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GENERALIZATIONS OF NUMBER-THEORETIC SUMS

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Abstract. For positive integers n and k, let $S_k(n)$ and $S'_k(n)$ be the sums of the elements in the finite sets $\left\{x^k : 1 \le x \le n, (x,n) = 1\right\}$ and $\left\{x^k : 1 \le x \le n/2, (x,n) = 1\right\}$, respectively. The formulae for both $S_k(n)$ and $S'_k(n)$ are established. The explicit formulae when $k = 1, 2, 3$ are also given.

1. Introduction

As usual (m,n) denotes the greatest common divisor of integers m and n. An *arithmetic function* f is a complex-valued function defined on the set of positive integers. There are many interesting examples of arithmetic function. Both of them are the Euler's phi-function,

$$\phi(n) = |\{x : 1 \le x \le n, (x,n) = 1\}|$$

and the Möbius function,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ where all } p_i \text{ are distinct primes.} \end{cases}$$

An arithmetic function f is said to be *multiplicative* [2, p. 107] if $f(mn) = f(m)f(n)$, whenever $(m,n) = 1$. It is well-known that ϕ is multiplicative ([2, p. 133], [4, p. 11], or [5, p. 69]) and so does μ ([2, p. 112], [4, p. 5], or [5, p. 193]). For positive integers n and k, define the following finite sets of positive integers:

$$R_k(n) = \left\{x^k : 1 \le x \le n, (x,n) = 1\right\},$$

$$R'_k(n) = \left\{x^k : 1 \le x \le \frac{n}{2}, (x,n) = 1\right\}.$$

Observe that $R_k(1) = R_k(2) = \{1\} = R'_k(2)$ and $R'_k(1) = \emptyset$. In this paper, $\sum A$ denotes the sum of the elements of a finite set A of positive integers. Then

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we let

$$S_k(n) = \sum R_k(n) \quad \text{and} \quad S'_k(n) = \sum R'_k(n).$$

It is clear that $S_k(1) = 1$, $S'_k(1) = 0$ and $S_k(2) = S'_k(2) = 1$. Note that the number of elements in $R_1(n)$ is $\phi(n)$ and it is a simple matter to compute $S_1(n)$. We have known in [2, p. 143] that

$$S_1(n) = \frac{n\phi(n)}{2} \quad (n > 1).$$

There is an exercise in [5, p. 196] to calculate $S_2(n)$ by the use of the Möbius inversion formula which asserts in the following theorem ([2, p. 113], [4, p. 6], or [5, p. 194]).

Theorem 1.1 (Möbius Inversion Formula)**.** *If F and f are arithmetic functions with $F(n) = \sum_{d|n} f(d)$ for every positive integer n, then*

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) \qquad (n \geq 1),$$

where the sum $\sum_{d|n}$ is over all divisors d of n.

The formula for $S_2(n)$ is given in [5, p. 196] that

$$S_2(n) = \frac{n^2}{6} \sum_{d|n} \mu(d) \left(\frac{2n}{d} + 3 + \frac{d}{n}\right) \qquad (n \geq 1). \tag{1}$$

From the following facts in [2, p. 144], [2, p. 113], and [2, p. 116], we have

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n} \qquad (n \geq 1), \tag{2}$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{3}$$

$$\sum_{d|n} \mu(d) d = \psi(n) \qquad (n \geq 1), \tag{4}$$

respectively, where $\psi(1) = 1$ and $\psi(n) = \prod_{p|n}(1 - p)$ for $n > 1$, the product is over the prime divisors of n. The formula (1) can be rewritten as

$$S_2(n) = \frac{2n^2\phi(n) + n\psi(n)}{6} \qquad (n > 1).$$

In another direction, Baum [1] provided the formula for $S'_1(n)$ as follows:

$$S'_1(n) = \frac{1}{8}\left(n\phi(n) - |r|\psi(n)\right) \qquad (n > 2),$$

where $n \equiv r \pmod 4$ with $r \in \{-1, 0, 1, 2\}$, and he advised the reader to prove

$$S_2'(n) = \begin{cases} \dfrac{n^2\phi(n) + 2n\psi(n)}{24} & \text{if } n \equiv 0 \pmod 4, \\ \dfrac{n^2\phi(n) - n\psi(n)}{24} & \text{if } n \equiv \pm 1 \pmod 4, \\ \dfrac{n^2\phi(n) - 4n\psi(n)}{24} & \text{if } n \equiv 2 \pmod 4, \end{cases} \qquad (n > 2)$$

as an exercise. However, there is no any general formula for $S_k(n)$ or $S_k'(n)$. So we are interested in establishing that for $S_k(n)$ and $S_k'(n)$ for all positive integers n and k.

In the present work, we establish the general formulae for both $S_k(n)$ and $S_k'(n)$ by the use of the Möbius inversion formula. We also confirm that the known results for $k = 1, 2$ are the special cases of our results. Moreover, we give the explicit formulae for $S_3(n)$ and $S_3'(n)$.

2. Main results

For convenience, we define $g_k(n) = 1^k + 2^k + \cdots + n^k$ for positive integers n and k. It is well known that

$$g_1(n) = \frac{n(n+1)}{2},$$

$$g_2(n) = \frac{n(n+1)(2n+1)}{6},$$

$$g_3(n) = \frac{n^2(n+1)^2}{4},$$

and (see [6])

$$g_k(n) = \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^{j-i} i^k \binom{j}{i} \binom{n+1}{j+1}$$

for all positive integers n and k. Any other version of the formula for $g_k(n)$ can be found in [3] or [7, p. 123].

First, we establish the formula for $S_k(n)$ in the following theorem.

Theorem 2.1. *For any positive integer k, we have*

$$S_k(n) = \sum_{d|n} \mu(d) d^k g_k\left(\frac{n}{d}\right)$$

for $n \geq 1$.

Proof. Let n and k be positive integers. For a positive divisor d of n which is denoted by $d|n$, define

$$A_d = \{x^k : 1 \leq x \leq n, (x, n) = d\}.$$

Note that $A_d \neq \emptyset$ since $d^k \in A_d$. Clearly, $\cup_{d|n} A_d = \{1^k, 2^k, \ldots, n^k\}$ and $A_{d_1} \cap A_{d_2} = \emptyset$ for $d_1 \neq d_2$. It follows that

$$g_k(n) = \sum_{i=1}^{n} i^k = \sum_{d|n} \sum A_d. \tag{5}$$

We next show that

$$A_d = d^k R_k\left(\frac{n}{d}\right). \tag{6}$$

If $x^k \in A_d$, then $1 \leq x/d \leq n/d$, $x/d \in \mathbb{N}$, and $(x/d, n/d) = 1$. Consequently, $(x/d)^k \in R_k(n/d)$ and so $x^k \in d^k R_k(n/d)$. If $y^k \in R_k(n/d)$, then $1 \leq y \leq n/d$ and $(y, n/d) = 1$. It follows that $d \leq dy \leq n$ and $(dy, n) = d$. This shows that $(dy)^k \in A_d$.

By (6), we have for $d|n$,

$$\sum A_d = \sum d^k R_k\left(\frac{n}{d}\right) = d^k S_k\left(\frac{n}{d}\right).$$

It follows by (5) that

$$g_k(n) = \sum_{d|n} d^k S_k\left(\frac{n}{d}\right) = \sum_{d|n} \left(\frac{n}{d}\right)^k S_k(d).$$

By the Möbius inversion formula with $f(n) = S_k(n)/n^k$ and $F(n) = g_k(n)/n^k$, we get

$$\frac{S_k(n)}{n^k} = \sum_{d|n} \mu(d) \frac{d^k}{n^k} g_k\left(\frac{n}{d}\right),$$

as desired. $\square$

We observe that the formula for $S_k(n)$ does not depend upon the form of n. However, we prove in the following theorem that the formula for $S'_k(n)$ does. The residue modulo 4 of n together with the formula for $S_k(n)$ in Theorem 2.1 determines the formula for $S'_k(n)$ as follows:

Theorem 2.2. *For any positive integer k, we have*

$$S'_k(n) = \begin{cases} \displaystyle\sum_{d|(n/2)} \mu(d) d^k g_k\left(\frac{n}{2d}\right) & \text{if } n \equiv 0 \pmod 4, \\ \displaystyle\sum_{d|n} \mu(d) d^k g_k\left(\frac{n/d - 1}{2}\right) & \text{if } n \equiv \pm 1 \pmod 4, \\ \displaystyle\sum_{d|(n/2)} \mu(d) d^k \left(g_k\left(\frac{n}{2d}\right) - 2^k g_k\left(\frac{n/2d - 1}{2}\right)\right) & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

for all $n > 2$.

Proof. We prove this formula by considering three cases.
Case I: $n \equiv 0 \pmod 4$. Then n and $n/2$ are even. It follows that $(x,n)=1$ if and only if $(x,n/2)=1$ for any positive integer x. From Theorem 2.1, we have

$$\begin{aligned} S'_k(n) &= \sum \left\{x^k : 1 \le x \le \frac{n}{2}, (x,n)=1\right\} \\ &= \sum \left\{x^k : 1 \le x \le \frac{n}{2}, \left(x, \frac{n}{2}\right)=1\right\} \\ &= S_k\left(\frac{n}{2}\right) = \sum_{d|(n/2)} \mu(d) d^k g_k\left(\frac{n}{2d}\right). \end{aligned}$$

Case II: $n \equiv \pm 1 \pmod 4$. For $d|n$, define

$$B_d = \left\{x^k : 1 \le x \le \frac{n}{2}, (x,n)=d\right\}.$$

Note that $B_d = \emptyset$ if and only if $d = n$. Clearly, $\cup_{d|n} B_d = \{1^k, 2^k, \ldots, ((n-1)/2)^k\}$ and $B_{d_1} \cap B_{d_2} = \emptyset$ for $d_1 \ne d_2$, so we have

$$g_k\left(\frac{n-1}{2}\right) = \sum_{i=1}^{\frac{n-1}{2}} i^k = \sum_{d|n} \sum B_d. \tag{7}$$

Next, we show that

$$B_d = d^k R'_k\left(\frac{n}{d}\right). \tag{8}$$

Observe that $R'_k(n/d) = \emptyset$ if and only if $d = n$. If $x^k \in B_d$, then $1 \le x/d \le n/2d$ and $(x/d, n/d) = 1$. It follows that $(x/d)^k \in R'_k(n/d)$. If $y^k \in R'_k(n/d)$, then $d \le dy \le n/2$ and $(dy, n) = d$, that is $(dy)^k \in B_d$.

By (8), we obtain

$$\sum B_d = d^k S'_k\left(\frac{n}{d}\right).$$

It follows by (7) that

$$g_k\left(\frac{n-1}{2}\right) = \sum_{d|n} d^k S'_k\left(\frac{n}{d}\right) = \sum_{d|n} \left(\frac{n}{d}\right)^k S'_k(d).$$

Rewrite the above equation to get

$$\sum_{d|n} \frac{S'_k(d)}{d^k} = \frac{1}{n^k} g_k\left(\frac{n-1}{2}\right).$$

Applying the Möbius inversion formula with $f(n) = S'_k(n)/n^k$ and $F(n) = g_k((n-1)/2)/n^k$, we have the desired result

$$\frac{S'_k(n)}{n^k} = \sum_{d|n} \mu(d) \frac{d^k}{n^k} g_k\left(\frac{n/d - 1}{2}\right).$$

Case III: $n \equiv 2 \pmod 4$. Then we can write $n = 2m$ for some odd integer m. Thus for any positive integer x, we have $(x,n)=1$ if and only if $(x,m)=1$

and x is odd. We also observe that for any positive integer y, $(2y, m) = 1$ if and only if $(y, m) = 1$. By using Theorem 2.1 and Case II, we have

$$\begin{aligned}
S'_k(n) &= \sum \left\{x^k : 1 \le x \le \frac{n}{2}, (x,n) = 1\right\} \\
&= \sum \left\{x^k : 1 \le x \le m, (x,m) = 1, x \text{ is odd}\right\} \\
&= \sum \Big(\left\{x^k : 1 \le x \le m, (x,m) = 1\right\} \\
&\qquad \setminus \left\{x^k : 1 \le x \le m, (x,m) = 1, x \text{ is even}\right\}\Big) \\
&= \sum \Big(\left\{x^k : 1 \le x \le m, (x,m) = 1\right\} \\
&\qquad \setminus \left\{(2y)^k : 1 \le 2y \le m, (2y,m) = 1\right\}\Big) \\
&= \sum \Big(\left\{x^k : 1 \le x \le m, (x,m) = 1\right\} \\
&\qquad \setminus 2^k \left\{y^k : 1 \le y \le \frac{m}{2}, (y,m) = 1\right\}\Big) \\
&= S_k(m) - 2^k S'_k(m) \\
&= \sum_{d|m} \mu(d) d^k g_k\left(\frac{m}{d}\right) - 2^k \sum_{d|m} \mu(d) d^k g_k\left(\frac{m/d - 1}{2}\right) \\
&= \sum_{d|(n/2)} \mu(d) d^k \left(g_k\left(\frac{n}{2d}\right) - 2^k g_k\left(\frac{n/2d - 1}{2}\right)\right),
\end{aligned}$$

as desired. □

3. Explicit formulae

In this section, we provide the explicit formulae for $S_k(n)$ and $S'_k(n)$, where $k = 1, 2,$ and 3, by using the following lemmas. The first lemma is verified by the fact that if f is a multiplicative function and F is an arithmetic function defined by $F(n) = \sum_{d|n} f(d)$, then F is also multiplicative [2, p. 109].

Lemma 3.1. *For positive integers m and n, we have*

$$\sum_{d|n} \mu(d) d^m = \psi_m(n), \tag{9}$$

where $\psi_m(1) = 1$ and $\psi_m(n) = \prod_{p|n} (1 - p^m)$ for $n > 1$.

Proof. If $n = 1$, then we are done. For $n \ge 2$, we write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ as its prime factorization. Since μ is multiplicative, the function f defined by $f(n) = \mu(n) n^m$ is multiplicative and so the function F defined by $F(n) = \sum_{d|n} \mu(d) d^m$ is also multiplicative. Since

$$F(p_i^{k_i}) = \sum_{d|p_i^{k_i}} \mu(d) d^m = \mu(1) + \mu(p_i) p_i^m = 1 - p_i^m$$

for all $1 \le i \le r$, we obtain

$$\sum_{d|n} \mu(d)d^m = F(n) = \prod_{1\le i\le r} F(p_i^{k_i}) = \prod_{p|n}(1-p^m) = \psi_m(n),$$

as desired. □

Note that $\psi_1 = \psi$ and so the equation (4) is a special case of Lemma 3.1.

Lemma 3.2. *For even positive integer $n > 2$, we have*

(i) $\phi\left(\frac{n}{2}\right) = \begin{cases} \phi(n)/2 & \text{if } n \equiv 0 \pmod 4, \\ \phi(n) & \text{if } n \equiv 2 \pmod 4, \end{cases}$

(ii) $\psi_m\left(\frac{n}{2}\right) = \begin{cases} \psi_m(n) & \text{if } n \equiv 0 \pmod 4, \\ \psi_m(n)/(1-2^m) & \text{if } n \equiv 2 \pmod 4 \end{cases}$

for $m \ge 1$.

Proof. If $n \equiv 0 \pmod 4$, then we can write $n = 2^r t$ for some positive integers r and t such that $r \ge 2$ and t is odd. Since ϕ is multiplicative, we obtain

$$\phi\left(\frac{n}{2}\right) = \phi(2^{r-1})\phi(t) = \frac{\phi(2^r)\phi(t)}{2} = \frac{\phi(n)}{2},$$

$$\psi_m\left(\frac{n}{2}\right) = \prod_{p|2^{r-1}t}(1-p^m) = \prod_{p|2^r t}(1-p^m) = \psi_m(n).$$

If $n \equiv 2 \pmod 4$, then we can write $n = 2t$ for some odd integer t and so

$$\phi\left(\frac{n}{2}\right) = \phi(t) = \phi(2)\phi(t) = \phi(2t) = \phi(n),$$

$$\psi_m\left(\frac{n}{2}\right) = \prod_{p|t}(1-p^m) = \frac{\prod_{p|2t}(1-p^m)}{1-2^m} = \frac{\psi_m(n)}{1-2^m}.$$

This completes the proof. □

The following example shows that the known formulae for $S_1(n)$ in [2, p. 143] and $S_1'(n)$ in [1] follow from our formulae in Theorem 2.1 and Theorem 2.2, respectively.

Example 3.3. Recall that $g_1(n) = n(n+1)/2$ for all $n \ge 1$. By using Theorem 2.1, (2), and (3), we get

$$\begin{aligned} S_1(n) &= \sum_{d|n} \mu(d) d g_1\left(\frac{n}{d}\right) \\ &= \frac{1}{2}\sum_{d|n}\mu(d)\left(\frac{n^2}{d} + n\right) \\ &= \frac{n\phi(n)}{2} \end{aligned}$$

for $n > 1$ as desired. To calculate the explicit formula for $S_1'(n)$ by using Theorem 2.2, we consider three cases for $n > 2$ as follows:
Case I: $n \equiv 0 \pmod 4$.

$$\begin{aligned}
S_1'(n) &= \sum_{d|(n/2)} \mu(d) d g_1\left(\frac{n}{2d}\right) \\
&= \frac{1}{8} \sum_{d|(n/2)} \mu(d)\left(\frac{n^2}{d} + 2n\right) \\
&= \frac{1}{8}\left(\frac{n^2\phi(n/2)}{n/2}\right) \qquad \text{by (2) and (3)} \\
&= \frac{n\phi(n)}{8} \qquad \text{by Lemma 3.2(i).}
\end{aligned}$$

Case II: $n \equiv \pm 1 \pmod 4$.

$$\begin{aligned}
S_1'(n) &= \sum_{d|n} \mu(d) d g_1\left(\frac{n/d - 1}{2}\right) \\
&= \frac{1}{8} \sum_{d|n} \mu(d)\left(\frac{n^2}{d} - d\right) \\
&= \frac{1}{8}\left(\frac{n^2\phi(n)}{n} - \psi(n)\right) \qquad \text{by (2) and (4)} \\
&= \frac{1}{8}\left(n\phi(n) - \psi(n)\right).
\end{aligned}$$

Case III: $n \equiv 2 \pmod 4$.

$$\begin{aligned}
S_1'(n) &= \sum_{d|(n/2)} \mu(d) d \left[g_1\left(\frac{n}{2d}\right) - 2g_1\left(\frac{n/2d - 1}{2}\right)\right] \\
&= \frac{1}{8} \sum_{d|(n/2)} \mu(d)\left(\frac{n^2}{2d} + 2n + 2d\right) \\
&= \frac{1}{8}\left(\frac{n^2\phi(n/2)}{2(n/2)} + 2\psi\left(\frac{n}{2}\right)\right) \qquad \text{by (2), (3), and (4)} \\
&= \frac{1}{8}\left(n\phi(n) - 2\psi(n)\right) \qquad \text{by Lemma 3.2(i), (ii).}
\end{aligned}$$

The next example confirms that the formulae in Theorem 2.1 and Theorem 2.2 are generalization of $S_2(n)$ in [5, p. 196] and $S_2'(n)$ in [1], respectively.

Example 3.4. Recall that $g_2(n) = n(n+1)(2n+1)/6$ for all $n \geq 1$. By using Theorem 2.1, (2), (3), and (4), we get

$$S_2(n) = \sum_{d|n} \mu(d) d^2 g_2\left(\frac{n}{d}\right)$$

$$= \frac{1}{6}\sum_{d|n}\mu(d)\left(\frac{2n^3}{d}+3n^2+nd\right)$$
$$= \frac{2n^2\phi(n)+n\psi(n)}{6}$$

for $n > 1$ as desired. To calculate the explicit formula for $S_2'(n)$ by using Theorem 2.2, we consider three cases for $n > 2$ as follows:
Case I: $n \equiv 0 \pmod 4$.

$$\begin{aligned} S_2'(n) &= \sum_{d|(n/2)}\mu(d)d^2 g_2\left(\frac{n}{2d}\right) \\ &= \frac{1}{24}\sum_{d|(n/2)}\mu(d)\left(\frac{n^3}{d}+3n^2+2nd\right) \\ &= \frac{1}{24}\left(\frac{n^3\phi(n/2)}{n/2}+2n\psi\left(\frac{n}{2}\right)\right) \qquad \text{by (2), (3), and (4)} \\ &= \frac{n^2\phi(n)+2n\psi(n)}{24} \qquad \text{by Lemma 3.2(i), (ii) with } \psi_1=\psi. \end{aligned}$$

Case II: $n \equiv \pm 1 \pmod 4$.

$$\begin{aligned} S_2'(n) &= \sum_{d|n}\mu(d)d^2 g_2\left(\frac{n/d-1}{2}\right) \\ &= \frac{1}{24}\sum_{d|n}\mu(d)\left(\frac{n^3}{d}-nd\right) \\ &= \frac{n^2\phi(n)-n\psi(n)}{24} \qquad \text{by (2) and (4).} \end{aligned}$$

Case III: $n \equiv 2 \pmod 4$.

$$\begin{aligned} S_2'(n) &= \sum_{d|(n/2)}\mu(d)d^2\left[g_2\left(\frac{n}{2d}\right)-2^2 g_2\left(\frac{n/2d-1}{2}\right)\right] \\ &= \frac{1}{24}\sum_{d|(n/2)}\mu(d)\left(\frac{n^3}{2d}+3n^2+4nd\right) \\ &= \frac{1}{24}\left(\frac{n^3\phi(n/2)}{2(n/2)}+4n\psi\left(\frac{n}{2}\right)\right) \qquad \text{by (2), (3), and (4)} \\ &= \frac{n^2\phi(n)-4n\psi(n)}{24} \qquad \text{by Lemma 3.2(i), (ii) with } \psi_1=\psi. \end{aligned}$$

Finally, we give the formulae for $S_3(n)$ and $S_3'(n)$ as in the following example.

Example 3.5. We show that

$$S_3(n) = \frac{n^3\phi(n)+n^2\psi(n)}{4} \qquad (n>1)$$

and

$$S_3'(n) = \begin{cases} \dfrac{n^3\phi(n) + 4n^2\psi(n)}{64} & \text{if } n \equiv 0 \pmod 4, \\ \dfrac{n^3\phi(n) - 2n^2\psi(n) + \psi_3(n)}{64} & \text{if } n \equiv \pm 1 \pmod 4, \quad (n > 2) \\ \dfrac{n^3\phi(n) - 8n^2\psi(n) + 8\psi_3(n)/7}{64} & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Recall that $g_3(n) = n^2(n+1)^2/4$ for all $n \geq 1$. By using Theorem 2.1, (2), (3), and (4), we get

$$\begin{aligned} S_3(n) &= \sum_{d|n} \mu(d) d^3 g_3\left(\frac{n}{d}\right) \\ &= \frac{1}{4} \sum_{d|n} \mu(d) \left(\frac{n^4}{d} + 2n^3 + n^2 d\right) \\ &= \frac{n^3\phi(n) + n^2\psi(n)}{4} \end{aligned}$$

for $n > 1$ as desired. To calculate the explicit formula for $S_3'(n)$ by using Theorem 2.2, we consider three cases for $n > 2$ as follows:

Case I: $n \equiv 0 \pmod 4$.

$$\begin{aligned} S_3'(n) &= \sum_{d|(n/2)} \mu(d) d^3 g_3\left(\frac{n}{2d}\right) \\ &= \frac{1}{64} \sum_{d|(n/2)} \mu(d) \left(\frac{n^4}{d} + 4n^3 + 4n^2 d\right) \\ &= \frac{1}{64} \left(\frac{n^4\phi(n/2)}{n/2} + 4n^2\psi\left(\frac{n}{2}\right)\right) && \text{by (2), (3), and (4)} \\ &= \frac{n^3\phi(n) + 4n^2\psi(n)}{64} && \text{by Lemma 3.2(i), (ii).} \end{aligned}$$

Case II: $n \equiv \pm 1 \pmod 4$.

$$\begin{aligned} S_3'(n) &= \sum_{d|n} \mu(d) d^3 g_3\left(\frac{n/d - 1}{2}\right) \\ &= \frac{1}{64} \sum_{d|n} \mu(d) \left(\frac{n^4}{d} - 2n^2 d + d^3\right) \\ &= \frac{n^3\phi(n) - 2n^2\psi(n) + \psi_3(n)}{64} && \text{by (2), (4), and (9).} \end{aligned}$$

Case III: $n \equiv 2 \pmod 4$.

$$S_3'(n) = \sum_{d|(n/2)} \mu(d) d^3 \left[g_3\left(\frac{n}{2d}\right) - 2^3 g_3\left(\frac{n/2d - 1}{2}\right)\right]$$

$$= \frac{1}{64} \sum_{d|(n/2)} \mu(d) \left(\frac{n^4}{2d} + 4n^3 + 8n^2 d - 8d^3 \right)$$
$$= \frac{1}{64} \left(\frac{n^4 \phi(n/2)}{2(n/2)} + 8n^2 \psi \left(\frac{n}{2} \right) - 8\psi_3 \left(\frac{n}{2} \right) \right) \quad \text{by (2), (3), (4), and (9)}$$
$$= \frac{n^3 \phi(n) - 8n^2 \psi(n) + 8\psi_3(n)/7}{64} \qquad \text{by Lemma 3.2(i), (ii).}$$

For the last case, we observe that $7|\psi_3(n)$ since n is even.

By the same way, the reader can verify any other explicit formula for $S_k(n)$ and $S'_k(n)$ $(k > 3)$ by using the formula for $g_k(n)$, Lemma 3.1, and Lemma 3.2 as an exercise.

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