Model and the space

$$\left\{\phi_v : \phi_v(x) = W_v\left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right] : W_v \in \mathbf{W}, x \in F^\times\right\}$$

is called the Kirillov model of (π, V_π).

Let $v \in V_\pi$ and ϕ_v be an element of the Kirillov model. We associate an integral

$$Z(s, \phi_v) = \int_{F^\times} \phi_v(x)|x|^{s-1/2}\, d^\times y,$$

where $d^\times y$ denotes the Haar measure on $F^\times$. By Proposition 4.7.5 of [1] this integral is convergent for $Re(s)$ large enough and has a meromorphic continuation to all s. Also there exists a polynomial p_{ϕ_v} such that $Z(s, \phi_v) = p_{\phi_v}(q^{-s})L(s, \pi)$, where $L(s, \pi)$ is called the local L-factor and it is an element of $\mathbb{C}[q^s, q^{-s}]$.

To compute $L(s, \pi)$ one needs to find the asymptotic behavior of ϕ_v. In [1], this is done by

1) finding the constituents of the Jacquet module (Theorem 4.5.4 of [1]).
2) determining the splitting type of the Jacquet module (Theorem 4.5.4 of [1]).
3) finding the asymptotic behavior of the Kirillov model up to constants by using (1) and (2) (Proposition 4.7.4 of [1]).
4) determining the space $\{v \in V_\pi : \phi_v = 0\}$ (Proposition 4.4.7).
5) determining the space $\{v \in V_\pi : \phi_v \in C_c^\infty(F^\times)\}$ (Theorem 4.7.1).
6) finding the asymptotic behavior of the Kirillov model explicitly by using all the information above from 1 to 5 (Theorem 4.7.2 [1]).

In this paper, we show that behaviour of Krillov model can be found without full knowledge of the structure of the Jacquet module. By induction our results can be generalized for the higher rank cases. This can be done without the second step above and using combinatorial methods. The results we obtain are not new; they can be found in Proposition 4.7.4 of [1]. The general formula for induced representation of an arbitrary quasi-split group is given in the equation (3.4.2) of [2]. For the matrix group $GSp_4(F)$, the asymptotic behaviour of Bessel model is given but not proved in [11] and proved in [3], [4], [5] and [6] in a similar way for non-split case. However, we have found that our method is very simple.

By Theorem 4.5.4 of [1], the constituents of the Jacquet module, which has length two are known. Without determining, which constituent is subrepresentation or subquotient we will find the asymptotic behavior of the Kirillov model. Our method can be easily generalized to the representations of $GL_n(F)$ and $GSp_4(F)$, even for the Bessel model of the latter one because, considering one more character in a higher rank case, results in two cases, depending on whether the new character already exists or not. By induction, these two cases can be handled. In contrary, there is no simple induction argument to find the full structure of the Jacquet module in a higher rank case. In such a case, the methods in Section 6.3 of the unpublished book Introduction to the Theory of Admissible Representations of p-adic Reductive Groups by William Casselman should be applied as in Section 3.2 of [4].

2. Definitions and preliminaries

In this paper, F will always denote non-archimedean local field of odd characteristic, $\mathcal{O}$ its ring of integers, v_F its valuation, $\mathcal{P}$ the unique maximal prime ideal of $\mathcal{O}$ and ϖ a fixed generator of $\mathcal{P}$. Let q be the cardinality of the residue field of F.

Let

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in F^x, b \in F \right\}$$

be the Borel subgroup of $GL_2(F)$,

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in F^x \right\}$$

be the split torus of $GL_2(F)$ and

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in F \right\}$$

be the unipotent subgroup of $GL_2(F)$.

Let χ_1, χ_2 be characters of $F^\times$ and V be the space of functions $f : GL_2(F) \to \mathbb{C}$ such that

$$f\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right] = \left|\frac{a}{d}\right|^{1/2} \chi_1(a)\chi_2(d)f(g),$$

where $g \in GL_2(F)$ and $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$, and invariant under some open compact subgroup of $GL_2(F)$.

We can define an action of $GL_2(F)$ on V by right translation and denote it by π. Let $V_N = span\{\pi(n)v - v : n \in N, v \in V\}$. V_N is invariant under T and $J(V) = V/V_N$ is a T module. Let π_N be the action of T on $J(V)$. $(\pi_N, J(V))$ is a smooth representation and $J(V)$ is called the Jacquet module of V.

By Theorem 4.5.4 of [1], constituents of $J(V)$ are the characters $\delta^{1/2}\chi$ and $\delta^{1/2}\chi'$ of T, where $\delta\left[\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\right] = \left|\frac{a}{d}\right|^{1/2}$,

$$\chi\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] = \chi_1(a)\chi_2(d) \quad \text{and} \quad \chi'\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] = \chi_1(d)\chi_2(a).$$

3. Asymptotic behavior of Kirillov model

By Theorem 4.5.4 of [1], let $U_1 := \delta^{1/2}\chi$ and $U_2 := \delta^{1/2}\chi'$ be the constituents of $J(V)$. By Frobenius Theorem $\delta^{1/2}\chi'$ is a subrepresentation and $\delta^{1/2}\chi$ is a subquotient, however we will not need that information. We will only use that

U_1 and U_2 are the only constituents of (semisimplification) $J(V)$. Hence we have

$$0 \to U_1 \to J(V) \to U_2 \to 0 \tag{1}$$

or

$$0 \to U_2 \to J(V) \to U_1 \to 0. \tag{2}$$

For $v \in V$, let $\overline{v}$ be the image of v in $J(V)$, $\overline{\overline{v}}$ be the image of v in $J(V)/U_1 \cong U_2$ and $\overline{\pi}_N$ be the representation on $J(V)/U_1$. Also let $h(x) := \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$. Hence by exact equation (1) we have

$$\overline{\pi}_N[h(x)]\overline{\overline{v}} = \delta^{1/2}\chi'[h(x)]\overline{\overline{v}}$$

and

$$\pi_N[h(x)]\overline{v} - \delta^{1/2}\chi'[h(x)]\overline{v} \in U_1$$

so

$$\pi_N[h(x)]\{\pi_N[h(x)]\overline{v} - \delta^{1/2}\chi'[h(x)]\overline{v}\} = \delta^{1/2}\chi[h(x)]\{\pi_N[h(x)]\overline{v} - \delta^{1/2}\chi'[h(x)]\overline{v}\}$$

and we get

$$\pi_N[h(x^2)]\overline{v} - |x|^{1/2}\chi_2(x)\pi_N[h(x)]\overline{v} = |x|^{1/2}\chi_1(x)\pi_N[h(x)]\overline{v} - |x|\chi_1(x)\chi_2(x)\overline{v}.$$

Hence

$$\pi[h(x^2)]v - |x|^{1/2}[\chi_1(x) + \chi_2(x)]\pi[h(x)]v + |x|\chi_1(x)\chi_2(x)v \in V_N. \tag{3}$$

If we follow the same steps above for the exact equation (2) we will also reach to the equation (3). This is the reason why we do not need to know which constituent is the subrepresentation.

As mentioned in the introduction ϕ_v is an element of the Kirillov model of (π, V_π). By equation (3) and a similar proof to that of Proposition 4.7.4 of [1] we have

$$\phi_{\pi[h(x^2)]v-|x|^{1/2}[\chi_1(x)+\chi_2(x)]\pi[h(x)]v+|x|\chi_1(x)\chi_2(x)v}(y) = 0$$

for $x \in \{\varpi\mathcal{O}^\times, \varpi^2\mathcal{O}^\times\}$ and $|y| \leq q^{-m}$ for some $m \in \mathbb{Z}$ depend on v.

Above equation is also equivalent to

$$\phi_v(x^2y) - |x|^{1/2}[\chi_1(x) + \chi_2(x)]\phi_v(xy) + |x|\chi_1(x)\chi_2(x)\phi_v(y) = 0. \tag{4}$$

If $x = \varpi, y = \varpi^k u$ for $k \geq m$ and $u \in \mathcal{O}^\times$, then

$$\phi_v(\varpi^{k+2}u) = q^{-1/2}[\chi_1(\varpi) + \chi_2(\varpi)]\phi_v(\varpi^{k+1}u) - q^{-1}\chi_1(\varpi)\chi_2(\varpi)\phi_v(\varpi^k u).$$

If we fix u and treat $\phi_v(\varpi^k u)$ as a sequence of k, then the equation above is a linear homogeneous recurrence relation with constant coefficients. Hence by Section 2.1 of [8], to find ϕ_v for fixed u we need to look at the roots of the equation

$$T^2 - q^{-1/2}[\chi_1(\varpi) + \chi_2(\varpi)]T + q^{-1}\chi_1(\varpi)\chi_2(\varpi) = 0,$$

which are $q^{-1/2}\chi_1(\varpi)$ and $q^{-1/2}\chi_2(\varpi)$. For simplicity let $A := q^{-1/2}\chi_1(\varpi)$ and $B := q^{-1/2}\chi_2(\varpi)$. Now there are two cases: If $\chi_1(\varpi) \neq \chi_2(\varpi)$, then we have two different roots and

$$\phi_v(\varpi^k u) = C_1^v(u)A^k + C_2^v(u)B^k. \tag{5}$$

If $\chi_1(\varpi) = \chi_2(\varpi)$, then we have a double root and

$$\phi_v(\varpi^k u) = [C_1^v(u) + C_2^v(u)k]A^k, \tag{6}$$

where $C_1^v(u)$ and $C_2^v(u)$ are constants depend on u and v. Now we will consider these two cases separately.

3.1. Case: $\chi_1(\varpi) \neq \chi_2(\varpi)$

In this section, we will prove the following theorem which is a refinement of the equation (5) and the main result of this paper in this case.

Theorem 3.1. *If $\chi_1(\varpi) \neq \chi_2(\varpi)$, then $C_1^v(u) = C_1^v\chi_1(u)$ and $C_2^v(u) = C_2^v\chi_2(u)$ where C_1^v and C_2^v are constants depend only on v. Hence*

$$\phi_v(x) = C_1^v|x|^{1/2}\chi_1(x) + C_2^v|x|^{1/2}\chi_2(x)$$

for small enough $|x|$.

Proof. Since the proofs are similar, we will prove the theorem only for $C_1^v(u)$. Lemma 3.2 provides the general behaviour of C_1^v and $C_1^v(u) = C_1^v\chi_1(u)$ will be proved case by case as in Table 1.

TABLE 1. Case by case proof of Theorem 3.1

Lemma 3.2	general behaviour of C_1^v
Prop. 3.3	$\chi_1(\varpi) \neq \pm\chi_2(\varpi)$
Prop. 3.4	$\chi_1(\varpi) = -\chi_2(\varpi)$ and $\chi_1(u) \neq \chi_2(u)$
Prop. 3.5(i)	$\chi_1(\varpi) = -\chi_2(\varpi)$, $\chi_1(u) = \chi_2(u)$ and $\exists\alpha \in \mathcal{O}^\times$ s.t. $\chi_1(\alpha) \neq \pm\chi_2(\alpha)$
Prop. 3.5(ii)	$\chi_1(\varpi) = -\chi_2(\varpi)$, $\chi_1(u) = \chi_2(u)$ and $\exists\alpha \in \mathcal{O}^\times$ s.t. $\chi_1(\alpha) = -\chi_2(\alpha)$
Prop. 3.6	$\chi_1(\varpi) = -\chi_2(\varpi)$ and $\chi_1\|_{\mathcal{O}^x} = \chi_2\|_{\mathcal{O}^x}$

□

The lemma and propositions mentioned in Table 1 will be proved in the rest of this case.

Lemma 3.2. *If $\chi_1(\varpi) \neq \chi_2(\varpi)$, then for $u_0, u_1 \in \mathcal{O}^\times$ we have*

$$C_1^v(u_0^2u_1) = \left[\frac{\chi_1(\varpi u_0) + \chi_2(\varpi u_0)}{\chi_1(\varpi)}\right] C_1^v(u_0u_1) - \left[\frac{\chi_1(u_0)\chi_2(\varpi u_0)}{\chi_1(\varpi)}\right] C_1^v(u_1).$$

Proof. By equation (5), for $k \geq m$ we have

$$\phi_v(\varpi^{k+2}u_0^2u_1) = C_1^v(u_0^2u_1)A^{k+2} + C_2^v(u_0^2u_1)B^{k+2}. \tag{7}$$

In equation (4) if we choose $x = \varpi u_0, y = \varpi^k u_1$ for $k \geq m$ and use the equation (5) we get

$$\phi_v(\varpi^{k+2}u_0^2u_1) = q^{-1/2}[\chi_1(\varpi u_0) + \chi_2(\varpi u_0)][C_1^v(u_0u_1)A^{k+1} + C_2^v(u_0u_1)B^{k+1}] \tag{8}$$
$$- q^{-1}\chi_1\chi_2(\varpi u_0)[C_1^v(u_1)A^k + C_2^v(u_1)B^k].$$

If we set the equations (7) and (8) equal to each other, then the some of

$$A^k[C_1^v(u_0^2u_1)A^2 - q^{-1/2}[\chi_1(\varpi u_0)+\chi_2(\varpi u_0)]C_1^v(u_0u_1)A + q^{-1}\chi_1\chi_2(\varpi u_0)C_1^v(u_1)]$$

and

$$B^k[C_2^v(u_0^2u_1)B^2 - q^{-1/2}[\chi_1(\varpi u_0)+\chi_2(\varpi u_0)]C_2^v(u_0u_1)B + q^{-1}\chi_1\chi_2(\varpi u_0)C_2^v(u_1)]$$

is zero. Since this is correct for all $k \geq m$ we have

$$C_1^v(u_0^2u_1)A^2 - q^{-1/2}[\chi_1(\varpi u_0)+\chi_2(\varpi u_0)]C_1^v(u_0u_1)A + q^{-1}\chi_1\chi_2(\varpi u_0)C_1^v(u_1) = 0.$$

Now, the result follows once we substitute $q^{-1/2}\chi_1(\varpi)$ for A. □

Proposition 3.3. *If $\chi_1(\varpi) \neq \pm\chi_2(\varpi)$, then $C_1^v(u) = C_1^v\chi_1(u)$.*

Proof. By using Lemma 3.2, a system of two linear equations of $C_1^v(u)$ and $C_1^v(u^{-1})$ will be obtained. If we let $u_0 = u$ and $u_1 = u^{-1}$, then by Lemma 3.2, we have

$$C_1^v(u) = \left[\frac{\chi_1(\varpi u) + \chi_2(\varpi u)}{\chi_1(\varpi)}\right] C_1^v(1) - \left[\frac{\chi_1(u)\chi_2(\varpi u)}{\chi_1(\varpi)}\right] C_1^v(u^{-1}). \tag{9}$$

If we let $u_0 = u^{-1}$ and $u_1 = u$, then similarly we have

$$C_1^v(u^{-1}) = \left[\frac{\chi_1(\varpi u^{-1}) + \chi_2(\varpi u^{-1})}{\chi_1(\varpi)}\right] C_1^v(1) - \left[\frac{\chi_1(u^{-1})\chi_2(\varpi u^{-1})}{\chi_1(\varpi)}\right] C_1^v(u). \tag{10}$$

By equations (9) and (10) we have

$$\begin{aligned} C_1^v(u) = & \left[\frac{\chi_1(\varpi u) + \chi_2(\varpi u)}{\chi_1(\varpi)}\right] C_1^v(1) \\ & - \left[\frac{\chi_1(u)\chi_2(\varpi u)}{\chi_1(\varpi)}\right] \left\{ \left[\frac{\chi_1(\varpi u^{-1}) + \chi_2(\varpi u^{-1})}{\chi_1(\varpi)}\right] C_1^v(1) \right. \\ & \left. - \left[\frac{\chi_1(u^{-1})\chi_2(\varpi u^{-1})}{\chi_1(\varpi)}\right] C_1^v(u) \right\}. \end{aligned}$$

Hence

$$\left[\chi_1(\varpi)^2 - \chi_2(\varpi)^2\right] C_1^v(u) = \left[\chi_1(\varpi)^2 - \chi_2(\varpi)^2\right] \chi_1(u) C_1^v(1)$$

and the result follows. □

Now we will consider the cases for which the left hand side of the above equation is zero.

Proposition 3.4. *If $\chi_1(\varpi) = -\chi_2(\varpi)$ and $\chi_1(u) \neq \chi_2(u)$, then $C_1^v(u) = C_1^v\chi_1(u)$.*

Proof. If $\chi_{\mathbf{1}}(\varpi) = -\chi_{\mathbf{2}}(\varpi)$, then Lemma 3.2 is equivalent to

$$C_1^v(u_0^2u_1) = [\chi_1(u_0) - \chi_2(u_0)]C_1^v(u_0u_1) + \chi_1\chi_2(u_0)C_1^v(u_1). \tag{11}$$

By using this relation, two equal expressions for $C_1^v(u^4)$ will be obtained but first we need to find $C_1^v(u^2)$ and $C_1^v(u^3)$.

By equation (11), if $u_0 = u$ and $u_1 = 1$, then

$$C_1^v(u^2) = [\chi_1(u) - \chi_2(u)]\, C_1^v(u) + [\chi_1\chi_2(u)]\, C_1^v(1); \tag{12}$$

if $u_0 = u$ and $u_1 = u$, then

$$C_1^v(u^3) = [\chi_1(u) - \chi_2(u)]\, C_1^v(u^2) + [\chi_1\chi_2(u)]\, C_1^v(u); \tag{13}$$

if $u_0 = u$ and $u_1 = u^2$, then

$$C_1^v(u^4) = [\chi_1(u) - \chi_2(u)]\, C_1^v(u^3) + [\chi_1\chi_2(u)]\, C_1^v(u^2); \tag{14}$$

if $u_0 = u^2$ and $u_1 = 1$, then

$$C_1^v(u^4) = \left[\chi_1(u^2) - \chi_2(u^2)\right] C_1^v(u^2) + \left[\chi_1\chi_2(u^2)\right] C_1^v(1). \tag{15}$$

If we set the equations (14) and (15) equal to each other we get

$$\begin{aligned} &[\chi_1(u) - \chi_2(u)]\, C_1^v(u^3) \\ = \ &\left[\chi_1(u^2) - \chi_2(u^2) - \chi_1\chi_2(u)\right] C_1^v(u^2) + \left[\chi_1\chi_2(u^2)\right] C_1^v(1). \end{aligned}$$

By equation (13)

$$[2\chi_2(u) - \chi_1(u)]\, C_1^v(u^2) + \chi_1(u)\, [\chi_1(u) - \chi_2(u)]\, C_1^v(u) = \left[\chi_1(u^2)\chi_2(u)\right] C_1^v(1)$$

and by equation (12), we have

$$\begin{aligned} &\left[2\chi_2(u)\chi_1(u) - 2\chi_2(u)^2 - \chi_1(u)^2 + \chi_1\chi_2(u) + \chi_1(u)^2 - \chi_1(u)\chi_2(u)\right] C_1^v(u) \\ = \ &\left[\chi_1(u)^2\chi_2(u) - 2\chi_1(u)\chi_2(u)^2 + \chi_1(u)^2\chi_2(u)\right] C_1^v(1). \end{aligned}$$

Hence

$$[\chi_1(u) - \chi_2(u)]\, C_1^v(u) = \chi_1(u)\, [\chi_1(u) - \chi_2(u)]\, C_1^v(1)$$

and the result follows. □

Proposition 3.5. *If $\chi_1(\varpi) = -\chi_2(\varpi)$, $\chi_1(u) = \chi_2(u)$ and*

i) *there exists $\alpha \in \mathcal{O}^\times$ such that $\chi_1(\alpha) \neq \pm\chi_2(\alpha)$, then $C_1^v(u) = C_1^v\chi_1(u)$.*

ii) *there exists $\alpha \in \mathcal{O}^\times$ such that $\chi_1(\alpha) = -\chi_2(\alpha)$, then $C_1^v(u) = C_1^v\chi_1(u)$.*

Proof. By using equation (11), and results of the previous proposition two equal expressions for $C_1^v(\alpha^2 u)$ will be obtained.

i) Since $\chi_1(\alpha u) \neq \chi_2(\alpha u)$ and $\chi_1(\alpha^2 u) \neq \chi_2(\alpha^2 u)$ from the previous proposition we obtain

$$C_1^v(\alpha u) = \chi_1(\alpha u)C_1^v(1) \tag{16}$$

and

$$C_1^v(\alpha^2 u) = \chi_1(\alpha^2 u)C_1^v(1). \tag{17}$$

By equation (11) we have

$$C_1^v(\alpha^2 u) = [\chi_1(\alpha) - \chi_2(\alpha)]C_1^v(\alpha u) + \chi_1\chi_2(\alpha)C_1^v(u)$$

and by equations (16) and (17),

$$\chi_1(\alpha^2 u)C_1^v(1) = [\chi_1(\alpha) - \chi_2(\alpha)]\chi_1(\alpha u)C_1^v(1) + \chi_1\chi_2(\alpha)C_1^v(u).$$

Hence $\chi_1(\alpha u)\chi_2(\alpha)C_1^v(1) = \chi_1\chi_2(\alpha)C_1^v(u)$ and the result follows.

ii) Since $\chi_1(\alpha^3 y) \neq \chi_2(\alpha^3 y)$ by Proposition 3.4 we have

$$C_1^v(\alpha^3 u) = \chi_1(\alpha^3 u)C_1^v(1).$$

By equation (11), we have $C_1^v(\alpha^3 u) = 2\chi_1(\alpha)C_1(\alpha^2 u) - \chi_1(\alpha^2)C_1(\alpha u)$, hence

$$\chi_1(\alpha^3 u)C_1^v(1) = 2\chi_1(\alpha)C_1(\alpha^2 u) - \chi_1(\alpha^2)C_1(\alpha u)$$

by previous proposition and this is equivalent to

$$\chi_1(\alpha^3 u)C_1(1) = 2\chi_1(\alpha)C_1(\alpha^2 u) - \chi_1(\alpha^2)\chi_1(\alpha u)C_1^v(1).$$

So we have

$$C_1^v(\alpha^2 u) = \chi_1(\alpha^2 u)C_1^v(1).$$

Now use the equation (11) one more time to get

$$C_1^v(\alpha^2 u) = 2\chi_1(\alpha)C_1^v(\alpha u) - \chi_1(\alpha^2)C_1^v(u).$$

Hence

$$\chi_1(\alpha^2 u)C_1^v(1) = 2\chi_1(\alpha)\chi_1(\alpha u)C_1^v(1) - \chi_1(\alpha^2)C_1^v(u)$$

and the proposition follows. □

Proposition 3.6. *If* $\chi_1(\varpi) = -\chi_2(\varpi)$ *and* $\chi_1|_{\mathcal{O}^x} = \chi_2|_{\mathcal{O}^x}$, *then* $C_1^v(u) = C_1^v\chi_1(u)$.

Proof. By equation (4), two equal expressions for $\phi_v(\varpi^{k+4}u^2)$ will be obtained. If $x = \varpi u$ and $y = \varpi^{k+2}$ for $k \geq m$, then we have

$$\phi_v(\varpi^{k+4}u^2) = q^{-1}\chi_1(\varpi^2 u^2)\phi_v(\varpi^{k+2})$$

and for $x = \varpi^2 u$ and $y = \varpi^k$ we have

$$\phi_v(\varpi^{k+4}u^2) = q^{-1}2\chi_1(\varpi^2 u)\phi_v(\varpi^{k+2}u) - q^{-2}\chi_1(\varpi^4 u^2)\phi_v(\varpi^k).$$

If we set the right hand sides of the two equations above equal to each other and divide by $q^{-1}\chi_1(\varpi^2 u)$ we get

$$\chi_1(u)\phi_v(\varpi^{k+2}) = 2\phi_v(\varpi^{k+2}u) - q^{-1}\chi_1(\varpi^2 u)\phi_v(\varpi^k).$$

Also by equation (5), we have

$$\begin{aligned} &\chi_1(u)[C_1^v(1)A^{k+2} + C_2^v(1)B^{k+2}] \\ &= 2[C_1^v(u)A^{k+2} + C_2^v(u)B^{k+2}] - q^{-1}\chi_1(\varpi^2 u)[C_1^v(1)A^k + C_2^v(1)B^k]. \end{aligned}$$

Since $A = -B = q^{-1/2}\chi_1(\varpi)$ we get

$$\begin{aligned} &A^{k+2}\chi_1(u)[C_1^v(1) + C_2^v(1)(-1)^k] \\ &= 2A^{k+2}[C_1^v(u) + C_2^v(u)(-1)^k] - \chi_1(u)A^2A^k[C_1^v(1) + C_2^v(1)(-1)^k]. \end{aligned}$$

Hence

$$\begin{aligned}&\chi_1(u)[C_1^v(1)+C_2^v(1)(-1)^k]\\ =\ &2[C_1^v(u)+C_2^v(u)(-1)^k]-\chi_1(u)[C_1^v(1)+C_2^v(1)(-1)^k]\end{aligned}$$

and

$$\chi_1(u)C_1^v(1)-C_1^v(u)=[C_2^v(u)-\chi_1(u)C_2^v(1)](-1)^k.$$

Since the equation above is correct for $k \geq m$, both sides of it should be zero and the proposition follows. □

3.2. Case: $\chi_1(\varpi)=\chi_2(\varpi)$

In this section, we will prove the following theorem which gives the asymptotic behaviour of Krillov model and the main result of this paper in this case.

Theorem 3.7. *Let* $\chi_1(\varpi)=\chi_2(\varpi)$.

i) *If* $\chi_1=\chi_2$, *then* $C_1^v(u)=\chi_1(u)C_1^v(1)$, $C_2^v(u)=\chi_2(u)C_2^v(1)$ *and*

$$\phi_v(x)=C_1^v|x|^{1/2}\chi_1(x)+C_2^v|x|^{1/2}v_k(x)\chi_2(x)$$

for small enough $|x|$.

ii) *If* $\chi_1 \neq \chi_2$, *then* $C_1^v(u)=D_1^v\chi_1(u)+D_2^v\chi_2(u)$, $C_2^v(u)=0$ *for some constants* D_1^v *and* D_2^v *depend only on* v *and*

$$\phi_v(x)=D_1^v|x|^{1/2}\chi_1(x)+D_2^v|x|^{1/2}\chi_2(x)$$

for small enough $|x|$.

Proof. Results about $C_2^v(u)$ and C_1^v are proved by, case by case consideration as given in Tables 2 and 3, respectively.

TABLE 2. Proof of $C_2^v(u)$ part of Theorem 3.7

Prop. 3.10(i)	$\chi_1=\chi_2$	$C_2^v(u)=\chi_1(u)C_2^v(1)$
Prop. 3.10(ii)	$\chi_1(u^4)\neq\pm\chi_2(u^4)$	$C_2^v=0$
Prop. 3.10(iii)	$\chi_1(u^4)=-\chi_2(u^4)$	$C_2^v=0$
Prop. 3.11(i)	$\chi_1(u)=-\chi_2(u)$	$C_2^v=0$
Prop. 3.11(ii)	$\chi_1(u^2)=-\chi_2(u^2)$	$C_2^v=0$
Proposition 3.12	$\chi_1(u)=\chi_2(u)$ and $\chi_1\neq\chi_2$	$C_2^v=0$

TABLE 3. Proof of $C_1^v(u)$ part of Theorem 3.7

Prop. 3.14	$\chi_1=\chi_2$
Prop. 3.15	$\chi_1\neq\chi_2$ and $\chi_1(u)=\chi_2(u)$
Prop. 3.16	$\chi_1(u)\neq\pm\chi_2(u)$
Proposition 3.17	$\chi_1(u)=-\chi_2(u)$

□

The lemmas and propositions mentioned in Tables 2 and 3 will be proved in the rest of this case. The first lemma provides the general behaviour of C_1^v and C_2^v.

Lemma 3.8. *If $\chi_1(\varpi) = \chi_2(\varpi)$, then for $u_0, u_1 \in \mathcal{O}^\times$ we have*

$$C_1^v(u_0^2u_1) = [\chi_1(u_0) + \chi_2(u_0)]C_1^v(u_0u_1) - \chi_1\chi_2(u_0)C_1^v(u_1) - [\chi_1(u_0) + \chi_2(u_0)]C_2^v(u_0u_1) + 2\chi_1\chi_2(u_0)C_2^v(u_1)$$

and

$$C_2^v(u_0^2u_1) = [\chi_1(u_0) + \chi_2(u_0)]C_2^v(u_0u_1) - \chi_1\chi_2(u_0)C_2^v(u_1). \tag{18}$$

Proof. By equation (6), we have

$$\phi(\varpi^{k+2}u_0^2u_1) = [C_1^v(u_0^2u_1) + C_2^v(u_0^2u_1)(k+2)]A^{k+2}. \tag{19}$$

In equation (4), if we choose $x = \varpi u_0$ and $y = \varpi^k u_1$ for $k \geq m$ and use the equation (6), then we get

$$\begin{aligned}\phi_v&(\varpi^{k+2}u_0^2u_1)\\ &= q^{-1/2}\chi_1(\varpi)[\chi_1(u_0) + \chi_2(u_0)][C_1^v(u_0u_1) + C_2^v(u_0u_1)(k+1)]A^{k+1}\\ &\quad - q^{-1}\chi_1(\varpi)^2\chi_1(u_0)\chi_2(u_0)[C_1^v(u_1) + C_2^v(u_1)k]A^k.\end{aligned} \tag{20}$$

Now if we set the equations (19) and (20) equal to each other, then we have

$$\begin{aligned}&[C_1(u_0^2u_1) + C_2(u_0^2u_1)(k+2)]A^{k+2}\\ &= A[\chi_1(u_0) + \chi_2(u_0)][C_1^v(u_0u_1) + C_2^v(u_0u_1)(k+1)]A^{k+1}\\ &\quad - A^2\chi_1\chi_2(u_0)[C_1^v(u_1) + C_2^v(u_1)k]A^k.\end{aligned}$$

Hence the sum of

$$[C_1^v(u_0^2u_1) + 2C_2^v(u_0^2u_1)] - [\chi_1(u_0) + \chi_2(u_0)][C_1^v(u_0u_1) + C_2^v(u_0u_1)] + \chi_1(u_0)\chi_2(u_0)C_1^v(u_1)$$

and

$$k[C_2^v(u_0^2u_1) - [\chi_1(u_0) + \chi_2(u_0)]C_2^v(u_0u_1) + \chi_1\chi_2(u_0)C_2^v(u_1)]$$

is zero. Since this holds for all $k \geq m$, the result follows. □

Now we will find a more simple expression for $C_2^v(u)$.

Lemma 3.9.

$$[\chi_1(u) + \chi_2(u)]\left\{[\chi_1(u^2) + \chi_2(u^2)]C_2^v(u) - [\chi_1(u) + \chi_2(u)]\chi_1\chi_2(u)C_2^v(1)\right\} = 0.$$

Proof. We will find two equal expressions for $C_1^v(u^4)$. In Lemma 3.8, if $u_0 = u$ and $u_1 = 1$, then

$$C_2^v(u^2) = [\chi_1(u) + \chi_2(u)]C_2^v(u) - \chi_1\chi_2(u)C_2^v(1), \tag{21}$$

$$\begin{aligned}C_1^v(u^2) &= [\chi_1(u) + \chi_2(u)]C_1^v(u) - \chi_1\chi_2(u)C_1^v(1)\\ &\quad - [\chi_1(u) + \chi_2(u)]C_2^v(u) + 2\chi_1\chi_2(u)C_2^v(1);\end{aligned} \tag{22}$$

if $u_0 = u$ and $u_1 = u$, then

$$C_2^v(u^3) = [\chi_1(u) + \chi_2(u)]C_2^v(u^2) - \chi_1\chi_2(u)C_2^v(u), \tag{23}$$

$$\begin{aligned} C_1^v(u^3) = {} & [\chi_1(u) + \chi_2(u)]C_1^v(u^2) - \chi_1\chi_2(u)C_1^v(u) \\ & - [\chi_1(u) + \chi_2(u)]C_2^v(u^2) + 2\chi_1\chi_2(u)C_2^v(u); \end{aligned} \tag{24}$$

if $u_0 = u$ and $u_1 = u^2$, then

$$\begin{aligned} C_1^v(u^4) = {} & [\chi_1(u) + \chi_2(u)]C_1^v(u^3) - \chi_1\chi_2(u)C_1^v(u^2) \\ & - [\chi_1(u) + \chi_2(u)]C_2^v(u^3) + 2\chi_1\chi_2(u)C_2^v(u^2); \end{aligned} \tag{25}$$

if $u_0 = u^2$ and $u_1 = 1$, then

$$\begin{aligned} C_1^v(u^4) = {} & [\chi_1(u^2) + \chi_2(u^2)]C_1^v(u^2) - \chi_1\chi_2(u^2)C_1^v(1) \\ & - [\chi_1(u^2) + \chi_2(u^2)]C_2^v(u^2) + 2\chi_1\chi_2(u^2)C_2^v(1). \end{aligned} \tag{26}$$

If we set the equations (25) and (26) equal to each other, then we get

$$\begin{aligned} & [\chi_1(u) + \chi_2(u)]C_1^v(u^3) - [\chi_1(u^2) + \chi_1\chi_2(u) + \chi_2(u^2)]C_1^v(u^2) \\ & \quad + \chi_1\chi_2(u^2)C_1^v(1) \\ & = [\chi_1(u) + \chi_2(u)]C_2^v(u^3) - [\chi_1(u) + \chi_2(u)]^2C_2^v(u^2) + 2\chi_1\chi_2(u^2)C_2^v(1). \end{aligned} \tag{27}$$

If we use the equations (24), (22) and (21) respectively, then the left hand side of the equation (27) becomes

$$\begin{aligned} & \left\{\chi_1\chi_2(u)[\chi_1(u) + \chi_2(u)] - [\chi_1(u) + \chi_2(u)]^3\right\} C_2^v(u) \\ & + \left\{[\chi_1(u) + \chi_2(u)]^2\chi_1(u)\chi_2(u) + 2\chi_1\chi_2(u^2)\right\} C_2^v(1). \end{aligned}$$

By equations (23) and (21), the right hand side of the equation (27) becomes

$$-\chi_1\chi_2(u)[\chi_1(u) + \chi_2(u)]C_2^v(u) + 2\chi_1\chi_2(u^2)C_2^v(1).$$

If we set the left and right hand sides of (27) equal to each other, then the result follows. □

The following proposition deals with the cases for which the coefficient of C_2^v in Lemma 3.9 is nonzero.

Proposition 3.10. i) *If* $\chi_1(u) = \chi_2(u)$, *then* $C_2^v(u) = \chi_1(u)C_2^v(1)$.
ii) *If* $\chi_1(u^4) \neq \pm\chi_2(u^4)$, *then* $C_2^v(u) = 0$.
iii) *If* $\chi_1(u^4) = -\chi_2(u^4)$, *then* $C_2^v(u) = 0$.

Proof. i) The result follows from Lemma 3.9.
ii) Since $\chi_1(u^2) \neq \pm\chi_2(u^2)$, by Lemma 3.9 we have

$$C_2^v(u) = \frac{[\chi_1(u) + \chi_2(u)]\chi_1\chi_2(u)}{\chi_1(u^2) + \chi_2(u^2)}C_2^v(1) \tag{28}$$

and since $\chi_1(u^4) \neq \pm\chi_2(u^4)$,

$$C_2^v(u^2) = \frac{[\chi_1(u^2) + \chi_2(u^2)]\chi_1\chi_2(u^2)}{\chi_1(u^4) + \chi_2(u^4)}C_2^v(1).$$

By equation (21), we have

$$\left\{\frac{[\chi_1(u^2)+\chi_2(u^2)]\chi_1\chi_2(u^2)}{\chi_1(u^4)+\chi_2(u^4)}+\chi_1\chi_2(u)-\frac{[\chi_1(u)+\chi_2(u)]^2\chi_1\chi_2(u)}{\chi_1(u^2)+\chi_2(u^2)}\right\}C_2^v(1)=0.$$

Hence $[\chi_1(u^2)-\chi_2(u^2)]C_2^v(1)=0$ and the second part follows.

iii) Since $\chi_1(u^4)=-\chi_2(u^4)$ we have $\chi_1(u^2)\neq\pm\chi_(u^2)$. By Lemma 3.9 for u^2 we get $C_2^v(1)=0$. Hence from the equation (28) we have $C_2^v(u)=0$. □

Now we are in the case $\chi_1(u^4)=\chi_2(u^4)$.

Proposition 3.11. i) *If* $\chi_1(u)=-\chi_2(u)$, *then* $C_2^v(u)=0$.
ii) *If* $\chi_1(u^2)=-\chi_2(u^2)$, *then* $C_2^v(u)=0$.

Proof. i) We will find two equal expressions for $C_2^v(u^7)$. By equation (23),

$$C_2^v(u^3)=\chi_1(u^2)C_2^v(u)$$

and by Lemma 3.8 and the equation above if $u_0=u$ and $u_1=u^3$, then

$$C_2^v(u^5)=\chi_1(u^2)C_2^v(u^3)=\chi_1(u^4)C_2^v(u). \tag{29}$$

By equation (24),

$$C_1^v(u^3)=\chi_1(u^2)[C_1^v(u)-2C_2^v(u)]$$

and by Lemma 3.8 and the equation above if $u_0=u$ and $u_1=u^3$, then

$$\begin{aligned}C_1^v(u^5)&=C_1^v[(u)^2u^3]\\&=\chi_1(u^2)C_1^v(u^3)-2\chi_1(u^2)C_2^v(u^3)\\&=\chi_1(u^4)[C_1^v(u)-2C_2^v(u)]-2\chi_1(u^4)C_2^v(u)\\&=\chi_1(u^4)[C_1^v(u)-4C_2^v(u)].\end{aligned} \tag{30}$$

By Lemma 3.8, if $u_0=u^3$ and $u_1=u$, then

$$\begin{aligned}C_2^v(u^7)&=C_1^v[(u^3)^2u]\\&=\chi_1(u^6)C_1^v(u)-2\chi_1(u^6)C_2^v(u)\\&=\chi_1(u^6)[C_1^v(u)-2C_2^v(u)].\end{aligned} \tag{31}$$

Similarly, by Lemma 3.8 and the equation 29 if $u_0=u$ and $u_1=u^5$, then

$$\begin{aligned}C_2^v(u^7)&=C_1^v[(u)^2u^5]\\&=\chi_1(u^2)\chi_1(u^4)[C_1^v(u)-4C_2^v(u)]-2\chi_1(u^2)\chi_1(u^4)C_2^v(u)\\&=\chi_1(u^6)[C_1^v(u)-6C_2^v(u)].\end{aligned} \tag{32}$$

Therefore $C_2^v(u)=0$ by equations (31) and (32).

ii) First note that by Lemma 3.9, $C_2^v(1)=0$ and from the first part, $C_2^v(u^2)=0$. Hence by equation (21) $C_2^v(u)=0$. □

Now we will show that $C_2^v(u)=0$ when $\chi_1\neq\chi_2$ and $\chi_1(u)=\chi_2(u)$.

Proposition 3.12. i) *If* $\chi_1(u) = \chi_2(u)$ *and there exists* $\alpha \in \mathcal{O}^\times$ *such that* $\chi_1(\alpha) \neq \pm\chi_2(\alpha)$*, then* $C_2^v(u) = 0$.

ii) *If* $\chi_1(u) = \chi_2(u)$, $\chi_1 = \pm\chi_2$ *and there exists* $\alpha \in \mathcal{O}^\times$ *such that* $\chi_1(\alpha) = -\chi_2(\alpha)$*, then* $C_2^v(u) = 0$.

Proof. i) By Propositions 3.10(ii), (iii), 3.11(ii) and Lemma 3.9 $C_2^v(\alpha) = 0$ and we have $C_2^v(1) = 0$. Now the result follows from Proposition 3.10(i).

ii) In equation (4), if we take $x = \varpi\alpha$ and $y = \varpi^{k+2}u$, then

$$\phi_v(\varpi^{k+4}\alpha^2 u) = q^{-1}\chi_1(\varpi^2\alpha^2)\phi_v(\varpi^{k+2}u) \tag{33}$$

and $x = \varpi^2\alpha$ and $y = \varpi^k u$, then

$$\phi_v(\varpi^{k+4}\alpha^2 u) = q^{-2}\chi_1(\varpi^4\alpha^2)\phi_v(\varpi^k u). \tag{34}$$

If we set the equations (33) and (34) equal to each other we get

$$\phi_v(\varpi^{k+2}u) = q^{-1}\chi_1(\varpi^2)\phi_v(\varpi^k u).$$

Now by equation (6), the equation above is

$$[C_1^v(u) + (k+2)C_2^v(u)]A^{k+2} = A^2[C_1^v(u) + kC_2^v(u)]A^k.$$

Hence $C_2^v(u) = 0$. □

Now we will prove the results of Theorem 3.7 about C_1^v. The following lemma provides the general behaviour of C_1^v.

Lemma 3.13. *If* $u_0, u_1 \in \mathcal{O}^\times$*, then we have*

$$C_1^v(u_0^2 u_1) = [\chi_1(u_0) + \chi_2(u_0)]C_1^v(u_0u_1) - \chi_1\chi_2(u_0)C_1^v(u_1).$$

Proof. By Lemma 3.8, we need to show that

$$-[\chi_1(u_0) + \chi_2(u_0)]C_2^v(u_0u_1) + 2\chi_1\chi_2(u_0)C_2^v(u_1) = 0. \tag{35}$$

From the first part of Theorem 3.7, if $\chi_1 \neq \chi_2$, then $C_2^v = 0$ and if $\chi_1 = \chi_2$, then $C_2^v(u) = \chi_1(u)C_2^v(u)$. Hence the result follows for both cases. □

Now, we will find $C_1^v(u)$ by case by case consideration.

Proposition 3.14. *If* $\chi_1 = \chi_2$*, then we have* $C_1^v(u) = \chi_1(u)C_1^v(1)$.

Proof. We will find two equal expressions for $C_1^v(u_0^2u_1^2)$. By Lemma 3.13, for $u_0, u_1 \in \mathcal{O}^\times$ we have

$$\begin{aligned}
C_1^v(u_0^2u_1^2) &= C_1^v[(u_0)^2u_1^2] \\
&= 2\chi_1(u_0)C_1^v(u_0u_1^2) - \chi_1(u_0^2)C_1^v(u_1^2) \\
&= 2\chi_1(u_0)[2\chi_1(u_1)C_1^v(u_0u_1) - \chi_1(u_1^2)C_1^v(u_0)] \\
&\quad - \chi_1(u_0^2)[2\chi_1(u_1)C_1^v(u_1) - \chi_1(u_1^2)C_1^v(1)] \\
&= 4\chi_1(u_0u_1)C_1^v(u_0u_1) - 2\chi_1(u_0u_1^2)C_1^v(u_0) - 2\chi_1(u_0^2u_1)C_1^v(u_1) \\
&\quad + \chi_1(u_0^2u_1^2)C_1^v(1)
\end{aligned} \tag{36}$$

and

$$C_1^v(u_0^2u_1^2) = C_1^v((u_0)^2u_1^2) = 2\chi_1(u_0u_1)C_1^v(u_0u_1) - \chi_1(u_0^2u_1^2)C_1^v(1). \tag{37}$$

If we set the equations (36) and (37) equal to each other we get

$$2\chi_1(u_0u_1)C_1^v(u_0u_1) = 2\chi_1(u_0u_1^2)C_1^v(u_0) + 2\chi_1(u_0^2u_1)C_1^v(u_1) - 2\chi_1(u_0^2u_1^2)C_1^v(1).$$

Now if we divide this equation by $2\chi_1(u_0^2u_1^2)$, then we have

$$\frac{C_1^v(u_0u_1)}{\chi_1(u_0u_1)} = \frac{C_1^v(u_0)}{\chi_1(u_0)} + \frac{C_1^v(u_1)}{\chi_1(u_1)} - C_1^v(1). \tag{38}$$

If we define

$$l : \mathcal{O}^\times \to \mathbb{C}, \quad l(x) = \frac{C_1^v(x)}{\chi_1(x)} - C_1^v(1),$$

then l is a continuous (locally constant) homomorphism by equation (38). $\mathcal{O}^\times$ is a compact group hence the image of l is a compact subgroup of $\mathbb{C}$. The only compact subgroup of $\mathbb{C}$ is $\{0\}$ so $l = 0$ and $C_1^v(u) = \chi_1(u)C_1^v(1)$. □

Now we are in the case of $\chi_1 \neq \chi_2$ hence there exists $r \in \mathcal{O}^\times$ such that $\chi_1(r) \neq \chi_2(r)$. We define

$$D_1^v = \frac{C_1^v(r) - \chi_2(r)C_1^v(1)}{\chi_1(r) - \chi_2(r)} \text{ and } D_2^v = \frac{\chi_1(r)C_1^v(1) - C_1^v(r)}{\chi_1(r) - \chi_2(r)}.$$

Define $H(x) := C_1^v(x) - \chi_1(x)D_1^v - \chi_2(x)D_2^v$. Hence to prove the results in Theorem 3.7 about C_1^v we need to show that $H(u) = 0$.

Proposition 3.15. *If $\chi_1 \neq \chi_2$ and $\chi_1(u) = \chi_2(u)$, then $H(u) = 0$.*

Proof. Let $S = \{u \in \mathcal{O}^\times : \frac{\chi_1}{\chi_2}(u) = 1\}$ then this set is compact and as in the proof of the previous proposition we can find a continuous homomorphism $(l|_S)$ from S to $\mathbb{C}$ and we get if $u \in S$, then $C_1^v(u) = \chi_1(u)C_1^v(1)$ and so $H(u) = 0$. □

Proposition 3.16. *If $\chi_1(u) \neq \pm\chi_2(u)$, then $H(u) = 0$.*

Proof. By Lemma 3.13, find two equal expressions for $C_1^v(u^2r^2)$.

$$\begin{aligned} C_1^v(u^2r^2) &= C_1^v[(u)^2r^2] \\ &= [\chi_1(u) + \chi_2(u)]C_1^v(ur^2) - \chi_1\chi_2(u)C_1^v(r^2) \\ &= [\chi_1(u) + \chi_2(u)]\{[\chi_1(r) + \chi_2(r)]C_1^v(ur) - \chi_1\chi_2(r)C_1^v(u)\} \\ &\quad - \chi_1\chi_2(u)\{[\chi_1(r) + \chi_2(r)]C_1^v(r) - \chi_1\chi_2(r)C_1^v(1)\} \\ &= [\chi_1(u) + \chi_2(u)][\chi_1(r) + \chi_2(r)]C_1^v(ur) \\ &\quad - \chi_1\chi_2(r)[\chi_1(u) + \chi_2(u)]C_1^v(u) \\ &\quad - \chi_1\chi_2(u)[\chi_1(r) + \chi_2(r)]C_1^v(r) + \chi_1\chi_2(ur)C_1^v(1)] \end{aligned} \tag{39}$$

and

$$C_1^v(u^2r^2) = C_1^v[(ur)^2] = [\chi_1(ur) + \chi_2(ur)]C_1^v(ur) - \chi_1\chi_2(ur)C_1^v(1). \tag{40}$$

If we set the equations (39) and (40) equal to each other we get

$$[\chi_1(u)\chi_2(r)+\chi_1(r)\chi_2(u)]C_1^v(ur)-\chi_1\chi_2(r)[\chi_1(u)+\chi_2(u)]C_1^v(u)$$
$$-\chi_1\chi_2(u)[\chi_1(r)+\chi_2(r]C_1^v(r)+2\chi_1\chi_2(ur)C_1^v(1)]=0.$$

Now divide the each term by $\chi_1\chi_2(u)$ then

$$\left[\chi_1\left(\frac{r}{u}\right)+\chi_2\left(\frac{r}{u}\right)\right]C_1^v(ur)-\chi_1\chi_2(r)\left[\frac{1}{\chi_1(u)}+\frac{1}{\chi_2(u)}\right]C_1^v(u)$$
$$-\chi_1(r)[C_1^v(r)-\chi_2(r)C_1^v(1)]+\chi_2(r)[\chi_1(r)C_1^v(1)-C_1^v(r)]=0.$$

Hence

$$\chi_1\left(\frac{r}{u}\right)\{C_1^v(ur)-\chi_2(r)C_1^v(u)-\chi_1(u)[C_1^v(r)-\chi_2(r)C_1^v(1)]\}$$
$$+\chi_2\left(\frac{r}{u}\right)\{C_1^v(ur)-\chi_1(r)C_1^v(u)+\chi_2(u)[\chi_1(r)C_1^v(1)-C_1^v(r)]\}=0$$

and

$$\chi_1\left(\frac{r}{u}\right)\{C_1^v(ur)-\chi_2(r)C_1^v(u)-\chi_1(u)D_1^v[\chi_1(r)-\chi_2(r)]\}$$
$$+\chi_2\left(\frac{r}{u}\right)\{C_1^v(ur)-\chi_1(r)C_1^v(u)+\chi_2(u)D_2^v[\chi_1(r)-\chi_2(r)]\}=0.$$

So we have

$$\chi_1\left(\frac{r}{u}\right)[H(ur)-\chi_2(r)H(u)]+\chi_2\left(\frac{r}{u}\right)[H(ur)-\chi_1(r)H(u)]=0 \tag{41}$$

and

$$\left[\chi_1\left(\frac{r}{u}\right)+\chi_2\left(\frac{r}{u}\right)\right]H(ur)=\chi_1\chi_2(r)\left[\frac{1}{\chi_1(u)}+\frac{1}{\chi_2(u)}\right]H(u). \tag{42}$$

Since the equation (42) is symmetric with respect to u and r we also get

$$\left[\chi_1\left(\frac{u}{r}\right)+\chi_2\left(\frac{u}{r}\right)\right]H(ur)=\chi_1\chi_2(u)\left[\frac{1}{\chi_1(r)}+\frac{1}{\chi_2(r)}\right]H(r). \tag{43}$$

Note that

$$\begin{aligned}H(r)&=C_1^v(r)-\chi_1(r)D_1^v-\chi_2(r)D_2^v\\&=C_1^v(r)-\chi_1(r)\left[\frac{C_1^v(r)-\chi_2(r)C_1^v(1)}{\chi_1(r)-\chi_2(r)}\right]-\chi_2(r)\left[\frac{\chi_1(r)C_1^v(1)-C_1^v(r)}{\chi_1(r)-\chi_2(r)}\right]\\&=0.\end{aligned}$$

Hence by equation (43), $H(ur)=0$ and by equation (42), $H(u)=0$. □

Proposition 3.17. i) *If $\chi_1(u)=-\chi_2(u)$ and $\chi_1(r)=-\chi(r)$, then $H(u)=0$.*
ii) *If $\chi_1(u)=-\chi_2(u)$ and $\chi_1(r)\neq\pm\chi(r)$, then $H(u)=0$.*

Proof. i) By Lemma 3.13, find two equal expressions for $C_1^v(u^2r^3)$.

$$\begin{aligned}C_1^v(u^2r^3)&=C_1^v[(u)^2r^3]\\&=\chi_1(u^2)C_1^v(r^3)=\chi_1(u^2)C_1^v[(r)^2r]=\chi_1(u^2r^2)C_1^v(r)\end{aligned} \tag{44}$$

and

$$\begin{aligned} C_1^v(u^2r^3) &= C_1^v[(ur)^2r] \\ &= 2\chi_1(ur)C_1^v[u(r)^2] - \chi_1(u^2r^2)C_1^v(r) \\ &= 2\chi_1(ur)[\chi_1(r^2)C_1^v(u)] - \chi_1(u^2r^2)C_1^v(r) \\ &= 2\chi_1(ur^3)C_1^v(u) - \chi_1(u^2r^2)C_1^v(r). \end{aligned} \tag{45}$$

If we set the equations (44) and (45) equal to each other, then we get the result.

ii) Since $\chi_1(ur) \neq \pm\chi_2(ur)$ and $\chi_1(ur^3) \neq \pm\chi_2(ur^3)$, by Proposition 3.16 $H(ur)$ and $H(ur^3)$ are zero. Hence

$$C_1^v(ur) = \chi_1(ur)D_1^v + \chi_2(ur)D_2^v \tag{46}$$

and

$$C_1^v(ur^3) = \chi_1(ur^3)D_1^v + \chi_2(ur^3)D_2^v. \tag{47}$$

By Lemma 3.13, we have

$$\begin{aligned} &C_1^v(ur^3) \\ &= [\chi_1(r) + \chi_2(r)]C_1^v(ur^2) - \chi_1\chi_2(r)C_1^v(ur) \\ &= [\chi_1(r) + \chi_2(r)]\{[\chi_1(r) + \chi_2(r)]C_1^v(ur) - \chi_1\chi_2(r)C_1^v(u)\} - \chi_1\chi_2(r)C_1^v(ur) \\ &= [\chi_1(r^2) + \chi_2(r^2) + \chi_1\chi_2(r)]C_1^v(ur) - \chi_1\chi_2(r)[\chi_1(r) + \chi_2(r)]C_1^v(u) \end{aligned}$$

and by equations (46) and (47) we get the result. □

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Yusuf Danisman
Department of Mathematics and Computer Science
Queensborugh Community College
CUNY, USA
Email address: yusufdanisman@gmail.com