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STRONG *P*-CLEANNESS OF TRIVIAL MORITA CONTEXTS

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Abstract. Let R be a ring with identity and $P(R)$ denote the prime radical of R. An element r of a ring R is called strongly P-clean, if there exists an idempotent e such that $r - e = p \in P(R)$ with $ep = pe$. In this paper, we determine necessary and sufficient conditions for an element of a trivial Morita context to be strongly P-clean.

1. Introduction

Throughout this paper rings are associative with $1 \neq 0$ and modules are unitary. The Jacobson radical, the set of nilpotent elements, the group of units, the set of idempotents of a ring R denoted by $J(R)$, $nil(R)$, $U(R)$, $Id(R)$, respectively. Also, $T_2(R)$ denotes a triangular matrix ring over R.

Strongly π-regular rings are introduced by Kaplansky in [9] as a common generalization of algebraic algebras and artinian rings. A ring R is said to be *strongly π-regular* if for every $r \in R$ there exist a positive integer n and $s \in R$ such that $r^n = r^{n+1}s$. The notion of strong cleanness has been studied in [13] by Nicholson as a generalization of strongly π-regularity. An element r of a ring R is called *strongly clean* if $r = e + u$ and $eu = ue$ where $e^2 = e \in R$ and $u \in U(R)$. If every element of a ring R is strongly clean, then R is *strongly clean*. There are many papers about strong cleanness in the literature [6,10]. In with parallel to these papers, Borooah et al. [1,2] develop the theory of strongly clean rings to matrix rings and triangular matrix rings over local rings. Also the notion of strong cleanness of matrix rings is handled for integral domains in [4, 7]. Motivated by these papers, Diesl [8] introduced a generalization of strongly clean rings. An element $r \in R$ is called *strongly nil clean*, if $r = e + n$ and $en = ne$ where $e^2 = e \in R$ for some $n \in nil(R)$. Similarly, a ring R is said to be *strongly nil clean*, if every element of R is strongly nil clean. The notion of strongly nil cleanness has been extended to matrices by Diesl. In [5], Chen et al. presented strong cleanness of rings related to prime radicals (i.e., *prime radical* of a ring R is the intersection of all prime ideals of R and denoted by

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$P(R)$). It is clear that $P(R)$ contains every nil ideal of R. In the commutative case, prime radical of a ring R is also equal to the set of nilpotent element of R. An element r of a ring R is said to be *strongly P-clean*, if $r = e + p$, $e^2 = e \in R$ and $ep = pe$ for some $p \in P(R)$, and also a ring R is called *strongly P-clean*, if every element of R is strongly P-clean.

Recently, the notion of strong cleanness is extended to generalized matrix rings which is special case of a Morita context [16]. Given a ring R and a central element $s \in R$, the 4-tuple $\left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right)$ is a ring denoted by $K_s(R)$ with formal matrix addition and multiplication defined by

$$\begin{pmatrix} r_1 & v_1 \\ w_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & v_2 \\ w_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + sv_1w_2 & r_1v_2 + v_1s_2 \\ w_1r_2 + s_1w_2 & sw_1v_2 + s_1s_2 \end{pmatrix}.$$

A *Morita context* is a 4-tuple (R, V, W, S), where R, S are rings, ${}_RV_S$ and ${}_SW_R$ are bi-modules, and there exist context products $V \times W \to R$ and $W \times V \to S$ written multiplicatively as $(v, w) \mapsto vw$ and $(w, v) \mapsto wv$, such that $\left(\begin{smallmatrix} R & V \\ W & S \end{smallmatrix}\right)$ is an associative ring with the trivial matrix operations. A Morita context $\left(\begin{smallmatrix} R & V \\ W & S \end{smallmatrix}\right)$ is called *trivial* if the context products are trivial, i.e., $WV = 0$ and $VW = 0$ (see for details [12] and [15]).

Morita contexts appeared in the work of Morita [11] and they play an important role in the category of rings and modules that described equivalences between full categories of modules over rings. This concept appears in disguise in the literature of the ring theory. A Morita context forms a very large class of rings generalizing matrix rings. Formal triangular matrix rings are some of the obvious examples of Morita contexts.

Motivated by aforementioned papers, we consider the strong P-cleanness in a Morita context. We investigate strong P-cleanness of a trivial Morita context where context products are equal to 0. Necessary and sufficient conditions are determined for an element in a trivial Morita context over a local ring to be strongly P-clean. Moreover, strong P-cleanness is investigated for triangular matrix rings and formal matrix rings over local rings in that paper. In Section 3, we define determinant and trace for an element of a trivial Morita context over a commutative ring. Then we exhibit some equal conditions to be strongly P-clean element in a trivial Morita context over a commutative local ring.

2. Noncommutative case

In this section, we study strong P-cleanness in a trivial Morita context $T = (R, V, W, S)$ where R and S are rings and ${}_RV_S$ and ${}_SW_R$ are bi-modules. Now we give some useful results related to some properties of a trivial Morita context and the notion of strong P-cleanness in a trivial Morita context.

We begin with the following results for the sake of completeness.

Lemma 2.1 ([3, Theorem 2.9]). *If $T = (R, V, W, S)$ is a trivial Morita context, then* $P(T) = \begin{pmatrix} P(R) & V \\ W & P(S) \end{pmatrix}$.

Lemma 2.2 ([15, Lemma 3.1]). *Let $T = (R, V, W, S)$ be a trivial Morita context. Then we have*

(1) $J(T) = \begin{pmatrix} J(R) & V \\ W & J(S) \end{pmatrix}$,
(2) $U(T) = \{\begin{pmatrix} r & v \\ w & s \end{pmatrix} : r \in U(R), s \in U(S)\}$.

Lemma 2.3. *Let R, S be local rings and $T = (R, V, W, S)$ a trivial Morita context. Then $Id(T) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & v \\ w & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ w & 1 \end{pmatrix}\}$ for any $v \in V$ and $w \in W$.*

Proof. It is straightforward. □

For elements $a, b \in R$, *a is equivalent to b*, if there exist $u_1, u_2 \in U(R)$ such that $b = u_1 a u_2$. Similarly, *a is similar to b*, if there exists $u \in U(R)$ such that $b = u^{-1} a u$.

Lemma 2.4. *Let R, S be local rings, $T = (R, V, W, S)$ a trivial Morita context. Then every non-trivial idempotent of T is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof. Let $E^2 = E \in T$ be non-trivial. By Lemma 2.3, E has the form $\begin{pmatrix} 1 & v \\ w & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & v \\ w & 1 \end{pmatrix}$ for $v \in V$ and $w \in W$. Assume that $E = \begin{pmatrix} 1 & v \\ w & 0 \end{pmatrix}$ for some $v \in V$ and $w \in W$. Then,

$$\begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ w & 0 \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and so E is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Similarly, if E has the form $\begin{pmatrix} 0 & v \\ w & 1 \end{pmatrix}$, it can be shown that E is equivalent to the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. □

Lemma 2.5. *In a trivial Morita context, if a non-trivial idempotent is equivalent to a diagonal matrix, then it is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.*

Proof. It can be shown similarly to [14, Theorem 4]. □

Lemma 2.6. *Let $T = (R, V, W, S)$ be a Morita context and $A \in T$. Then A is strongly P-clean if and only if UAU^{-1} is strongly P-clean for any $U \in U(T)$.*

Proof. Let A be strongly P-clean. Then we have $A = E + P$ and $EP = PE$ where $E^2 = E \in T$ and $P \in P(T)$. Put $E_u = UEU^{-1}$ and $P_u = UPU^{-1}$. It is clear that ${E_u}^2 = E_u$ and $P_u \in P(T)$. Also we have $E_u P_u = P_u E_u$. Hence $UAU^{-1} = E_u + P_u$, as asserted. The converse is clear. □

Lemma 2.7. *Let R, S be rings and $T = (R, V, W, S)$ a Morita context. If $U \in U(T)$ is strongly P-clean, then $1_T - U \in P(T)$.*

Proof. Assume that $U \in U(T)$ is strongly P-clean. Then, there exists an idempotent $E \in T$ such that $U - E = P \in P(T)$ and $EP = PE$. Then, we have $U^{-1}U - U^{-1}E = 1_T - U^{-1}E \in P(T) \subseteq J(T)$. Hence, $U^{-1}E \in U(T)$ and so $E \in U(T)$. So E is identity matrix. This completes the proof. □

Lemma 2.8. *Let R, S be rings and $T = (R, V, W, S)$ a Morita context. Then $A \in T$ is strongly P-clean if and only if $1_T - A \in T$ strongly P-clean.*

Proof. It is clear. □

The next result is useful to determine whether an element is strongly P-clean or not, in a trivial Morita context over local rings.

Theorem 2.9. *Let R, S be local rings, $T = (R, V, W, S)$ a trivial Morita context. Then, A is strongly P-clean if and only if one of the following holds.*

(1) $A \in P(T)$.
(2) $1_T - A \in P(T)$.
(3) *A is similar to* $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ *such that* $r \in P(R)$ *and* $s \in 1 + P(S)$.
(4) *A is similar to* $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ *such that* $r \in 1 + P(R)$ *and* $s \in P(S)$.

Proof. Assume that A is strongly P-clean, also neither $A \in P(T)$, nor $1_T - A \in P(T)$. Then, there exists a non-trivial idempotent E such that $A - E = P \in P(T)$ and $PE = EP$. Without loss of generality we may assume E has the form $\left(\begin{smallmatrix} 1 & v \\ w & 0 \end{smallmatrix}\right)$. Therefore, E is similar to $E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ by Lemma 2.5. There exists $U \in U(T)$ such that $UEU^{-1} = E_1$. Hence, $UAU^{-1} = E_1 + UPU^{-1}$. Also we have $UPU^{-1}E_1 = E_1UPU^{-1}$. Thus, $UPU^{-1} = \left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ with $r \in P(R)$ and $s \in P(S)$. Finally, A is similar to $\left(\begin{smallmatrix} 1+r & 0 \\ 0 & s \end{smallmatrix}\right)$ such that $r \in P(R)$ and $s \in P(S)$. Similarly, if the idempotent has the form $\left(\begin{smallmatrix} 0 & v \\ w & 1 \end{smallmatrix}\right)$, we conclude that A is similar to $\left(\begin{smallmatrix} r & 0 \\ 0 & 1+s \end{smallmatrix}\right)$ such that $r \in P(R)$ and $s \in P(S)$. The converse is straightforward. □

The next corollary is a direct consequence of Theorem 2.9.

Corollary 2.10 ([5, Proposition 3.6])**.** *Let R be a local ring and $A \in T_2(R)$. Then, A is strongly P-clean if and only if one of the following holds*:

(1) $A \in P(T_2(R))$.
(2) $1 - A \in P(T_2(R))$.
(3) *A is similar to* $\left(\begin{smallmatrix} r_1 & 0 \\ 0 & r_2 \end{smallmatrix}\right)$ *where* $r_1 \in P(R)$ *and* $r_2 \in 1 + P(R)$.
(4) *A is similar to* $\left(\begin{smallmatrix} r_1 & 0 \\ 0 & r_2 \end{smallmatrix}\right)$ *where* $r_1 \in 1 + P(R)$ *and* $r_2 \in P(R)$.

Now we give a result about strongly clean elements in a trivial Morita context over local rings.

Proposition 2.11. *Let R, S be local rings, $T = (R, V, W, S)$ a trivial Morita context. Then, A is strongly clean if and only if one of the following holds*:

(1) $A \in U(T)$.
(2) $1_T - A \in U(T)$.
(3) *A is similar to* $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ *such that* $r \in 1 + J(R)$ *and* $s \in J(S)$.
(4) *A is similar to* $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ *such that* $r \in J(R)$ *and* $s \in 1 + J(S)$.

Proof. If $A \in U(T)$ or $1_T - A \in U(T)$, then A is strongly clean. Assume that A is similar to $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ where $r \in 1 + J(R)$ and $s \in J(S)$. Then $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} r & 0 \\ 0 & s-1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is known that A is strongly clean if and only if UAU^{-1} is strongly clean for every $U \in U(T)$. So A is strongly clean. One can similarly show that A is strongly clean, in case of $r \in J(R)$ and $s \in 1 + J(S)$. Conversely, suppose that A is strongly clean, neither $A \in U(T)$, nor $1_T - A \in U(T)$. It can be shown that A is similar to $\begin{pmatrix} 1+r & 0 \\ 0 & s \end{pmatrix}$ or $\begin{pmatrix} r & 0 \\ 0 & 1+s \end{pmatrix}$ such that $r \in J(R)$ and $s \in J(S)$, as in the proof of Theorem 2.9. Hence the proof is completed. □

By using Proposition 2.11, we prove that every trivial Morita context over the commutative local ring is strongly clean.

Let R and S be rings, ${}_RV_S$ and ${}_SW_R$ bi-modules. For $a \in R$, $l_a : V \to V$ and $r_a : W \to W$ denote the abelian group endomorphisms of V and W, respectively, given by $l_a(v) = av$ and $r_a(w) = wa$ for all $v \in V$ and $w \in W$. Similarly, for $b \in S$, r_b and l_b are the abelian group homomorphisms of V and W, respectively.

We give new characterizations to determine whether an element of a trivial Morita context is strongly P-clean or not, via endomorphisms of modules which are components of a trivial Morita context.

Theorem 2.12. *Let R, S be local rings, $T = (R, V, W, S)$ a trivial Morita context. Then the following are equivalent for any $x \in 1 + P(R)$ and $t \in P(S)$.*

(1) *$l_x - r_t$ of V and $r_x - l_t$ of W are surjective.*
(2) *$A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is strongly P-clean in T such that $y \in V$, $z \in W$.*

Proof. Let $l_x - r_t$ of V and $r_x - l_t$ of W be surjective. Then, there exist $y \in V$ and $z \in W$ such that $(l_x - r_t)(e) = y$ and $(r_x - l_t)(f) = z$ for some $e \in V$ and $f \in W$. Take $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. Hence, we can write $A = \begin{pmatrix} 1 & e \\ f & 0 \end{pmatrix} + P$ such that $P \in P(T)$. Clearly, $E = \begin{pmatrix} 1 & e \\ f & 0 \end{pmatrix}$ is an idempotent and $PE = EP$. So A is strongly P-clean. For the converse, we claim that $l_x - r_t$ of V and $r_x - l_t$ of W are surjective. Let $y \in V$ and $z \in W$. As $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is strongly P-clean, there exists an idempotent $E = \begin{pmatrix} 1 & e \\ f & 0 \end{pmatrix} \in T$ such that $AE = EA$. Therefore we conclude that $xe - et = (l_x - r_t)(e) = y$ and $fx - tf = (l_x - r_t)(f) = z$. This completes the proof. □

Theorem 2.13. *Let R, S be local rings, $T = (R, V, W, S)$ a trivial Morita context. Then, the following are equivalent for any $x \in P(R)$ and $t \in 1 + P(S)$.*

(1) *$r_t - l_x$ of V and $l_t - r_x$ of W are surjective.*
(2) *$A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is strongly P-clean in T such that $y \in V$ and $w \in W$.*

Proof. It is similar to the proof of Theorem 2.12. □

The following are immediate from Theorem 2.12 and Theorem 2.13.

Corollary 2.14. *Let R, S be local rings, $T = (R, V, W, S)$ a trivial Morita context. If T is a strongly P-clean ring, then the following hold.*

(1) *For any $x \in 1 + P(R)$ and $t \in P(S)$, $r_t - l_x$ of V and $r_x - l_t$ of W are surjective.*
(2) *For any $x \in P(R)$ and $t \in 1 + P(S)$, $r_t - l_x$ of V and $r_x - l_t$ of W are surjective.*

Corollary 2.15. *Let R be a local ring. If $T_2(R)$ is a strongly P-clean ring, then the following hold.*

(1) *For any* $x \in 1 + P(R)$ *and* $t \in P(R)$, $r_t - l_x$ *and* $r_x - l_t$ *of* R *are surjective.*
(2) *For any* $x \in P(R)$ *and* $t \in 1 + P(R)$, $r_t - l_x$ *and* $r_x - l_t$ *of* R *are surjective.*

3. Characteristic criteria for commutative case

In this section we study on a trivial Morita context where $R = S$ is a commutative ring and ${}_RV_R$, ${}_RW_R$ are bi-modules. Firstly we define the determinant, the trace and the scalar multiplication for an element $A \in T = (R, V, W, R)$, afterwards we give a useful lemma without any proof.

Let R be a commutative ring and $T = (R, V, W, R)$ a trivial Morita context. Then we define

$$\det_T(A) = rs,\ tr_T(A) = r + s,\ x\left(\begin{smallmatrix} r & v \\ w & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} xr & xv \\ xw & xs \end{smallmatrix}\right)$$

for any $A = \left(\begin{smallmatrix} r & v \\ w & s \end{smallmatrix}\right) \in T$ and $x \in R$.

The next result is very useful for this paper.

Lemma 3.1. *Let* R *be a commutative ring and* $T = (R, V, W, R)$ *a trivial Morita context. Then the following hold for any* $A, B \in T$.

(1) $\det_T(AB) = \det_T(A)\det_T(B)$.
(2) $A \in U(T)$ *if and only if* $\det_T(A) \in U(R)$.
(3) *Let* $A = \left(\begin{smallmatrix} r & v \\ w & s \end{smallmatrix}\right) \in U(T)$. *Then* $A^{-1} = (\det_T(A))^{-1}\left(\begin{smallmatrix} s & -v \\ -w & r \end{smallmatrix}\right)$.
(4) *If* A *is similar to* B, *then* $\det_T(A) = \det_T(B)$ *and* $tr_T(A) = tr_T(B)$.

Let R be a commutative ring, $T = (R, V, W, R)$ a trivial Morita context. We say that A is a *non-trivial strongly* P*-clean element,* if A is strongly P-clean, $A \notin P(T)$ and $1_T - A \notin P(T)$.

Proposition 3.2. *Let* R *be a commutative local ring and* $T = (R, V, W, R)$ *a trivial Morita context. If* $A \in T$ *is strongly* P*-clean, then either* $A \in P(T)$, *or* $1_T - A \in P(T)$ *or* $tr_T(A) \in 1 + P(R)$ *and* $(tr_T(A))^2 - 4\det_T(A) = u^2$ *for some* $u \in 1 + P(R)$.

Proof. Let $A = \left(\begin{smallmatrix} r & v \\ w & s \end{smallmatrix}\right) \in T$. If A is a trivial strongly P-clean element of T, then $A \in P(T)$ or $1_T - A \in P(T)$. Assume that A is not a trivial strongly P-clean element of T. By Lemma 3.1 and Theorem 2.9, we have $r \in P(R)$ and $s \in 1 + P(R)$ or $r \in 1 + P(R)$ and $s \in P(R)$. Without loss of generality, we may assume $r \in P(R)$ and $s \in 1 + P(R)$. It is clear that $tr_T(A) \in 1 + P(R)$. Also $(tr_T(A))^2 - 4\det_T(A) = (r + s)^2 - 4rs = (r - s)^2$ and $r - s \in 1 + P(R)$. This completes the proof. □

Proposition 3.3. *Let* R *be a commutative ring,* $T = (R, V, W, R)$ *a trivial Morita context,* $A \in T \backslash P(T)$. *If* $\det_T(A) \in P(R)$ *and* $tr_T(A) \in P(R)$, *then* A *is not strongly* P*-clean.*

Proof. Suppose that $A \in T$ is strongly P-clean. Then, we have $1_T - A$ is strongly P-clean in T, by Lemma 2.8. It can be shown that $\det_T(1_T - A) = 1 - tr_T(A) + \det_T(A)$. So $\det_T(1_T - A) \in U(R)$, by Lemma 3.1. Therefore, $(1_T - A) - 1_T \in P(T)$, by Lemma 2.7. This is a contradiction. Hence, A is not strongly P-clean. □

Proposition 3.4. *Let R be a commutative local ring, $T = (R, V, W, R)$ a trivial Morita context and $A \in T$. Then $\det_T(A)$, $tr_T(A) \in P(R)$ and A is strongly P-clean if and only if $A \in P(T)$.*

Proof. Assume that $\det_T(A)$, $tr_T(A) \in P(R)$ and A is strongly P-clean. Then, there exists an idempotent $E \in T$ such that $A - E = P \in P(T)$ and $AE = EA$. If E is equal to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & v \\ w & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & v \\ w & 1 \end{smallmatrix}\right)$ for any $v \in V$ and $w \in W$, this contradicts to our assumptions. Hence, $E = 0$ as desired. The converse is clear. □

The next two theorems are beneficial to determine whether an element, in a trivial Morita context over a commutative local ring, is strongly P-clean or not.

Theorem 3.5. *Let R be a commutative local ring, $T = (R, V, W, R)$ a trivial Morita context. Then the following are equivalent for $A \in T$.*

(1) *A is a non-trivial strongly P-clean element in T.*
(2) *A is similar to $\left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right)$ such that $rs \in P(R)$ and $r + s \in 1 + P(R)$.*
(3) *$\det_T(A) \in P(R)$, $tr_T(A) \in 1 + P(R)$ and A is similar to a diagonal matrix.*
(4) *$\det_T(A) \in P(R)$, $tr_T(A) \in 1 + P(R)$ and the characteristic polynomial of A is solvable in R.*

Proof. (1) $\Leftrightarrow$ (2) It is clear, by Theorem 2.9.

(2) $\Rightarrow$ (3) It is straightforward, by Lemma 3.1.

(3) $\Rightarrow$ (4) Assume that A is similar to $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $\det_T(A) \in P(R)$ and $tr_T(A) \in 1 + P(R)$. By Lemma 3.1, we have $\det_T(A) = ab$ and $tr_T(A) = a + b$. Therefore, $a^2 - tr_T(A)a + ab = 0$ and $b^2 - tr_T(A)b + ab = 0$, as desired.

(4) $\Rightarrow$ (2) Assume that $\det_T(A) \in P(R)$, $tr_T(A) \in 1 + P(R)$ and the characteristic polynomial of A is solvable in R. Let $r^2 - (tr_T(A))r + \det_T(A) = 0$. Take $s = tr_T(A) - r$. It is clear that $s^2 - (tr_T(A))s + \det_T(A) = 0$. Hence, we have $tr_T(A) = r + s$ and $\det_T(A) = rs$. Since $tr_T(A) \in 1 + P(R)$, one of the r and s must be an element of $U(R)$. Without loss of generality, we may assume $r \in U(R)$ and $s \in P(R)$. Take $A = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$. We know that $x + t = tr_T(A) \in 1 + P(R)$. By a similar argument to above, $x \in U(R)$ and $t \in P(R)$ or $x \in P(R)$ and $t \in U(R)$. Assume that $x \in U(R)$ and $t \in P(R)$. Then, $x - s = t - r \in U(R)$. Choose $U = \begin{pmatrix} 1 & y(x-s)^{-1} \\ z(t-r)^{-1} & 1 \end{pmatrix}$. By Lemma 3.1, $U \in U(T)$. Therefore, $U^{-1}AU = \begin{pmatrix} x & \frac{y(s-t)(s-x)}{(s-x)^2} \\ \frac{z(r-x)(r-t)}{(r-t)2} & t \end{pmatrix} = \left(\begin{smallmatrix} x & 0 \\ 0 & t \end{smallmatrix}\right)$. In case of $x \in P(R)$ and $t \in U(R)$, the result is obtained via similar technique. □

Theorem 3.6. *Let R be a commutative local ring, $T = (R, V, W, R)$ a trivial Morita context. Then, the following are equivalent.*

(1) *$A \in T$ is strongly P-clean.*
(2) *$A - A^2 \in P(T)$.*
(3) *A is a trivial strongly P-clean element of T, or the characteristic polynomial of A has a root in $P(R)$ and a root in $1 + P(R)$.*

Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Let $A = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ and $A - A^2 = \begin{pmatrix} x-x^2 & * \\ * & t-t^2 \end{pmatrix} \in P(T)$. Then, $x - x^2 \in P(R)$ and $t - t^2 \in P(R)$, by Lemma 2.1. As R is a local ring, we have the following cases.

Case 1: Let $x \in U(R)$ and $t \in U(R)$. Therefore, $1 - x \in P(R)$ and $1 - t \in P(R)$. This implies that $1_T - A \in P(T)$, by Lemma 2.1.

Case 2: Let $x \in J(R)$ and $t \in U(R)$. Thus, $1 - x \in U(R)$. As $x - x^2 \in P(R)$, we have $x \in P(R)$. Also, it is clear that $t \in 1 + P(R)$. Hence, $t^2 - tr_T(A)t + \det_T(A) = t^2 - (x + t)t + xt = 0$ and $x^2 - tr_T(A)x + \det_T(A) = x^2 - (x + t)x + xt = 0$.

Case 3: Let $x \in U(R)$ and $t \in J(R)$. It can be obtained via similar proof to Case 2.

Case 4: Let $x \in J(R)$ and $t \in J(R)$. Then, $x, t \in P(R)$. Therefore, $A \in P(T)$, by Lemma 2.1.

(3) $\Rightarrow$ (1) Let $A = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in T$. Assume that the characteristic polynomial of A has a root in $P(R)$ and a root in $1 + P(R)$. That is, for $r \in P(R)$ and $s \in 1+P(R)$, let $r^2 - tr_T(A)r + \det_T(A) = 0$ and $s^2 - tr_T(A)s + \det_T(A) = 0$. Then, we have $tr_T(A) = x + t = r + s \in 1 + P(R)$ and $\det_T(A) = xt = rs \in P(R)$. By Theorem 3.5, A is strongly P-clean. Also, if $A \in P(T)$ or $1_T - A \in P(T)$, the result is clear. □

Now we determine when the converse of Proposition 3.2 is true. So we have the following by means of Theorem 3.5 and Theorem 3.6.

Corollary 3.7. *Let R be a commutative local ring, $T = (R, V, W, R)$ a trivial Morita context. If $2 \in U(R)$, then the following are equivalent.*

(1) *$A \in T$ is strongly P-clean.*
(2) *$A \in P(T)$, or $1_T - A \in P(T)$, or $tr_T(A) \in 1 + P(R)$ and $(tr_T(A))^2 - 4\det_T(A) = u^2$ for some $u \in 1 + P(R)$.*

Proof. (1) $\Rightarrow$ (2) It is clear, by Proposition 3.2.

(2) $\Rightarrow$ (1) It can be obtained from Theorem 3.6 and Theorem 3.5. □

We complete this section with the following results which are related to strong cleanness of a trivial Morita context over a commutative local ring.

Proposition 3.8. *Let R be a commutative local ring, $T = (R, V, W, R)$ a trivial Morita context. Then, T is strongly clean.*

Proof. Let $A = \begin{pmatrix} r & v \\ w & s \end{pmatrix} \in T$. We will complete the proof case by case.

Case 1: Let $r, s \in U(R)$. It is clear that A is strongly clean.

Case 2: Let $r, s \in J(R)$. Then, $1_T - A \in U(T)$, by Lemma 2.2. So A is strongly clean, by Proposition 2.11.

Case 3: Let $r \in J(R)$ and $s \in U(R)$. Then, $r - s \in U(R)$. Take $U = \begin{pmatrix} 1 & v(r-s)^{-1} \\ w(r-s)^{-1} & 1 \end{pmatrix}$. Hence, $UAU^{-1} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$. Therefore, A is strongly clean, by Proposition 2.11.

Case 4: Let $r \in U(R)$ and $s \in J(R)$. It is similar to Case 3.

Hence, T is a strongly clean ring. □

The following result is a direct consequence of Proposition 3.8.

Corollary 3.9. *Let R be a commutative local ring. Then we have the followings.*

(1) $K_0(R)$ *is strongly clean.*
(2) $T_2(R)$ *is strongly clean.*

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