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CENTRALIZING AND COMMUTING INVOLUTION IN RINGS WITH DERIVATIONS

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Abstract. In [1], Ali and Dar proved the $*$-version of classical theorem due to Posner [15, Theorem] with involution of the second kind. The main objective of this paper is to improve the above mentioned result without the condition of the second kind involution. Moreover, a related result has been discussed.

1. Introduction and results

This research has been motivated by the works done by Ali and Dar [1], and Nejjar et al. [14]. In what follows, $\mathscr{R}$ is a prime ring with involution $*$, $\mathcal{Z}(\mathscr{R})$ the center of $\mathscr{R}$, $\mathscr{C}$ the extended centroid of $\mathscr{R}$, $\mathcal{H}(\mathscr{R})$ the set of hermitian elements of $\mathscr{R}$ and $\mathcal{S}(\mathscr{R})$ the set of skew-hermitian elements $\mathscr{R}$. The involution is said to be of the first kind if $\mathcal{Z}(\mathscr{R}) \subseteq \mathcal{H}(\mathscr{R})$, otherwise it is said to be of the second kind. In the later case it is worthwhile to see that $\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$. A ring $\mathscr{R}$ is called normal if $[x, x^*] = 0$ for all $x \in \mathscr{R}$. We denote s_4 as the standard identity with four variables. We refer the reader to [11] for justification and amplification for the above mentioned notations and key definitions.

An additive mapping $\delta : \mathscr{R} \to \mathscr{R}$ is said to be a derivation on $\mathscr{R}$ if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathscr{R}$. A derivation δ is said to be inner if there exists $a \in \mathscr{R}$ such that $\delta_a(x) = ax - xa$ for all $x \in \mathscr{R}$. A mapping f of $\mathscr{R}$ into itself is known as centralizing if $[f(x), x] \in \mathcal{Z}(\mathscr{R})$ holds for all $x \in \mathscr{R}$; in the special case when $[f(x), x] = 0$ for all $x \in \mathscr{R}$, f is called commuting. The history of centralizing and commuting maps goes back to 1955 when Divinsky [10] established that a simple artinian rings is commutative if it has commuting non-trivial automorphisms. After two years, Posner [15] proved that the existence of nonzero centralizing derivation on a prime ring forces the ring to be commutative. The main concepts arising directly from Posner's result are the study of centralizing (resp. commuting) derivations, centralizing (resp. commuting) additive maps, centralizing (resp. commuting) traces of multiadditive maps, and various generalizations of the notion of a centralizing (resp. commuting) maps, with many

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applications to different areas like Lie theory. For more details of said work see [7] and references therein. In 1993, Brešar [6] characterized additive centralizing maps. In particular, he established the following: Let R be a prime ring. If an additive map f of $\mathscr{R}$ is commuting in $\mathscr{R}$, then there exist $\lambda \in \mathscr{C}$ and an additive mapping $\xi : \mathscr{R} \to \mathscr{C}$, such that $f(x) = \lambda x + \xi(x)$ for all $x \in \mathscr{R}$.

In the last couple of years, there has been a lot of work concerning involution, specially of the second kind, with differential identities; see, for instance [1–4, 8, 9, 13, 14]. In 2014, Ali and Dar [1, Main Theorem] established the following result: Let $\mathscr{R}$ be a prime ring with involution $*$ such that $char(\mathscr{R}) \neq 2$. Let δ be a derivation of $\mathscr{R}$ such that $[\delta(x), x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in \mathscr{R}$ and $\delta(\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})) \neq (0)$. Then $\mathscr{R}$ is commutative. Later, Nejjar et al. [14, Theorem 3.7] improved Ali and Dar's result by removing the condition of $\delta(\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})) \neq (0)$ on prime rings with involution of the second kind. In view of above mentioned results, it is natural to ask a question, what happens if we take prime ring without the condition of the second kind involution in these results? In the present manuscript, we give an affirmative answer to the above mentioned question and establish the following theorem:

Theorem 1.1. *Let $\mathscr{R}$ be a prime ring with involution such that $char(\mathscr{R}) \neq 2$. If $\mathscr{R}$ admits a nonzero derivation $\delta : \mathscr{R} \to \mathscr{R}$ such that $[\delta(x), x^*] \in \mathcal{Z}(\mathscr{R})$ for all $x \in \mathscr{R}$, then the involution is of the first kind or $\mathscr{R}$ satisfies s_4.*

In [16], Vukman states that if $\mathscr{R}$ is a prime ring of characteristic not two and δ is a derivation of $\mathscr{R}$ such that $[[\delta(x), x], x] = 0$ for all $x \in \mathscr{R}$, then either $\mathscr{R}$ is commutative or $\delta = 0$. This result has been extended for Lie ideals by Awtar [5]. Our next theorem is motivated by the these results. In fact, we prove the following theorem:

Theorem 1.2. *Let $\mathscr{R}$ be a prime ring with involution such that $char(\mathscr{R}) \neq 2$. If $\mathscr{R}$ admits a nonzero derivation $\delta : \mathscr{R} \to \mathscr{R}$ such that $[[\delta(x), x], x^*] = 0$ for all $x \in \mathscr{R}$, then the involution is of the first kind or $\mathscr{R}$ satisfies s_4.*

In [1], Ali and Dar demonstrated that a prime ring is commutative if it has commuting involution of the second kind. In fact, they proved the following lemma: Let $\mathscr{R}$ be a prime ring with involution such that $char(\mathscr{R}) \neq 2$. If $(\mathcal{S}(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})) \neq (0)$ and $[x, x^*] = 0$ for all $x \in \mathscr{R}$, then $\mathscr{R}$ is commutative. We prove this result in more general context as:

Lemma 1.1. *Let $\mathscr{R}$ be a prime ring with involution such that $char(\mathscr{R}) \neq 2$. If $[x, x^*] = 0$ for all $x \in \mathscr{R}$, then the involution is of the first kind or $\mathscr{R}$ satisfies s_4.*

Proof. By the hypothesis, we have

$$[x, x^*] = 0 \tag{1.1}$$

for all $x \in \mathscr{R}$. In view of [6, Theorem 3.2], there exist $\lambda \in \mathscr{C}$ and an additive map $\xi : \mathscr{R} \to \mathscr{C}$ such that

$$x^* = \lambda x + \xi(x) \tag{1.2}$$

for all $x \in \mathscr{R}$. Taking $x^* = x$ in (1.2), we get

$$x = \lambda x^* + \xi(x^*) \tag{1.3}$$

for all $x \in \mathscr{R}$. Application of (1.2) yields

$$x = \lambda(\lambda x + \xi(x)) + \xi(x^*) = \lambda^2 x + \lambda\xi(x) + \xi(x^*) \tag{1.4}$$

for all $x \in \mathscr{R}$. By commuting above relation with y, it follows

$$[x, y] = \lambda^2 [x, y] \tag{1.5}$$

for all $x, y \in \mathscr{R}$. Now commuting (1.2) with y, we obtain

$$[x^*, y] = \lambda[x, y] \tag{1.6}$$

for all $x, y \in \mathscr{R}$. Left multiplication of (1.6) with $\lambda \in \mathscr{C}$ and application of (1.5) yields

$$\lambda[x^*, y] = \lambda^2 [x, y] = [x, y] \tag{1.7}$$

for all $x, y \in \mathscr{R}$. Replace x by yx in (1.7), we obtain

$$\lambda[x^*, y]y^* = y[x, y] \tag{1.8}$$

for all $x, y \in \mathscr{R}$. Right multiplication of (1.7) with y^* yields

$$\lambda[x^*, y]y^* = [x, y]y^* \tag{1.9}$$

for all $x, y \in \mathscr{R}$. In view of (1.8) and (1.9), we have

$$[x, y]y^* = y[x, y] \tag{1.10}$$

for all $x, y \in \mathscr{R}$. Again replacing x by y^*x in (1.7) and proceeding as above, we arrive at

$$[x, y]y = y^*[x, y] \tag{1.11}$$

for all $x, y \in \mathscr{R}$. Combination of (1.10) with (1.11) gives

$$[x, y](y^* + y) = (y^* + y)[x, y]. \tag{1.12}$$

Replacement of y by $h + k$, where $h \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$, in above relation yields

$$[[x, h], h] + [[x, k], h]] = 0 \tag{1.13}$$

for all $x \in \mathscr{R}$, $h \in \mathcal{H}(\mathscr{R})$ and $k \in \mathcal{S}(\mathscr{R})$. Replacing h by $-h$ in above expression and combining with (1.13), we get

$$[[x, h], h]] = 0 \tag{1.14}$$

for all $x \in \mathscr{R}$ and $h \in \mathcal{H}(\mathscr{R})$. It can be easily seen that $\delta_x(h) = [x, h]$ is a derivation on $\mathscr{R}$. If $\delta_x(h)$ is nonzero, then it follows from [12, Theorem 2] that $\mathscr{R}$ satisfies s_4. On the other hand, if $\delta_x(h) = 0$, then the involution is of the first kind. This completes the proof. □

Now, we are ready to prove our main results:

Proof of Theorem 1.1. By the assumption $[\delta(x), x^*] \in Z(R)$ for all $x \in \mathscr{R}$. This further implies that $[\delta(x)^*, x] \in \mathcal{Z}(\mathscr{R})$ for all $x \in \mathscr{R}$. Therefore, in view of [6, Theorem 3.2], there exist $\lambda \in \mathscr{C}$ and an additive map $\xi : \mathscr{R} \to \mathscr{C}$ such that

$$\delta(x) = \lambda^* x^* + (\xi(x))^* \tag{1.15}$$

for all $x \in \mathscr{R}$. Choose $\beta \in \mathcal{Z}(\mathscr{R})$ and $x \in \mathscr{R}$. Then by (1.15), we have

$$\delta(\beta x) = \beta^* \lambda^* x^* + (\xi(\beta x))^* \tag{1.16}$$

for all $x \in \mathscr{R}$. Thus,

$$\delta(\beta x) = \delta(\beta)x + \beta\delta(x) = \delta(\beta)x + \beta\lambda^* x^* + (\xi(\beta x))^* \tag{1.17}$$

for all $x \in \mathscr{R}$. In view of (1.16) and (1.17), we have

$$\delta(\beta)x + (\beta - \beta^*)\lambda^* x^* = 0 \tag{1.18}$$

for all $x \in \mathscr{R}$. Commuting above relation with y gives

$$\delta(\beta)[x, y] + (\beta - \beta^*)\lambda^*[x^*, y] = 0 \tag{1.19}$$

for all $x, y \in \mathscr{R}$. For $y = x$, the above relation reduces to

$$(\beta - \beta^*)\lambda^*[x^*, x] = 0 \tag{1.20}$$

for all $x \in \mathscr{R}$. Since the center of the prime ring is free from zero devisors, so either $\beta = \beta^*$ for all $\beta \in \mathcal{Z}(\mathscr{R})$ or $[x, x^*] = 0$ for all $x \in \mathscr{R}$. Therefore in view of Lemma 1.1, we get the required result. This completes the proof of the theorem. □

Proof of Theorem 1.2. We have

$$[[\delta(x), x], x^*] = 0 \tag{1.21}$$

for all $x \in \mathscr{R}$. A linearization of (1.21) gives

$$\begin{aligned} 0 = {} & [[\delta(y), x], x^*] + [[\delta(x), y], x^*] + [[\delta(x), x], y^*] \\ & + [[\delta(y), y], x^*] + [[\delta(x), y], y^*] + [[\delta(y), x], y^*] \end{aligned} \tag{1.22}$$

for all $x, y \in \mathscr{R}$. Replacing y by $-y$ in above expression and using (1.21), we obtain

$$\begin{aligned} 0 = {} & -[[\delta(y), x], x^*] - [[\delta(x), y], x^*] - [[\delta(x), x], y^*] \\ & + [[\delta(y), y], x^*] + [[\delta(x), y], y^*] + [[\delta(y), x], y^*] \end{aligned} \tag{1.23}$$

for all $x, y \in \mathscr{R}$. Combination of (1.22) with (1.23) yields

$$[[\delta(y), x], x^*] + [[\delta(x), y], x^*] + [[\delta(x), x], y^*] = 0 \tag{1.24}$$

for all $x, y \in \mathscr{R}$. Taking $y = k$ in (1.24), where $k \in \mathcal{S}(\mathscr{R})$, we get

$$[[\delta(k), x], x^*] + [[\delta(x), k], x^*] - [[\delta(x), x], k] = 0 \tag{1.25}$$

for all $x \in \mathscr{R}$ and $k \in \mathcal{S}(\mathscr{R})$. Substitution of x by $x+k$ in above equation, where $k \in \mathcal{S}(\mathscr{R})$, gives

$$[[\delta(k),k],x^*]-[[\delta(k),x],k]+[[\delta(x),k],k]=0 \tag{1.26}$$

for all $x \in \mathscr{R}$ and $k \in \mathcal{S}(\mathscr{R})$. Replacement of x by βx, where $\beta \in \mathcal{Z}(\mathscr{R})$, in (1.26) yields

$$\begin{aligned} 0 = \beta^*[[\delta(k),k],x^*] - \delta(\beta)[[k,x],k] \\ - \beta[[\delta(k),x],k] + \beta[[\delta(x),k],k] \end{aligned} \tag{1.27}$$

for all $x, z \in \mathscr{R}$ and $k \in \mathcal{S}(\mathscr{R})$. Multiply (1.26) by β and combine with (1.27), we have

$$(\beta-\beta^*)[[\delta(k),k],x^*]+\delta(\beta)[[x,k],k]=0 \tag{1.28}$$

for all $x \in \mathscr{R}$, $k \in \mathcal{S}(\mathscr{R})$ and $\beta \in \mathcal{Z}(\mathscr{R})$. Another combination of (1.22) with (1.23) gives

$$[[\delta(y),y],x^*]+[[\delta(x),y],y^*]+[[\delta(y),x],y^*]=0 \tag{1.29}$$

for all $x, y \in \mathscr{R}$. Take $y=k$, where $k \in \mathcal{S}(\mathscr{R})$, we get

$$[[\delta(k),k],x^*]-[[\delta(x),k],k]-[[\delta(k),x],k]=0 \tag{1.30}$$

for all $x \in \mathscr{R}$ and $k \in \mathcal{S}(\mathscr{R})$. In view of (1.26) and (1.30), we have

$$[[\delta(x),k],k]=0 \tag{1.31}$$

for all $x \in \mathscr{R}$ and $k \in \mathcal{S}(\mathscr{R})$. Replacing x by βx yields that

$$\delta(\beta)[[x,k],k]=0 \tag{1.32}$$

for all $x \in \mathscr{R}$, $k \in \mathcal{S}(\mathscr{R})$ and $\beta \in \mathcal{Z}(\mathscr{R})$. Using above relation in (1.28), we get $(\beta-\beta^*)[[\delta(k),k],x^*]=0$ for all $x \in \mathscr{R}$, $k \in \mathcal{S}(\mathscr{R})$ and $\beta \in \mathcal{Z}(\mathscr{R})$. This implies either $\beta=\beta^*$ for all $\beta \in \mathcal{Z}(\mathscr{R})$ or $[\delta(k),k] \in \mathcal{Z}(\mathscr{R})$ for all $x \in \mathscr{R}$ and $k \in \mathcal{S}(\mathscr{R})$. Therefore the involution is of the first kind or $\mathscr{R}$ satisfies s_4 from [12, Theorem 7]. This completes the proof of the theorem. □

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