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SEMISTAR G-GCD DOMAINS

Wafa Gmiza and Sana Hizem

Abstract. Let $\star$ be a semistar operation on the integral domain D. In this paper, we prove that D is a G-$\widetilde{\star}$-GCD domain if and only if $D[X]$ is a G-$\star_1$-GCD domain if and only if the Nagata ring of D with respect to the semistar operation $\widetilde{\star}$, $Na(D,\star_f)$ is a G-GCD domain if and only if $Na(D,\star_f)$ is a GCD domain, where $\star_1$ is the semistar operation on $D[X]$ introduced by G. Picozza [12].

1. Introduction

Let D be an integral domain with quotient field K. Let $\overline{F}(D)$ be the set of all nonzero D-submodules of K, $F(D)$ be the set of all nonzero fractional ideals of D and $f(D)$ be the set of all nonzero finitely generated D-submodules of K.

Semistar operations were first defined in 1994 by A. Okabe and R. Matsuda [10] as an extension of the classical star operations.

A semistar operation on D is a map $\star : \overline{F}(D) \to \overline{F}(D)$; $E \mapsto E^\star$ such that for all $x \in K \setminus \{0\}$ and for all E, $F \in \overline{F}(D)$, the following properties are satisfied:

(1) $(xE)^\star = xE^\star$.
(2) If $E \subseteq F$, then $E^\star \subseteq F^\star$.
(3) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

For every $E \in \overline{F}(D)$, set $E^{\star_f} = \cup\{F^\star \mid F \in f(D) \text{ and } F \subseteq E\}$, $\star_f$ is a semistar operation on D called the semistar operation of finite type associated to $\star$. A semistar operation is said to be of finite type whenever $\star = \star_f$. Let $*_1$ and $*_2$ be two semistar operations on D, we say that $*_1 \leqslant *_2$ if $E^{*_1} \subseteq E^{*_2}$ for each $E \in \overline{F}(D)$, or, equivalently, if $(E^{*_1})^{*_2} = (E^{*_2})^{*_1} = E^{*_2}$. Let $\star$ be a semistar operation on D and I be a nonzero ideal of D, we say that I is a quasi-$\star$-ideal if $I = I^\star \cap D$ and we say that I is a quasi-$\star$-maximal ideal if I is a maximal element in the set of proper quasi-$\star$-ideals. We denote by $M(\star)$ the set of quasi-$\star$-maximal ideals of D. If $\star$ is a non trivial semistar operation ($D^\star \neq K$)

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of finite type, then each proper quasi-$\star$-ideal is contained in a quasi-$\star$-maximal ideal [5, Lemma 4.20].

Let $\star$ be a semistar operation on D, we denote by $\widetilde{\star}$, the semistar operation defined by $\widetilde{\star} : \overline{F}(D) \to \overline{F}(D)$; $E \mapsto E^{\widetilde{\star}} := \cup\{E : J \mid J^{\star_f} = D^{\star_f}\}$. Let $I \in F(D)$, we denote by $I^{-1} = \{x \in K \mid xI \subseteq D\}$ and $I_v = (I^{-1})^{-1}$. If $\star$ is a semistar operation on D, we say that I is $\star$-invertible if $(II^{-1})^\star = D^\star$ and I is called $\star_f$-locally principal if for each $M \in M(\star_f)$ there exists $x \in D$ such that $ID_M = xD_M$.

Let I be a nonzero fractional ideal of D, we say that I is a $\star$-principal ideal if there exists $x \in K$ such that $I^\star = xD^\star$.

Let $\star$ be a semistar operation on the integral domain D. By [4], we say that D is $\star$-GCD if for each $a, b \in D\backslash\{0\}$, $(a,b)_v$ is $\widetilde{\star}$-principal and we say that D is G-$\star$-GCD if for each $a, b \in D\backslash\{0\}$, $aD \cap bD$ is $\star_f$-invertible.

For a semistar operation $\star$ on D, S. El. Baghdadi in [4], proved the analogues of classical properties of GCD rings and G-GCD rings. He proved that D is $\star$-GCD if and only if for all $I \in f(D)$, I_v is a $\widetilde{\star}$-principal ideal and D is G-$\star$-GCD if and only if for all $I \in f(D)$, I_v is a $\star_f$-invertible ideal.

In Section 2 of this paper, we show that D is G-$\star$-GCD if and only if $D[X]$ is G-$\star_1$-GCD, where $\star_1$ is the semistar operation on $D[X]$ introduced by G. Picozza [12]. We generalize some classical results in the context of semistar operations. We prove among others, that if $\star$ is a semistar operation on D, $I \in f(D)$ and if $D^\star$ is integrally closed, then $(I : I)^\star = D^\star$, and if L is a localizing system of D, $f, g \in K[X]\backslash\{0\}$ and if $D^{\star_L}$ is integrally closed, then $(D : c_D(f)c_D(g))^{\star_L} = (D : c_D(fg))^{\star_L}$, where $\star_L$ is the semistar operation on D associated to L [5, Proposition 2.4]. Let (H) be the following property: for every family $(I_\lambda)_{\lambda\in\Lambda}$ of fractional ideals of D with nonzero intersection, we have $(\bigcap_{\lambda\in\Lambda} I_\lambda)^{\widetilde{\star}} = \bigcap_{\lambda\in\Lambda} I_\lambda^{\widetilde{\star}}$. We prove that if $D^{\widetilde{\star}}$ is integrally closed and D satisfies the property (H), then for each $I \in f(D[X])$ there exist $g \in D[X]\backslash\{0\}$ and $N \in f(D)$ such that $(I_v)^{\star_1} = g(N[X]_v)^{\star_1} = g(N_v)^{\widetilde{\star}}[X]$. As a consequence, we get the main result of this paper: if $\star$ is a semistar operation satisfying the property (H), then D is G-$\widetilde{\star}$-GCD if and only if $D[X]$ is G-$\star_1$-GCD.

In Section 3, we prove that D is G-$\widetilde{\star}$-GCD if and only if $Na(D, \star_f)$ is G-GCD if and only if $Na(D, \star_f)$ is GCD, where $Na(D, \star_f)$ is the Nagata ring associated to $\star_f$.

2. G-$\star$-GCD polynomial rings

We recall some definitions and properties related to semistar operations. It is clear that any semistar operation satisfies the following axioms: for all $E, F \in \overline{F}(D)$

(1) $(EF)^\star = (EF^\star)^\star = (E^\star F)^\star = (E^\star F^\star)^\star$.
(2) $(E + F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star$.

(3) For every subset $(E_\alpha)_{\alpha\in\wedge} \subseteq \overline{F}(D)$, $\underset{\alpha\in\wedge}{\cap} E_\alpha^\star = (\underset{\alpha\in\wedge}{\cap} \mathrm{E}_\alpha^\star)^\star$, if $\underset{\alpha\in\wedge}{\cap} E_\alpha^\star \neq (0)$.

The identity is a semistar operation on D, denoted by d_D. The map

$$\begin{aligned} \star \quad : \quad & \overline{F}(D) \longrightarrow \overline{F}(D) \\ & E \longmapsto E^e = K \end{aligned}$$

is a semistar operation called the trivial semistar operation.

Let $\star$ be a semistar operation on D. An ideal I of D is called a quasi-$\star$-ideal of D if $I = I^\star \cap D$, it is easy to see that, for any ideal I of D, the ideal $I^\star \cap D$ is a quasi-$\star$-ideal. An ideal is said to be a quasi-$\star$-prime, if it is prime and a quasi-$\star$-ideal.

A quasi-$\star$-maximal ideal is an ideal that is a maximal element in the set of quasi-$\star$-prime ideals. If $\star$ is a non trivial semistar operation of finite type, then each proper quasi-$\star$-ideal is contained in a quasi-$\star$-maximal ideal [5, Lemma 4.20].

Recall from [5], that a localizing system of D is a family L of ideals of D such that:

(LS_1) If $I \in L$ and J is an ideal of D such that $I \subseteq J$, then $J \in L$.

(LS_2) If $I \in L$ and J is an ideal of D such that $(J :_D iD) \in L$ for each $i \in I$, then $J \in L$.

A localizing system L is finitely generated if for each $I \in L$, there exists a finitely generated ideal $J \in L$ such that $J \subseteq I$. If L is a localizing system, and $I, J \in L$, then $I \cap J \in L$ and $IJ \in L$.

A semistar operation $\star$ is stable if $(E\cap F)^\star = E^\star \cap F^\star$ for each $E, F \in \overline{F}(D)$. The relation between localizing systems and stable semistar operations has been investigated by M. Fontana and J. Huckaba in [5]. We recall the following results from [5]:

Proposition 2.1. *Let D be an integral domain.*

(1) *Let $\star$ be a semistar operation on D and $L^\star = \{I$ ideal of D such that $I^\star = D^\star\}$, then $L^\star$ is a localizing system (called the localizing system associated to $\star$).*

(2) *Let L be a localizing system. The map:*

$$\begin{aligned} \star_L \quad : \quad & \overline{F}(D) \longrightarrow \overline{F}(D) \\ & E \longmapsto E^{\star_L} = \cup\{E :_K J,\ \ J \in L\} \end{aligned}$$

is a stable semistar operation on D.

(3) *Let $\star$ be a semistar operation of finite type. Then $L^\star$ is a finitely generated localizing system.*

(4) *Let L be a finitely generated localizing system. Then $\star_L$ is a semistar operation of finite type.*

(5) *Let $\star$ be a semistar operation on D. Then $\star_{L^\star} = \star$ if and only if $\star$ is stable.*

If $\star$ is a semistar operation, the map $\widetilde{\star} := \star_{L^{\star_f}}$ is a semistar operation associated to the localizing system $L^{\star_f}$. $\widetilde{\star}$ is a stable semistar operation of finite type on D, and for $E \in \overline{F}(D)$, $E^{\widetilde{\star}} = \cap\{ED_M \mid M \in M(\star_f)\}$ [12].

By [5], $\star = \widetilde{\star}$ if and only if $\star$ is stable of finite type. Recall from [8], that if $E \in \overline{F}(D)$ we say that E is a $\star$-finite ideal if there exists $F \in f(D)$ such that $E^\star = F^\star$. In particular, if E is $\star_f$-finite, then it is $\star$-finite. We notice that, in the previous definition of a $\star$-finite ideal, we do not require that $F \subseteq E$. Notice that, E is $\star_f$-finite if and only if there exists $F \in f(D)$ and $F \subseteq E$ such that $F^\star = E^\star$. Let D be an integral domain, T be an overring of D, $i : D \to T$ be the canonical embedding of D in T and $\star$ be a semistar operation on D. By [6], the map $\star_i : \overline{F}(T) \to \overline{F}(T)$, $E \mapsto E^{\star_i} := E^\star$ is a semistar operation on T.

Lemma 2.2. *Let $\star$ be a semistar operation on the integral domain D, and let $I \in f(D)$. If $D^\star$ is integrally closed, then $(I : I)^\star = D^\star$.*

Proof. Because $D \subseteq I : I$, $D^\star \subseteq (I : I)^\star$. Conversely, since $D^\star$ is integrally closed, $D^\star = \cap\{V_\alpha \mid V_\alpha$ is a valuation overring of $D^\star\}$. Let $x \in I : I$ and V_α be a valuation overring of $D^\star$, then $xIV_\alpha \subseteq IV_\alpha$. Since $I \in f(D)$, there exists $a \in K\backslash\{0\}$ such that $IV_\alpha = aV_\alpha$. Hence $xaV_\alpha \subseteq aV_\alpha$ which implies that $x \in D^\star$. ☐

Lemma 2.3. *Let D be an integral domain, L be a localizing system of D, $I \in \overline{F}(D)$ and $J \in f(D)$. Then $(I : J)^{\star_L} = (I^{\star_L} : J)$.*

Proof. Let $x \in (I : J)^{\star_L}$, there exists $F \in L$ such that $xF \subseteq I : J$, so $x \in I^{\star_L} : J$. Conversely, let $x \in I^{\star_L} : J$. Since $J \in f(D)$, there exists $F \in L$ such that $xJF \subseteq I$ then $x \in (I : J)^{\star_L}$. ☐

Proposition 2.4 ([12])**.** *Let D be an integral domain and L be a localizing system of D. Let X be an indeterminate on D.*

(1) *$L[X] := \{I$ ideal of $D[X] \mid JD[X] \subset I$ for some $J \in L\}$ is a localizing system of $D[X]$ and $L[X] = \{I$ ideal of $D[X]$ such that $I \cap D \in L\}$.*
(2) *If L is a finitely generated localizing system of D, then $L[X]$ is a finitely generated localizing system of $D[X]$.*

Let D be an integral domain and L be a localizing system of D. Let X be an indeterminate on D. G. Picozza in [12], defined the following semistar operation on $D[X]$:

$$\begin{aligned} * \ : \ & \overline{F}(D[X]) \longrightarrow \overline{F}(D[X]) \\ & E \longmapsto (E)^* := \cup\{E : J[X] \mid J \in L\} \end{aligned}$$

It is clear that $*$ is a stable semistar operation on $D[X]$.

Remark 2.5. (1) Let $I \in \overline{F}(D)$ then $(I[X])^* = I^{\star_L}[X]$. Indeed, let $f \in (I[X])^*$, there exists $F \in L$ such that $fF \subseteq I[X]$ which implies that $f \in K[X]$. Set $f = \sum_{i=0}^n a_iX^i$ with $a_i \in K$ then $a_iF \subseteq I$ so, $a_i \in I^{\star_L}$ for each $i \in \{0, \dots, n\}$. Hence $f \in I^{\star_L}[X]$. Conversely, let $f = \sum_{i=0}^n a_iX^i \in I^{\star_L}[X] \subseteq K[X]$, there exists $F \in L$ such that $a_iF \subseteq I$ for each $i \in \{0, \dots, n\}$. Hence $a_iX^i \subseteq (I[X])^*$ and $f \in (I[X])^*$.

(2) If $\star$ is a semistar operation on the integral domain D, then $\star_1 = \star_{L^{\star_f}[X]}$ is a stable semistar operation of finite type on $D[X]$.

Lemma 2.6 ([13, Lemme 1])**.** *Let D be an integral domain and $f, g \in K[X]$. If D is integrally closed, then $(c(f)c(g))^{-1} = (c(fg))^{-1}$.*

Lemma 2.7. *Let D be an integral domain, L be a localizing system of D and $f, g \in K[X]$. If $D^{\star_L}$ is an integrally closed domain, then $(D : c_D(f)c_D(g))^{\star_L} = (D : c_D(fg))^{\star_L}$.*

Proof. Let $R = D^{\star_L}$. By Lemma 2.6, $(c_R(f)c_R(g))^{-1} = (c_R(fg))^{-1}$. But $c_R(f) = c_D(f)R$ implies that $(c_D(f)c_D(g)R)^{-1} = (c_D(fg)R)^{-1}$. That is to say $(D^{\star_L} : c_D(f)c_D(g)D^{\star_L}) = (D^{\star_L} : c_D(fg)D^{\star_L})$. So $(D^{\star_L} : c_D(f)c_D(g)) = (D^{\star_L} : c_D(fg))$ and by Lemma 2.3, $(D : c_D(f)c_D(g))^{\star_L} = (D : c_D(fg))^{\star_L}$. $\square$

Lemma 2.8 ([13, Lemme 3])**.** *Let I be a divisorial ideal of $D[X]$ such that $J = I \cap K \neq (0)$, let $B = D[X]$. Then $J = \cap\{d(D :_K c(g)) \mid I \subseteq Bcg^{-1}, d \in D\backslash\{0\}$ and $g \in B\}$.*

Lemma 2.9. *Let $\star$ be a semistar operation on the integral domain D satisfying the property (H) : whenever $(I_\alpha)_{\alpha\in\Lambda}$ is a family of fractional ideals of D with nonzero intersection, $(\bigcap_{\alpha\in\Lambda} I_\alpha)^{\widetilde{\star}} = \bigcap_{\alpha\in\Lambda} I_\alpha^{\widetilde{\star}}$. Let $I \in F(D)$. Then*

(1) $(I^{-1})^{\widetilde{\star}} = (I^{\widetilde{\star}})^{-1}$.
(2) $(I_v)^{\widetilde{\star}} = (I^{\widetilde{\star}})_v$.

Proof. (1) Let $x \in (I^{-1})^{\widetilde{\star}}$, there exists $F \in L^{\star_f}$ such that $xF \subseteq I^{-1}$. Hence $xI \subseteq D^{\widetilde{\star}}$ and $x \in (I^{\widetilde{\star}})^{-1}$. Conversely, since $\star$ satisfies the property (H), $(I^{-1})^{\widetilde{\star}} = \bigcap_{a\in I} a^{-1}D^{\widetilde{\star}}$ and $(I^{\widetilde{\star}})^{-1} = \bigcap_{a\in I^{\widetilde{\star}}} a^{-1}D^{\widetilde{\star}}$. As $I \subseteq I^{\widetilde{\star}}$ we have $(I^{-1})^{\widetilde{\star}} \supseteq (I^{\widetilde{\star}})^{-1}$.

(2) $(I_v)^{\widetilde{\star}} = ((I^{-1})^{-1})^{\widetilde{\star}} = ((I^{-1})^{\widetilde{\star}})^{-1} = ((I^{\widetilde{\star}})^{-1})^{-1} = (I^{\widetilde{\star}})_v$. $\square$

Examples 2.10. (1) Let D be an integral domain and e be the following semistar operation:

$$\begin{aligned} e \quad : \quad & \overline{F}(D) \longrightarrow \overline{F}(D) \\ & E \longmapsto E^e = K \end{aligned}$$

e is a stable semistar operation of finite type and satisfies the property (H).

(2) Recall from [14, Definition 4.1] that, if D is an integral domain and Θ is a set of overrings of D such that the quotient field of D is not in Θ, we say that Θ is a Jaffard family on D if for every integral ideal I of D,

- $D = \bigcap_{T\in\Theta} T$.
- Θ is locally finite. (i.e., if every $x \in D\backslash\{0\}$ is a nonunit in only finitely many $T \in \Theta$.)

- $I = \underset{T\in\Theta}{\cap}(IT \cap D)$.
- If $T \neq S$ are in Θ, then $(IT \cap D) + (IS \cap D) = D$.

Let D be an integral domain, Θ be a Jaffard family on D and $T \in \Theta$ such that $T \neq D$. As T is a flat overring of D, the following semistar operation

$$\begin{aligned} \star \ : \ \overline{F}(D) &\longrightarrow \overline{F}(D) \\ E &\longmapsto E^{\star} = ET \end{aligned}$$

is a stable semistar operation of finite type on D and $\star \neq d$. By [14, Proposition 4.5], for each family $(I_\alpha)_{\alpha\in\Lambda}$ of D-submodules of K with nonzero intersection, $(\underset{\alpha\in\Lambda}{\cap} I_\alpha)T = \underset{\alpha\in\Lambda}{\cap} I_\alpha T$. Hence $(\underset{\alpha\in\Lambda}{\cap} I_\alpha)^{\star} = \underset{\alpha\in\Lambda}{\cap}(I_\alpha^{\star})$.

(3) Recall from [11], that a domain D has finite character if each nonzero element of D is contained in at most finitely many maximal ideals of D. We say that D is h-local if D has finite character and each nonzero prime ideal of D is contained in a unique maximal ideal of D. By [11, Example 3.2], there exists a non local domain D such that D is h-local and every maximal ideal of D has height 2. By [14, Page 8], $\{D_M \mid M \in Max(D)\}$ is a Jaffard family. Let $N \in Max(D)$, the following semistar operation

$$\begin{aligned} \star_{\{D_N\}} \ : \ \overline{F}(D) &\longrightarrow \overline{F}(D) \\ E &\longmapsto E^{\star_{\{D_N\}}} = ED_N \end{aligned}$$

is a stable semistar operation of finite type, $\star_{\{D_N\}} \neq d$ and $\star_{\{D_N\}}$ satisfies the property (H).

Theorem 2.11. *Let $\star$ be a semistar operation on the integral domain D such that whenever $(I_\lambda)_{\lambda\in\Lambda}$ is a family of fractional ideals of D with nonzero intersection, we have $(\underset{\alpha\in\Lambda}{\cap} I_\alpha)^{\widetilde{\star}} = \underset{\alpha\in\Lambda}{\cap} I_\alpha^{\widetilde{\star}}$. Suppose that $D^{\widetilde{\star}}$ is integrally closed. Let $I \in f(D[X])$. Then there exist $g \in D[X]\backslash\{0\}$ and $N \in f(D)$ such that $(I_v)^{\star_1} = g((N[X])_v)^{\star_1} = g(N_v)^{\widetilde{\star}}[X]$.*

Proof. Since $I \in f(D[X])$, there exists $g \in D[X]\backslash\{0\}$ such that $gI^{-1} \subseteq D[X]$. Hence $1 \in (g^{-1}I)_v$. Let $J = (g^{-1}I)_v$, J is a divisorial ideal of $D[X]$ and $J \cap K \neq (0)$. By Lemma 2.8, $J \cap K = \cap\{d(D : c(h)) \mid J \subseteq Bdh^{-1}, d \in D\backslash\{0\}$ and $h \in B\}$, where $B = D[X]$. Let $H = \cap\{d(D : c(h)) \mid J \subseteq Bdh^{-1}, d \in D\backslash\{0\}$ and $h \in B\}$. H is a divisorial ideal of D. Indeed, $H \subseteq J$ which implies that $H[X] \subseteq J$. So $H_v[X] \subseteq J_v$ and again $H_v \subseteq J \cap K = H$. We prove that $J^{\star_1} = (HB)^{\star_1}$. As $H \subseteq J$, $HB \subseteq J$ hence $(HB)^{\star_1} \subseteq J^{\star_1}$. Conversely, let $f \in J$, $d \in D\backslash\{0\}$ and $h \in B$ such that $J \subseteq Bdh^{-1}$. Then $c(fh) \subseteq dD$ and $d^{-1} \in D : c(fh)$. Since $D^{\widetilde{\star}}$ is integrally closed, $(D : c(fh))^{\widetilde{\star}} = (D : c(f)c(h))^{\widetilde{\star}}$. So there exists $F \in L^{\star_f}$ such that $d^{-1}F \subseteq D : c(f)c(h)$ hence $c(f) \subseteq \cap\{d(D : c(h))^{\widetilde{\star}} \mid J \subseteq Bdh^{-1}, d \in D\backslash\{0\}\}$. By hypothesis, $c(f) \subseteq H^{\widetilde{\star}}$ and $f \in H^{\widetilde{\star}}B = (HB)^{\star_1}$. Consequently $g^{-1}I \subseteq (HB)^{\star_1}$. As I is a finitely generated submodule of B, there exist a finitely generated ideal F of D, $F \in L^{\star_f}$ and a finitely generated D-submodule N of K such that $N \subseteq H$ and $g^{-1}IF \subseteq NB$. So

$g^{-1}(IF)_v \subseteq (NB)_v$ which implies that $g^{-1}I_vF \subseteq (NB)_v$. Hence $g^{-1}I_v \subseteq ((NB)_v)^{\star_1}$, that is to say $J^{\star_1} \subseteq ((NB)_v)^{\star_1}$. Conversely, as $N \subseteq H$ then $(NB)_v \subseteq (HB)_v$. Since H is a divisorial ideal of D, HB is a divisorial ideal of B. Therefore $(NB)_v \subseteq HB$ which implies that $(g^{-1}I_v)^{\star_1} = J^{\star_1} = ((NB)_v)^{\star_1}$. Hence $(I_v)^{\star_1} = g((NB)_v)^{\star_1}$. □

Definition 2.12. Let $\star$ be a semistar operation on the integral domain D.

(1) An ideal I of D is called $\star$-invertible if $(II^{-1})^\star = D^\star$.
(2) We say that D is a generalized $\star$-GCD domain (G-$\star$-GCD) if the intersection of two principal ideals $aD \cap bD$ is $\star_f$-invertible for all $0 \neq a, b \in D$.

Theorem 2.13 ([4, Theorem 4.10])**.** *Let $\star$ be a semistar operation on the integral domain D. The following are equivalent*:

(1) *D is a G-$\star$-GCD domain, that is, $aD \cap bD$ is a $\star_f$-invertible ideal of D for all $a, b \in D\backslash\{0\}$.*
(2) *For all $I \in f(D)$, $(D : I)$ is a $\star_f$-invertible ideal of D.*
(3) *For all $I \in f(D)$, I_v is a $\star_f$-invertible ideal of D.*

Remark 2.14. (1) If D is a G-$\star$-GCD domain, then $D^{\widetilde{\star}}$ is an integrally closed domain. Indeed, since D is a G-$\star$-GCD domain, by [4, Remark 4.11(1)], D_M is a GCD domain for each $M \in M(\star_f)$. So, D_M is a G-GCD domain for each $M \in M(\star_f)$. By [2, Corollary 1], D_M is integrally closed. As $D^{\widetilde{\star}} = \cap\{D_P \mid P \in M(\star_f)\}$ then $D^{\widetilde{\star}}$ is an integrally closed domain.

(2) Let $*_1$ and $*_2$ be two semistar operations on D such that $*_1 \leqslant *_2$. If D is a G-$*_1$-GCD domain, then D is a G-$*_2$-GCD domain. Indeed, let $I \in f(D)$. Since D is a G-$*_1$-GCD domain, I_v is $*_1$-invertible so $(I_vI^{-1})^{*_1} = D^{*_1}$ and $D^{*_2} = (D^{*_1})^{*_2} = ((I_vI^{-1})^{*_1})^{*_2} = (I_vI^{-1})^{*_2}$.

Theorem 2.15. *Let $\star$ be a semistar operation satisfying the property (H). Then D is a G-$\widetilde{\star}$-GCD domain if and only if $D[X]$ is a G-$\star_1$-GCD domain.*

Proof. Suppose that D is a G-$\widetilde{\star}$-GCD domain, we prove that $D[X]$ is a G-$\star_1$-GCD domain. Let $I \in f(D[X])$. By Remark 2.14, $D^{\widetilde{\star}}$ is an integrally closed domain. By Theorem 2.11, there exist $N \in f(D)$ and $g \in D[X]\backslash\{0\}$ such that $(I_v)^{\star_1} = g(N^{\widetilde{\star}})_v[X]$. As $I \subseteq (I_v)^{\star_1}$ then $(g(N^{\widetilde{\star}})_vD[X])^{-1} \subseteq (ID^{\widetilde{\star}}[X])^{-1}$. But $(g(N^{\widetilde{\star}})_v[X])^{-1} = g^{-1}(N^{\widetilde{\star}})^{-1}[X]$. On the other hand, $(ID^{\widetilde{\star}}[X])^{-1} = (I^{-1})^{\star_1}$. Indeed, let $f \in D^{\widetilde{\star}}[X] : ID^{\widetilde{\star}}[X]$ then $fI \subseteq D^{\widetilde{\star}}[X] = (D[X])^{\star_1}$. Since I is a finitely generated submodule of $D[X]$, there exists $F \in L^{\star_f}$ such that $fIF \subseteq D[X]$. Hence $f \in (I^{-1})^{\star_1}$. Conversely, let $f \in (I^{-1})^{\star_1}$, there exists $F \in L^{\star_f}$ such that $fF \subseteq I^{-1}$ which implies that $f \in (ID^{\widetilde{\star}}[X])^{-1}$. Therefore $g^{-1}(N^{\widetilde{\star}})^{-1}[X] \subseteq (I^{-1})^{\star_1}$ and again $(I_vg^{-1}(N^{\widetilde{\star}})^{-1}[X])^{\star_1} \subseteq (I_vI^{-1})^{\star_1}$. As $(I_vg^{-1}(N^{\widetilde{\star}})^{-1}[X])^{\star_1} = ((g(N[X])_v)^{\star_1}g^{-1}N^{\widetilde{\star}})^{-1}[X])^{\star_1} = (N_vN^{-1}[X])^{\star_1} = (N_vN^{-1})^{\widetilde{\star}}[X]$ and D is a G-$\widetilde{\star}$-GCD domain, $(I_vI^{-1})^{\star_1} = (D[X])^{\star_1}$ that is to say I_v is $\star_1$-invertible in $D[X]$. So $D[X]$ is a G-$\star_1$-GCD domain.

Conversely (this implication does not require the hypothesis (H)). Suppose that $D[X]$ is a G-$\star_1$-GCD domain. We prove that D is a G-$\widetilde{\star}$-GCD domain. Let $I \in f(D)$ and $J = I[X] \in f(D[X])$ then J_v is $\star_1$-invertible. As $J_v = I_v[X]$, $D^{\widetilde{\star}}[X] = (J_v J^{-1})^{\star_1} = (I_v I^{-1})^{\widetilde{\star}}[X]$, this leads to $(I_v I^{-1})^{\widetilde{\star}} = D^{\widetilde{\star}}$. □

Corollary 2.16. *Let D be an integral domain.*

(1) *D is a G-GCD domain if and only if $D[X]$ is a G-GCD domain.*
(2) *D is a G-w-GCD domain if and only if $D[X]$ is a G-$w_{D[X]}$-GCD domain.*

Proof. (1) If $\star = d_D$, then $\star_1 = d_{D[X]}$. Indeed, the localizing system L^{d_D} associated to d_D is equal to $\{D\}$ then $L^{d_D}[X] = \{D[X]\}$. Let $E \in \overline{F}(D[X])$, we have $E^{(d_D)_1} = E : D[X] = E = E^{d_{D[X]}}$. So, $(d_D)_1 = d_{D[X]}$.

(2) If $\star = v$, then $\widetilde{\star} = w$, $v_f = t$ and $\star_1 = \star_{L^t[X]}$. If D is G-w-GCD, then $D[X]$ is G-$\star_1$-GCD domain. By [12], $\star_1 \lneq w_{D[X]}$ so, $D[X]$ is G-$w_{D[X]}$-GCD domain. By [3, Theorem 2.3] and the fact that $M(w_{D[X]}) = \{Q[X] \mid Q \in M(t)\} \cup \{Q \in Spec(D[X]) \mid Q \cap D = (0) \text{ and } c(Q)^t = D\}$ the converse holds. □

3. G-$\star$-GCD Nagata rings

Let $\star$ be a semistar operation on the integral domain D and let $N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0 \text{ and } c(h)^\star = D^\star\}$. $N(\star)$ is a saturated multiplicative subset of $D[X]$ and $N(\star) = N(\star_f)$. Let $Na(D,\star) := D[X]_{N(\star)} = \{\frac{f}{g} \mid f, g \in D[X]; g \neq 0, c(g)^\star = D^\star\}$ be the Nagata ring of D with respect to the semistar operation $\star$.

Proposition 3.1 ([7, Proposition 3.1])**.** *Let $\star$ be a semistar operation on the integral domain D. Then*:

(1) $Max(Na(D,\star)) = \{Q[X]_{N(\star)} \mid Q \in M(\star_f)\}$.
(2) $Na(D,\star) = \cap\{D_Q(X) \mid Q \in M(\star_f)\} = \cap\{D[X]_{Q[X]} \mid Q \in M(\star_f)\}$.
(3) *$E^{\widetilde{\star}} = ENa(D,\star) \cap K$ for each $E \in \overline{F}(D)$.*
(4) $Na(D,\star) = Na(D,\star_f) = Na(D,\widetilde{\star})$.

Lemma 3.2. *Let D be an integral domain, $P \in Spec(D)$ and E a nonzero subset of D. If $ED_P = aD_P$ with $a \in D$, then there exists $x \in E$ such that $ED_P = xD_P$.*

Proof. As $ED_P = aD_P$, there exist $n \in \mathbb{N}^*$, $a_i \in E$, $b_i \in D$ and $s \in D\backslash P$ such that $a = \frac{\sum_{i=1}^n a_i b_i}{s}$. Since $a_i \in E$ we get $a_i \in aD_P$ which implies that there exist $d_i \in D$ and $t \in D\backslash P$ such that $a_i = a\frac{d_i}{t}$. Hence $1 = \frac{\sum_{i=1}^n d_i b_i}{st}$. Since $st \notin P$, there exists $i_0 \in \{1, \dots, n\}$ such that $d_{i_0} b_{i_0} \notin P$. Therefore $\frac{d_{i_0}}{t} \in U(D_P)$ and $aD_P = a_{i_0} D_P$. □

Theorem 3.3 ([1, Theorem 7])**.** *Let $\star$ be a semistar operation on the integral domain D and $f \in D[X]\backslash\{0\}$ such that $c(f)$ is $\star_f$-locally principal. Then $c(f)Na(D,\star_f) = fNa(D,\star_f)$.*

Proof. (The proof uses arguments similar to those used in the proof of Theorem 7 of [1]. But the change of notation requires a new proof.)

Let $f = \sum_{i=0}^{n} a_i X^i$ with $a_i \in D$. Since $c(f)$ is $\star_f$-locally principal, for each $M \in M(\star_f)$ there exists $x \in D$ such that $c(f)D_M = xD_M$. By Lemma 3.2, there exists $i_0 \in \{0, \dots, n\}$ such that $c(f)D_M = a_{i_0}D_M$. As $a_i \in a_{i_0}D_M$, for each $i \in \{0, \dots, n\}$ there exists $\gamma_i \in D_M$ such that $a_i = a_{i_0}\gamma_i$. In particular, $\gamma_{i_0} = 1$. Let $h = \gamma_0 + \gamma_1 X + \cdots + \gamma_n X^n$ then $a_{i_0}h = f$ and $c(h)D_M = D_M$. Hence $hD[X]_{M[X]} = D[X]_{M[X]}$. We get $c(f)D[X]_{M[X]} = a_{i_0}hD[X]_{M[X]} = fD[X]_{M[X]}$. Consequently $c(f) \subseteq \cap\{fD[X]_{M[X]} \mid M \in M(\star_f)\} = fNa(D, \star_f)$. Conversely, since $f \in c(f)D[X] \subseteq c(f)Na(D, \star_f)$, we conclude that

$$fNa(D, \star_f) \subseteq c(f)Na(D, \star_f).$$

□

Theorem 3.4. *Let $\star$ be a semistar operation on the integral domain D and X be an indeterminate on D. Let $f \in D[X]$. Then the following statements are equivalent*:

(1) *$c(f)$ is $\star_f$-locally principal.*
(2) *$fNa(D, \star_f) = c(f)Na(D, \star_f)$.*
(3) *There exists an ideal I of D such that $fNa(D, \star_f) = INa(D, \star_f)$.*
(4) *$c(f)Na(D, \star_f)$ is a principal ideal of $Na(D, \star_f)$.*
(5) *$c(f)Na(D, \star_f)$ is a locally principal ideal of $Na(D, \star_f)$.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 3.3.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Suppose that $fNa(D, \star_f) = INa(D, \star_f)$ with I an ideal of D. Let $M \in M(\star_f)$. Then $ID[X]_{M[X]} = fD[X]_{M[X]}$. By Lemma 3.2, there exists $a \in I$ such that $fD[X]_{M[X]} = aD[X]_{M[X]}$. As $INa(D, \star_f) = fNa(D, \star_f) \subseteq c(f)Na(D, \star_f)$ then $I \subseteq c(f)Na(D, \star_f) \cap K = (c(f))^{\widetilde{\star}}$ and again $I \subseteq c(f)D_M$. So there exist $b \in c(f)$ and $s \in D\backslash M$ such that $a = \frac{b}{s}$. Hence $fD[X]_{M[X]} = bD[X]_{M[X]}$ which implies that $f \in bD[X]_{M[X]}$. There exist $g \in D[X]$ and $h \in D[X]\backslash M[X]$ such that $f = b\frac{g}{h}$. As $h \notin M[X]$ then $c(h)D_M = D_M$. By applying the Dedekind-Mertens lemma to f and h we get $c(h)^m c(fh) = c(h)^{m+1}c(f)$, where $m = \deg(f)$. Since $c(h)D_M = D_M$ then $c(f)D_M = c(fh)c(h)^m D_M = c(bg)D_M \subseteq bD_M \subseteq c(f)D_M$.

(2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1) Suppose that $c(f)Na(D, \star_f)$ is locally principal, we prove that $c(f)$ is $\star_f$-locally principal. Let $M \in M(\star_f)$ and $J = c(f)Na(D, \star_f)$. Then $JNa(D, \star_f)_{M[X]_{N(\star_f)}}$ is a principal ideal of $Na(D, \star_f)_{M[X]_{N(\star_f)}} = D[X]_{M[X]}$. So, $JNa(D, \star_f)_{M[X]_{N(\star_f)}} = c(f)D[X]_{M[X]}$ is a principal ideal of $D[X]_{M[X]}$. By Lemma 3.2, there exists $a \in c(f)$ such that $c(f)D[X]_{M[X]} = aD[X]_{M[X]}$. As $f \in c(f)D[X]_{M[X]}$, there exist $k \in D[X]$ and $h \in D[X]\backslash M[X]$ such that $f = a\frac{k}{h}$. By applying the Dedekind-Mertens lemma to f and $h \in D[X]$, we get $c(h)^m c(fh) = c(h)^{m+1}c(f)$, where $m = \deg(f)$. Since $c(h)D_M = D_M$, then $c(f)D_M = c(fh)D_M \subseteq aD_M \subseteq c(f)D_M$. □

Corollary 3.5. *Let $\star$ be a semistar operation on the integral domain D and I be an ideal of D. Then the following statements are equivalent*:

(1) *I is $\widetilde{\star}$-finite and $\star_f$-locally principal.*
(2) *$INa(D,\star_f)$ is a finitely generated, locally principal ideal.*
(3) *$INa(D,\star_f)$ is a principal ideal of $Na(D,\star_f)$.*

Proof. (1) $\Rightarrow$ (3) Since I is $\widetilde{\star}$-finite, $I^{\widetilde{\star}} = (a_0,\ldots,a_n)^{\widetilde{\star}} = c(f)^{\widetilde{\star}}$ with $f = \sum_{i=0}^n a_iX^i$ and $ID_M = c(f)D_M$ for each $M \in M(\star_f)$. As I is $\star_f$-locally principal then $c(f)$ is $\star_f$-locally principal. On the other hand $I^{\widetilde{\star}} = INa(D,\star_f)\cap K = c(f)^{\widetilde{\star}} = c(f)Na(D,\star_f)\cap K$ so, $c(f)Na(D,\star_f) = INa(D,\star_f)$. By Theorem 3.4, $c(f)Na(D,\star_f)$ is a principal ideal of $Na(D,\star_f)$.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Suppose that $INa(D,\star_f)$ is a finitely generated and locally principal ideal, there exist $f_1,\ldots,f_n \in D[X]$ such that

$$INa(D,\star_f) = (f_1,\ldots,f_n)Na(D,\star_f).$$

Since $f_i \in INa(D,\star_f)$, there exist $a_{i,j} \in I$, $f_{i,j} \in D[X]$ and $h_i \in N(\star_f)$ such that $f_i = \frac{\sum_{j=1}^{m_i} a_{i,j}f_{i,j}}{h_i}$. Let $J = (a_{i,1},\ldots,a_{i,m_i} \,|\, i \in \{1,\ldots,n\}) \subseteq I$ then $INa(D,\star_f) = JNa(D,\star_f)$. Hence $I^{\widetilde{\star}} = INa(D,\star_f)\cap K = JNa(D,\star_f)\cap K = J^{\widetilde{\star}}$. Since $J \subseteq I$, I is $\widetilde{\star}$-finite, let $f = \sum_{i=1}^n \sum_{j=1}^{m_i} b_iX^i$. As $INa(D,\star_f) = c(f)Na(D,\star_f)$ is a locally principal ideal, then by Theorem 3.4, $c(f)$ is $\star_f$-locally principal that is to say $JD_M = c(f)D_M$ is a principal ideal for each $M \in M(\star_f)$. Since $I^{\widetilde{\star}} = J^{\widetilde{\star}}$, for each $M \in M(\star_f)$, $ID_M = JD_M$ is a principal ideal which implies that I is a $\star_f$-locally principal ideal. $\square$

Proposition 3.6. *Let $\star$ be a semistar operation on the integral domain D and $N = \{f \in D[X,Y] \,|\, c_D(f)^{\star_f} = D^{\star_f}\}$. Then*

(1) *$N = D[X,Y]\setminus \cup\{M[X,Y] \,|\, M \in M(\star_f)\}$ is a saturated multiplicative subset of $D[X,Y]$.*
(2) *$D[X,Y]_N = Na(D,\star_f)(Y)$ where $Na(D,\star_f)(Y)$ is the Nagata ring of $Na(D,\star_f)$ associated to d.*

Proof. (1) Let $f \in N$ then $f \notin M[X,Y]$ for each $M \in M(\star_f)$ so, $f \in D[X,Y]\setminus\cup\{M[X,Y] \,|\, M \in M(\star_f)\}$. Conversely, let $f \in D[X,Y]\setminus \cup\{M[X,Y] \,|\, M \in M(\star_f)\}$. If $c(f)^{\star_f} \neq D^{\star_f}$, there exists $M \in M(\star_f)$ such that $c(f) \subseteq M$ which implies that $f \in M[X,Y]$ which is a contradiction. So $c(f)^{\star_f} = D^{\star_f}$ and $N = D[X,Y]\setminus\cup\{M[X,Y] \,|\, M \in M(\star_f)\}$. Hence N is a saturated multiplicative subset of $D[X,Y]$.

(2) Let $R = Na(D,\star_f)$ and $f \in R(Y)$ so $f = \frac{f_1}{f_2}$ with $f_1, f_2 \in R[Y]$, $f_2 \neq 0$ and $c_R(f_2) = R$. Let $f_1 = \frac{\sum_{i=0}^n f_{1,i}Y^i}{h_1}$ with $h_1 \in D[X]$, $c(h_1)^{\star_f} = D^{\star_f}$ and $f_{1,i} \in D[X]$. Let $f_2 = \frac{\sum_{j=0}^m f_{2,j}Y^j}{h_2}$ with $f_{2,j} \in D[X]$, $c(h_2)^{\star_f} = D^{\star_f}$ and $(f_{2,0},\ldots,f_{2,m})_{N(\star_f)} = c_R(f_2) = Na(D,\star_f)$. Let $g = \sum_{j=0}^m f_{2,j}Y^j \in D[X,Y]$, $c_{Na(D,\star_f)}(g) = Na(D,\star_f) = (f_{2,0},\ldots,f_{2,m})_{N(\star_f)}$, there exists $h \in N(\star_f)$

such that $h \in (f_{2,0}, \ldots, f_{2,m})D[X]$. So $c_D(h) \subseteq c_D(f_{2,0}) + \cdots + c_D(f_{2,m}) = c_D(g) \subseteq D$ which implies that $D^{\star_f} = c_D(h)^{\star_f} \subseteq c_D(g)^{\star_f} \subseteq D^{\star_f}$ and again $c_D(g)^{\star_f} = D^{\star_f}$. Then $gh_1 \in N$ therefore $f = \frac{f_1}{f_2} \in D[X,Y]_N$. Conversely, let $f = \frac{f_1}{f_2} \in D[X,Y]_N$ with $f_1 \in D[X,Y]$ and $f_2 \in N = D[X,Y] \setminus \cup \{M[X,Y] \mid M \in M(\star_f)\}$. Let $f_2 = \sum_{j=0}^{m} f_{2,j} Y^j$ with $f_{2,j} \in D[X]$ for each $j \in \{0, \ldots, m\}$. If $c_{Na(D,\star_f)}(f_2) \neq Na(D, \star_f)$, there exists $M \in M(\star_f)$ such that $c_{Na(D,\star_f)}(f_2) \subseteq M[X]_{N(\star_f)}$ which implies that $(f_{2,0}, \ldots, f_{2,m}) \subseteq M[X]$. Hence $f_2 \in M[X,Y]$ which is impossible. Then $c_{Na(D,\star_f)}(f_2) = Na(D, \star_f)$ so, $f = \frac{f_1}{f_2} \in Na(D, \star_f)(Y)$. □

Theorem 3.7. *Let $\star$ be a semistar operation on the integral domain D and X be an indeterminate on D. Then every nonzero finitely generated and locally principal ideal of $Na(D, \star_f)$ is a principal ideal.*

Proof. Let $I = (f_1, \ldots, f_n)Na(D, \star_f) \neq 0$ such that I is a locally principal ideal, where $f_i \in D[X]$ for each $i \in \{1, \ldots, n\}$. Let $g = f_1 + f_2Y + \cdots + f_nY^{n-1} \in D[X,Y]$, $R = Na(D, \star_f)$ and K_1 the quotient field of R. So, $c_R(g) = (f_1, \ldots, f_n)R = I$ is locally principal in R. By Theorem 3.3, $INa(D, \star_f)(Y) = c_R(g)R(Y) = gR(Y)$. By Proposition 3.6, $Na(D, \star_f)(Y) = D[X,Y]_N$ then $ID[X,Y]_N = gD[X,Y]_N$. Let $m_i = \deg(f_i)$ for each $i \in \{1, \ldots, n\}$ and $f = f_1 + f_2X^{m_1+1} + f_3X^{m_1+m_2+2} + \cdots + f_nX^{m_1+\cdots+m_{n-1}+(n-1)} \in D[X]$. Hence $c_D(f) = c_D(g)$ and $f \in ID[X,Y]_N = gD[X,Y]_N$, there exist $h_1 \in D[X,Y]$ and $h_2 \in N$ such that $f = g\frac{h_1}{h_2}$. We prove that $c_D(h_1)^{\star_f} = D^{\star_f}$. Suppose that $c_D(h_1)^{\star_f} \neq D^{\star_f}$, there exists $P \in M(\star_f)$ such that $c_D(h_1) \subseteq P$. Let (V, M) be a valuation overring of D such that $M \cap D = P$. If $c_V(h_2) \subseteq M$, then $c_D(h_2) \subseteq M \cap D = P$ which is impossible. As V is a valuation domain, $c_V(fh_2) = c_V(f)c_V(h_2)$. So $c_V(f) = c_V(f)c_V(h_2) = c_V(gh_1) = c_V(g)c_V(h_1) = c_V(f)c_V(h_1)$. By Nakayama's lemma, either $c_V(f) = (0)$ or $c_V(h_1) = V$ and since $f \neq 0$ then $c_V(h_1) = V$ which is impossible because $c_V(h_1) = c_D(h_1)V \subseteq PV \subseteq MV = M$. Then $c_D(h_1)^{\star_f} = D^{\star_f}$ and $fNa(D, \star_f)(Y) = fD[X,Y]_N = g\frac{h_1}{h_2}D[X,Y]_N = gNa(D, \star_f)(Y) = I(Y)$. Hence $fNa(D, \star_f) = I$. □

Proposition 3.8 ([12, Proposition 3.4 and Lemma 3.5])**.** *Let $\star$ be a semistar operation on the integral domain D. Let $* := \star_\triangle$ be the spectral semistar operation on $D[X]$ defined by the set $\triangle := \{P[X] \mid P \in M(\star_f)\}$ and let i be the canonical embedding of $D[X]$ in $Na(D, \star_f)$. Then*

(1) $\star_1 := \star_{L^{\star_f}[X]} \leq *$.
(2) $*_i = d_{Na(D,\star_f)}$.

Theorem 3.9. *Let $\star$ be a semistar operation satisfying the property (H). The following statements are equivalent*:

(1) *D is a G-$\widetilde{\star}$-GCD domain.*
(2) *$D[X]$ is a G-$\star_1$-GCD domain.*
(3) *$D[X]$ is a G-$*$-GCD domain.*

(4) *$Na(D,\star_f)$ is a G-GCD domain.*
(5) *$Na(D,\star_f)$ is a GCD domain.*

Proof. (1) $\iff$ (2) follows from Theorem 2.15.

(2) $\Longrightarrow$ (3) If $D[X]$ is a G-$\star_1$-GCD domain and since $\star_1 \leq *$, $D[X]$ is a G-$*$-GCD domain.

(3) $\Longrightarrow$ (4) If $D[X]$ is a G-$*$-GCD domain, by [4, Remark 4.11(3)], $(D[X])^{\widetilde{*}}$ is a G-$\widetilde{*}_i$-GCD domain. By [9, Lemma 3.8], $*$ is a stable semistar operation of finite type which implies by [12, Proposition 1.5], that $*_i$ is a stable semistar operation of finite type. Then $(D[X])^*$ is a G-$*_i$-GCD domain and by [12, Proposition 1.6(2)], $(D[X])^* = Na(D,\star_f)$. Hence $Na(D,\star_f)$ is a G-GCD domain.

(4) $\Longrightarrow$ (5) Let $I \in f(Na(D,\star_f))$. Since $Na(D,\star_f)$ is a G-GCD domain, I_v is an invertible ideal. So I_v is a finitely generated and locally principal ideal of $Na(D,\star_f)$. By Theorem 3.7, I_v is a principal ideal of $Na(D,\star_f)$ then $Na(D,\star_f)$ is a GCD domain.

(5) $\Longrightarrow$ (1) If $Na(D,\star_f)$ is a GCD domain, we prove that D is a G-$\widetilde{\star}$-GCD domain. Let $I \in f(D)$ then $(INa(D,\star_f))^{-1} = I^{-1}Na(D,\star_f)$. So $I_vNa(D,\star_f) = (INa(D,\star_f))_v$. In fact, let $f \in (INa(D,\star_f))_v$. Since $Na(D,\star_f)$ is a GCD domain and $I \in f(D)$, $(INa(D,\star_f))^{-1}$ is invertible in $Na(D,\star_f)$. Hence $(INa(D,\star_f))^{-1} \in f(Na(D,\star_f))$, there exists $g \in N(\star_f)$ such that $fg(INa(D,\star_f))^{-1} = fgI^{-1}Na(D,\star_f) \subseteq D[X]$ so, $I^{-1}c(fg) \subseteq D$ and $c(fg) \subseteq I_v$. $fg \subseteq I_vD[X]$ which implies that $f \in I_vNa(D,\star_f)$. Conversely, let $x \in I_v$ then $x(INa(D,\star_f))^{-1} \subseteq Na(D,\star_f)$. Hence $x \in (INa(D,\star_f))_v$. As $Na(D,\star_f)$ is a GCD domain, $(INa(D,\star_f))_v$ is invertible in $Na(D,\star_f)$. Hence

$$\begin{aligned} &I_vI^{-1}Na(D,\star_f) = Na(D,\star_f) \text{ and} \\ &\qquad (I_vI^{-1})^{\widetilde{\star}} = I_vI^{-1}Na(D,\star_f) \cap K = Na(D,\star_f) \cap K = D^{\widetilde{\star}}. \end{aligned} \qquad \square$$

If $\star = d$, then we recover the result of [2, Theorem 2].

Corollary 3.10. *Let D be an integral domain. The following statements are equivalent*:

(1) *D is a G-GCD domain.*
(2) *$D[X]$ is a G-GCD domain.*
(3) *$D(X)$ is a G-GCD domain.*
(4) *$D(X)$ is a GCD domain.*

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References

[1] D. D. Anderson, *Multiplication ideals, multiplication rings, and the ring $R(X)$*, Canad. J. Math. **28** (1976), no. 4, 760–768. https://doi.org/10.4153/CJM-1976-072-1

[2] D. D. Anderson and D. F. Anderson, *Generalized GCD domains*, Comment. Math. Univ. St. Paul. **28** (1980), no. 2, 215–221.

[3] G. W. Chang and M. Fontana, *Uppers to zero and semistar operations in polynomial rings*, J. Algebra **318** (2007), no. 1, 484–493. https://doi.org/10.1016/j.jalgebra.2007.06.010

[4] S. El-Baghdadi, *Semistar GCD domains*, Comm. Algebra **38** (2010), no. 8, 3029–3044. https://doi.org/10.1080/00927870903114961

[5] M. Fontana and J. A. Huckaba, *Localizing systems and semistar operations*, in Non-Noetherian commutative ring theory, 169–197, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[6] M. Fontana and K. A. Loper, *Kronecker function rings: a general approach*, in Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), 189–205, Lecture Notes in Pure and Appl. Math., 220, Dekker, New York, 2001.

[7] ———, *Nagata rings, Kronecker function rings, and related semistar operations*, Comm. Algebra **31** (2003), no. 10, 4775–4805. https://doi.org/10.1081/AGB-120023132

[8] M. Fontana and G. Picozza, *Semistar invertibility on integral domains*, Algebra Colloq. **12** (2005), no. 4, 645–664. https://doi.org/10.1142/S1005386705000611

[9] W. Gmiza and S. Hizem, *Semistar ascending chain conditions over polynomial rings*, Submitted.

[10] A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.

[11] B. Olberding, *Characterizations and constructions of h-local domains*, in Models, modules and abelian groups, 385–406, Walter de Gruyter, Berlin, 2008. https://doi.org/10.1515/9783110203035.385

[12] G. Picozza, *A note on semistar Noetherian domains*, Houston J. Math. **33** (2007), no. 2, 415–432.

[13] J. Querré, *Sur le groupe des classes de diviseurs*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 7, A397–A399.

[14] D. Spirito, *Jaffard families and localizations of star operations*, to appear in J. Commut. Algebra.

Wafa Gmiza
Department of Mathematics
Faculty of sciences
University of Monastir 5000
Tunisia
Email address: gmiza.wafa@yahoo.com

Sana Hizem
Department of Mathematics
Faculty of sciences
University of Monastir 5000
Tunisia
Email address: hizems@yahoo.fr