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THE QUASI-NEUTRAL LIMIT OF THE COMPRESSIBLE MAGNETOHYDRODYNAMIC FLOWS FOR IONIC DYNAMICS

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Abstract. In this paper we study the quasi-neutral limit of the compressible magnetohydrodynamic flows in the periodic domain $\mathbb{T}^3$ with the well-prepared initial data. We prove that the weak solution of the compressible magnetohydrodynamic flows governed by the Poisson equation converges to the strong solution of the compressible flow of magnetohydrodynamic flows as long as the latter exists.

1. Introduction

Magnetohydrodynamic flows arise in science and engineering in a variety of practical applications such as in plasma confinement, liquid-metal cooling of nuclear reactors, and electromagnetic casting. The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. The set of equations that describe MHD is a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. These differential equations must be solved simultaneously, either analytically or numerically.

As a physical model of fluids, we here consider the compressible magnetohydrodynamic flows governed by the Poisson equation in the periodic domain $\Omega = \mathbb{T}^3$ where $\mathbb{T}^3$ is the three dimensional periodic domain:

$$\partial_t \varrho_\epsilon + \operatorname{div}(\varrho_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1}$$

$$\partial_t[\varrho_\epsilon \mathbf{u}_\epsilon] + \operatorname{div}[\varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon] + \nabla P(\varrho_\epsilon) = \mu \Delta \mathbf{u}_\epsilon + (\mu+\lambda)\nabla \operatorname{div}\mathbf{u}_\epsilon + (\mathbf{H}_\epsilon \cdot \nabla)\mathbf{H}_\epsilon - \frac{1}{2}\nabla|\mathbf{H}_\epsilon|^2 - \varrho_\epsilon \nabla G_\epsilon, \tag{2}$$

$$\partial_t \mathbf{H}_\epsilon + (\operatorname{div}\mathbf{u}_\epsilon)\mathbf{H}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{H}_\epsilon - (\mathbf{H}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon = \nu \Delta \mathbf{H}_\epsilon, \tag{3}$$

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$$-\epsilon^2\Delta G_\epsilon = \varrho_\epsilon - \exp G_\epsilon, \tag{4}$$

where $\mathbf{u}_\epsilon$ is the vector field, $\gamma > \frac{3}{2}$, ϱ_ϵ is the density, the pressure $p(\varrho_\epsilon) = \varrho_\epsilon^\gamma$, $\mathbf{H}_\epsilon$ is the magnetic field, and G_ϵ is a potential function. Note that we assume that the viscosities μ, λ, ν do not depend on ϵ due to the good regularity of density, velocity, and magnetic field.

We first notice that the global-in-time existence solutions for system (1)-(4) has been studied by Hu, Wang [9]. However, we can follow the same spirit of the global existence for the system (1)-(4). In [1], Bresch, Desjardins, and Ducomet studied the quasi-neutral limit for the isentropic compressible Navier-Stokes-Poisson system for ions with capillary effect on $\mathbb{T}^3$. They established the existence of global weak solutions of the model and obtained that the weak solution the primitive model converges to the weak solutions of the compressible capillary Navier-Stokes equations in the torus $\mathbb{T}^3$. For the ionic Euler-Poisson system, the quasi-neutral limit was studied, for example, in [3, 7, 8, 15].

If we replace the Poisson equation (4) by

$$-\epsilon^2\Delta G = \varrho - D(x), \tag{5}$$

where $D(x)$ is a given function, we obtain the corresponding model for electrons and a few results on the the quasi-neutral limit are available [2, 6, 10, 13, 14]. Ju, Li, and Li [10] studied the quasi-neutral limit for local strong solutions to the Navier-Stokes-Fourier-Poisson system on $\mathbb{T}^3$. Chen, Donatelli, and Marcati [2] studied the quasi-neutral limit of a hydrodynamic model for charge-carrier transport in the framework of weak solutions. In [13], Ju and Li studied the combined quasi-neutral and zero-electron-mass limit of the Navier-Stokes-Fourier-Poisson system in the torus $\mathbb{T}^3$ and showed the limit is the the incompressible Navier-Stokes equations. In [6], Donatelli and Marcati gave some descriptions on the quasineutral limit for the full Navier-Stokes-Poisson system in $\mathbb{R}^3$. Very recently, Li, Ju, and Xu [14] improved the result in [13] to allow the temperature have a large variation. Also, they are many results on quasi-neutral limit to the electric Euler-Poisson and Navier-Stokes-Poisson system, among others, we mention [4, 5, 11, 12, 16, 17].

Motivated by the results in [1, 10, 13, 14], in this paper we want to study the quasi-neutral limit to the system (1)-(4). Formally, letting ϵ tend to 0 in (4), we get $\varrho = \exp(G)$. Thus, the term $\varrho\nabla G$ in (2) turns to $\nabla\varrho$. Hence we can expect that, as ϵ tend to 0, the limiting system is the following compressible magnetohydrodynamic flows:

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \tag{6}$$

$$\begin{aligned}\partial_t(\varrho\mathbf{u}) &+ \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\Pi(\varrho) \\ &= \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u} + (\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2,\end{aligned} \tag{7}$$

$$\partial_t\mathbf{H} + (\operatorname{div}\mathbf{u})\mathbf{H} + (\mathbf{u}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{H}, \tag{8}$$

where $\Pi(\varrho) = \varrho^\gamma + \varrho$.

The purpose of this paper is to give a rigorous proof of the above formal process for the well-prepared initial data case.

The remainder of this paper is arranged as follows. In Section 2 we define the weak solutions to the primitive system (1)-(4) and state our main result. In Section 3 we give the proof of it.

2. Main result

We now introduce the notion of weak solutions of the system (1)-(4).

Definition 2.1. We say that a quantity $\{\varrho, \mathbf{u}, \mathbf{H}, G\}$ is a weak solution of the magnetohydrodynamic flows (**MHD**) (1)-(4) supplemented with the initial data $\{\varrho_0, \mathbf{u}_0, \mathbf{H}_0, G_0\}$ provided that the following hold.

- The density ϱ is a non-negative function, $\varrho \in L^\infty(0,T;(L^\gamma + L^2)(\Omega))$, the velocity field $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, $\varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega))$ and $\mathbf{u}$ represents a renormalized solution of equation (1) on the $(0,T)\times\Omega$, that is, the integral identity

$$(9)\qquad \int_\Omega (\varrho + b(\varrho))\varphi(\tau,\cdot)\mathrm{d}x - \int_\Omega (\varrho_0 + b(\varrho_0))\varphi(0,\cdot)\mathrm{d}x$$
$$= \int_0^T\int_\Omega \Big((\varrho + b(\varrho))\partial_t\varphi + (\varrho + b(\varrho))\mathbf{u}\cdot\nabla\varphi + (b(\varrho) - b'(\varrho)\varrho)\mathrm{div}\mathbf{u}\varphi\Big)\mathrm{d}x\mathrm{d}t$$

holds for any test function $\varphi \in \mathcal{D}([0,T)\times\Omega)$ and any b such that

$$b \in C^1[0,\infty),\ b'(r) = 0 \text{ whenever } r \geq r_b.$$

- The balance of momentum holds in distributional sense, namely

$$(10)\qquad \int_\Omega \varrho\mathbf{u}\cdot\vec{\varphi}(\tau,\cdot)\mathrm{d}x - \int_\Omega (\varrho\mathbf{u})_0\cdot\vec{\varphi}(0,\cdot)\mathrm{d}x$$
$$= \int_0^T\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\vec{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\vec{\varphi} + \varrho^\gamma\ \mathrm{div}\vec{\varphi} + [(\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2]\cdot\vec{\varphi}\Big)\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T\int_\Omega \Big(\mu\nabla\mathbf{u} : \nabla\vec{\varphi} + (\mu+\lambda)\mathrm{div}\mathbf{u}\mathrm{div}\vec{\varphi} - \varrho\nabla G\cdot\vec{\varphi}\Big)\mathrm{d}x\mathrm{d}t$$

for any test function $\vec{\varphi} \in \mathcal{D}([0,T);\Omega)$.

- The total energy of the system holds,

$$(11)\qquad \int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2 + \frac{1}{(\gamma-1)}\varrho^\gamma + \frac{\epsilon^2}{2}|\nabla G|^2 + (G-1)\exp G\Big)(\tau,\cdot)\mathrm{d}x$$
$$+ \int_0^\tau\int_\Omega \Big(\mu|\nabla\mathbf{u}|^2 + (\mu+\lambda)(\mathrm{div}\mathbf{u})^2 + \nu|\nabla\mathbf{H}|^2\Big)\mathrm{d}x dt \leq E_{0,\epsilon}$$

holds for *a.e.* $\tau \in (0,T)$ where

$$E_0 = \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{2}|\mathbf{H}_0|^2 + \frac{1}{(\gamma-1)}\varrho_0^\gamma + \frac{\epsilon^2}{2}|\nabla G_0|^2 + (G_0-1)\exp G_0\Big)(\tau,\cdot)\mathrm{d}x.$$

- The Maxwell equation verifies

$$(12)\quad \int_\Omega \mathbf{H}\cdot\vec{\varphi}(\tau,\cdot)\mathrm{d}x - \int_\Omega (\mathbf{H})_0\cdot\vec{\varphi}(0,\cdot)\mathrm{d}x$$

$$= \int_0^T\int_\Omega \mathbf{B}\cdot\partial_t\overrightarrow{\varphi}\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_\Omega\Big(-\nu\nabla\mathbf{H}:\nabla\overrightarrow{\varphi}+(\mathbf{H}\cdot\nabla)\mathbf{u}\cdot\overrightarrow{\varphi}-(\mathbf{u}\cdot\nabla)\mathbf{H}\cdot\overrightarrow{\varphi}-(\mathbf{H}\cdot\overrightarrow{\varphi})\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t,$$

$$\int_0^T\int_\Omega \mathbf{B}\cdot\nabla\phi\mathrm{d}x\mathrm{d}t = 0$$

  for all $\overrightarrow{\varphi}\in[\mathcal{D}([0,T)\times\Omega)]^3$, and $\phi\in\mathcal{D}([0,T)\times\Omega)$.
- the equation (4) holds in $\mathcal{D}'((0,\infty)\times\Omega)$.

Remark 2.1. The existence of weak solutions in $(0,T)\times\Omega$ to the compressible magnetohydrodynamic flows (1)-(4) can be established by slightly modifying the arguments in [9]. Since we are mainly interested in the quasi-neutral limit, we omit the details on existence theory here.

Before stating our main results, we recall the local existence of smooth solutions to the problem (6)-(8). Since the system (6)-(8) is a parabolic-hyperbolic one, the results in [18] imply that

Proposition 2.1 ([18]). *Let $s>7/2$ be an integer and assume that the initial data $(\varrho_0,\mathbf{u}_0,\mathbf{H}_0)$ satisfy*

$$(13)\qquad \varrho_0,\mathbf{u}_0,\mathbf{H}_0\in H^{s+2}(\Omega),\quad 0<\bar{\rho}\le\rho_0(x),$$

for a positive constant $\bar{\rho}$. Then there exist positive constants T_* (the maximal time interval, $0<T_*\le+\infty$), and $\hat{\rho}$, such that the system (6)-(8) with initial data $(\varrho,\mathbf{u},\mathbf{H})|_{t=0}=(\varrho_0,\mathbf{u}_0,\mathbf{H}_0)$ has a unique classical solution $(\rho,\mathbf{u},\mathbf{H})$ satisfying

$$\rho\in C^l([0,T_*),H^{s+2-l}(\Omega)),\ \ \mathbf{u},\,\mathbf{H}\in C^l([0,T_*),H^{s+2-2l}(\Omega)),\quad l=0,1;$$
$$0<\hat{\rho}\le\rho(x,t).$$

Now we can state our main results as follows.

Theorem 2.1. *Let$(\varrho_\epsilon,\mathbf{u}_\epsilon,\mathbf{H}_\epsilon,G_\epsilon)$ the global weak solution of the system (1)-(4) with the initial data $(\varrho_{0,\epsilon},\mathbf{u}_{0,\epsilon},\mathbf{H}_{0,\epsilon},G_{0,\epsilon})$. Assume that $(\varrho_{0,\epsilon},\mathbf{u}_{0,\epsilon},\mathbf{H}_{0,\epsilon},G_{0,\epsilon})$ satisfy*

$$(14)\quad \int_\Omega|\varrho_{0,\epsilon}-\varrho_0|^2\mathrm{d}x+\int_\Omega|\mathbf{H}_{0,\epsilon}-\mathbf{H}_0|^2\mathrm{d}x+\int_\Omega|\exp(G_{0,\epsilon})-\varrho_0|^2\mathrm{d}x\le C\epsilon,$$

$$(15)\qquad \|\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon}-\sqrt{\varrho_0}\mathbf{u}_0\|^2_{L^2(\Omega)}\le C\epsilon,$$

$$(16)\qquad \epsilon\|\nabla G_{0,\epsilon}\|^2_{L^2(\Omega)}\le C,$$

where $(\varrho_0, \mathbf{u}_0, \mathbf{H}_0)$ *satisfy the conditions* (13). *Then, for* $0 < T < T^*$ *(defined in Proposition 2.1), one has*

$$\|\varrho_\epsilon - \varrho\|_{L^\infty(0,T;(L^2+L^\gamma))(\Omega)} \leq C\sqrt{\epsilon}, \tag{17}$$

$$\|\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon - \sqrt{\varrho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\epsilon}, \tag{18}$$

$$\|\mathbf{H}_\epsilon - \mathbf{H}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\epsilon}, \tag{19}$$

$$\|\sqrt{\exp(V_\epsilon)} - \sqrt{\varrho}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\epsilon}. \tag{20}$$

Here $(\varrho, \mathbf{u}, \mathbf{H})$ *is the solution to the system* (6)-(8) *constructed in Proposition 2.1.*

3. Proof of Theorem 2.1

In this section we are going to give a rigorous proof of Theorem 2.1 by applying and modifying the relative entropy method. The main difficulty here is that the target system is the compressible magnetohydrodynamic flows. Thus, we need to pay more an attention on construct the modified energy inequality and deal with the remainders.

Step I. Let us set

$$h(\varrho_\epsilon) = \frac{2}{\gamma - 1}(\varrho_\epsilon^\gamma - \varrho^\gamma - \gamma\varrho^{\gamma-1}(\varrho_\epsilon - \varrho))$$

and define the relative entropy:

$$\begin{aligned}\mathcal{E}_\epsilon(\tau) = \frac{1}{2}\int_\Omega \Big(&\varrho_\epsilon|\mathbf{u}_\epsilon - \mathbf{u}|^2 + h(\varrho_\epsilon) + |\mathbf{H}_\epsilon - \mathbf{H}|^2 + \epsilon^2|\nabla G_\epsilon|^2 \\ &+ m_\epsilon \ln\Big(\frac{m_\epsilon}{\varrho}\Big) - m_\epsilon + \varrho\Big)\mathrm{d}x,\end{aligned} \tag{21}$$

with $m_\epsilon = \exp G_\epsilon$ where $(\varrho_\epsilon, \mathbf{u}_\epsilon, \mathbf{H}_\epsilon, G_\epsilon)$ is a solution of the system (1)-(4) and $(\varrho, \mathbf{u}, \mathbf{H})$ is a solution of the system (6)-(8).

We remark that in (21), we have used the following computation:

$$\int_\Omega (G_\epsilon - 1)\exp G_\epsilon \mathrm{d}x = \int_\Omega (m_\epsilon \ln m_\epsilon - m_\epsilon)\mathrm{d}x$$

and

$$\begin{aligned}\int_\Omega m_\epsilon \ln(1/\varrho)\mathrm{d}x = &\int_\Omega m_{0,\epsilon}\ln(1/\varrho_0)\mathrm{d}x - \int_0^\tau\int_\Omega m_\epsilon\partial_t \ln\varrho \mathrm{d}x\mathrm{d}t \\ &+ \epsilon\int_0^\tau\int_\Omega \partial_t\Delta G_\epsilon \ln(1/\varrho)\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\ln\varrho \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{22}$$

where we have used (3).

Let us observe that

$$\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u} - \mathbf{u}_\epsilon)\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t = \int_0^\tau\int_\Omega m_\epsilon\mathbf{u}\cdot\nabla\ln\varrho - \epsilon\int_0^\tau\int_\Omega \Delta G_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \varrho_\epsilon \mathbf{u}_\epsilon\cdot\nabla\ln\varrho \mathrm{d}x\mathrm{d}t. \tag{23}$$

Using (23), (22) implies that

$$\begin{aligned}\int_\Omega m_\epsilon \ln(1/\varrho)\mathrm{d}x &= \int_\Omega m_{0,\epsilon}\ln(1/\varrho_0)\mathrm{d}x - \int_0^\tau\int_\Omega m_\epsilon\partial_t\ln\varrho \mathrm{d}x\mathrm{d}t\\ &\quad+\epsilon\int_0^\tau\int_\Omega \partial_t\Delta G_\epsilon\ln(1/\varrho)\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t\\ &\quad-\int_0^\tau\int_\Omega m_\epsilon\mathbf{u}\cdot\nabla\ln\varrho+\epsilon\int_0^\tau\int_\Omega\Delta G_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{24}$$

Taking $\frac{1}{2}|\mathbf{u}|^2$ and $p'(\varrho)$ with $p(\varrho)=\frac{1}{\gamma-1}\varrho^\gamma$ as a test function in (1), we get

$$\int_\Omega\frac{1}{2}\varrho_\epsilon|\mathbf{u}|^2\mathrm{d}x=\int_\Omega\frac{1}{2}\varrho_{0,\epsilon}|\mathbf{u}_0|^2\mathrm{d}x+\int_0^\tau\int_\Omega\Big(\varrho_\epsilon\mathbf{u}\cdot\partial_t\mathbf{u}+\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}\cdot\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t \tag{25}$$

and

$$\int_\Omega\frac{1}{2}\varrho_\epsilon p'(\varrho)\mathrm{d}x=\int_\Omega\frac{1}{2}\varrho_{0,\epsilon}p'(\varrho_0)\mathrm{d}x+\int_0^\tau\int_\Omega\Big(\varrho_\epsilon\partial_t p'(\varrho)+\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla p'(\varrho)\Big)\mathrm{d}x\mathrm{d}t. \tag{26}$$

We choose $\mathbf{u}$ as a test function to the moment equation (2) and it provides

$$\begin{aligned}&-\int_\Omega(\varrho_\epsilon\mathbf{u}_\epsilon\cdot\mathbf{u})(\tau)\mathrm{d}x \\ &= -\int_\Omega(\varrho_{0,\epsilon}\mathbf{u}_{0,\epsilon})\cdot\mathbf{u}_0\mathrm{d}x-\int_0^\tau\int_\Omega\varrho_\epsilon\mathbf{u}_\epsilon\cdot\partial_t\mathbf{u}\mathrm{d}x\mathrm{d}t\\ &\quad-\int_0^\tau\int_\Omega\Big(\varrho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon:\nabla\mathbf{u}+\varrho_\epsilon^\gamma\mathrm{div}\mathbf{u}-\mu\nabla\mathbf{u}_\epsilon:\nabla\mathbf{u}-(\mu+\nu)\mathrm{div}\mathbf{u}_\epsilon\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t\\ &\quad-\int_0^\tau\int_\Omega\Big(m_\epsilon\mathrm{div}\mathbf{u}-\epsilon^2\mathrm{D}\mathbf{u}:(\nabla G_\epsilon\otimes\nabla G_\epsilon)-\frac{\epsilon^2}{2}|\nabla G_\epsilon|^2\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t,\\ &\quad-\int_0^\tau\int_\Omega\Big((\mathbf{H}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon-\frac{1}{2}\nabla|\mathbf{H}_\epsilon|^2\Big)\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{27}$$

where $\mathrm{D}\mathbf{u}=\frac{1}{2}(\nabla\mathbf{u}+\nabla^T\mathbf{u})$ while the equation (4) together with using the integration by part provides that

$$\begin{aligned}&-\int_0^\tau\int_\Omega\varrho_\epsilon\nabla G_\epsilon\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t=-\int_0^\tau\int_\Omega(m_\epsilon-\epsilon^2\Delta G_\epsilon)\nabla G_\epsilon\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t\\ &=\int_0^\tau\int_\Omega m_\epsilon\mathrm{div}\mathbf{u}\mathrm{d}x\mathrm{d}t-\epsilon^2\int_0^\tau\int_\Omega\Big(\mathrm{D}\mathbf{u}:(\nabla G_\epsilon\otimes\nabla G_\epsilon)+\frac{1}{2}|\nabla G_\epsilon|^2\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We also get

$$p'(\varrho)\varrho-p(\varrho)=\varrho^\gamma.$$

Thus, we deduce, after adding (11), (22), (23), (25), (26), and (27), the following inequality:

$$\int_\Omega\Big(\frac{1}{2}\varrho_\epsilon|\mathbf{u}_\epsilon-\mathbf{u}|^2+p(\varrho_\epsilon)-\varrho_\epsilon p'(\varrho)+\frac{1}{2}|\mathbf{H}_\epsilon-\mathbf{H}|^2+\frac{1}{2}\epsilon^2|\nabla G_\epsilon|^2 \tag{28}$$

$$
\begin{aligned}
&\qquad + m_\epsilon \ln\Big(\frac{m_\epsilon}{\varrho}\Big) - m_\epsilon + \varrho\Big)\mathrm{d}x \\
&+ \int_0^\tau\int_\Omega \Big(\mu|\nabla \mathbf{u}_\epsilon - \nabla \mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}\mathbf{u}_\epsilon - \mathrm{div}\mathbf{u}|^2 + \nu|\nabla \mathbf{H}_\epsilon - \nabla \mathbf{H}|^2\Big)\mathrm{d}x\mathrm{d}t \\
\le & \int_\Omega \Big(\frac{1}{2}\varrho_{0,\epsilon}|\mathbf{u}_{0,\epsilon} - \mathbf{u}_0|^2 + p(\varrho_{0,\epsilon}) - \varrho_{0,\epsilon}p'(\varrho_0) + \frac{1}{2}\epsilon^2|\nabla G_{0,\epsilon}|^2 \\
&\qquad + m_{0,\epsilon}\ln\Big(\frac{m_{0,\epsilon}}{\varrho_0}\Big) - m_{0,\epsilon} + \varrho_0\Big)\mathrm{d}x \\
&+ \int_0^\tau\int_\Omega \Big(\varrho_\epsilon(\partial_t \mathbf{u} + \mathbf{u}_\epsilon\cdot\nabla\mathbf{u})\cdot(\mathbf{u}-\mathbf{u}_\epsilon)\Big)\mathrm{d}x\mathrm{d}t \\
&+ \mu\int_0^\tau\int_\Omega \nabla\mathbf{u} : \nabla(\mathbf{u}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t + (\mu+\nu)\int_0^\tau\int_\Omega \mathrm{div}\mathbf{u}(\mathrm{div}\mathbf{u} - \mathrm{div}\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau\int_\Omega (\varrho_\epsilon\partial_t p(\varrho) + \varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla p'(\varrho))\mathrm{d}x\mathrm{d}t - \int_0^\tau\int_\Omega \varrho_\epsilon^\gamma \mathrm{div}\mathbf{u}\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau\int_\Omega m_\epsilon\partial_t \ln\varrho\mathrm{d}x\mathrm{d}t + \epsilon\int_0^\tau\int_\Omega \partial_t\Delta G_\epsilon \ln(1/\varrho)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t - \int_0^\tau\int_\Omega m_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t \\
&+ \epsilon\int_0^\tau\int_\Omega \Delta G_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \Big(m_\epsilon\mathrm{div}\mathbf{u} - \epsilon^2\mathrm{D}\mathbf{u} : (\nabla G_\epsilon\otimes\nabla G_\epsilon) - \frac{\epsilon^2}{2}|\nabla G_\epsilon|^2\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \Big[((\mathbf{H}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon - \frac{1}{2}\nabla|\mathbf{H}_\epsilon|^2)\cdot\mathbf{u} + 2\nu\nabla\mathbf{H}_\epsilon : \nabla\mathbf{H} - \nu|\nabla\mathbf{H}|^2\Big]\mathrm{d}x\mathrm{d}t, \\
&- \int_\Omega \mathbf{H}_\epsilon\cdot\mathbf{H} - \frac{1}{2}|\mathbf{H}|^2\mathrm{d}x.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_\Omega \varrho^\gamma\mathrm{d}x - \int_\Omega \varrho_0^\gamma\mathrm{d}x &= \int_0^\tau\int_\Omega \partial_t\varrho^\gamma\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau\int_\Omega \Big(\varrho\partial_t p'(\varrho) + \varrho\nabla p'(\varrho)\cdot\mathbf{u} + \varrho^\gamma\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t. \qquad (29)
\end{aligned}
$$

Using the identity (29), the relative entropy in (28) can be written as follows:

(30)

$$
\begin{aligned}
&\Big[\mathcal{E}_\epsilon(t)\Big]_0^\tau + \int_0^\tau\int_\Omega \Big(\mu|\nabla\mathbf{u}_\epsilon - \nabla\mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}\mathbf{u}_\epsilon - \mathrm{div}\mathbf{u}|^2 + \nu|\nabla\mathbf{H}_\epsilon - \nabla\mathbf{H}|^2\Big)\mathrm{d}x\mathrm{d}t \\
&\le \sum_{j=1}^{7} A_\epsilon^j,
\end{aligned}
$$

where

$$A_\epsilon^1 = \int_0^\tau\int_\Omega \Big(\varrho_\epsilon\Big(\partial_t\mathbf{u} + \mathbf{u}_\epsilon\cdot\nabla\mathbf{u} + \nabla\ln\varrho\Big)\cdot(\mathbf{u}-\mathbf{u}_\epsilon)\Big)\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^2 = \mu\int_0^\tau\int_\Omega \nabla\mathbf{u}:\nabla(\mathbf{u}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t + (\mu+\nu)\int_0^\tau\int_\Omega \mathrm{div}\mathbf{u}(\mathrm{div}\mathbf{u}-\mathrm{div}\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^3 = \int_0^\tau\int_\Omega \Big((\varrho-\varrho_\epsilon)\partial_t p'(\varrho) + \nabla p'(\varrho)\cdot(\varrho\mathbf{u}-\varrho_\epsilon\mathbf{u}_\epsilon) - \mathrm{div}\mathbf{u}(\varrho_\epsilon^\gamma - \varrho^\gamma)\Big)\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^4 = -\int_0^\tau\int_\Omega m_\epsilon(\partial_t\ln\varrho + \mathrm{div}\mathbf{u} + \mathbf{u}\cdot\nabla\ln\varrho)\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^5 = \epsilon\int_0^\tau\int_\Omega \partial_t\Delta G_\epsilon\ln(1/\varrho)\mathrm{d}x\mathrm{d}t + \epsilon\int_0^\tau\int_\Omega \Delta G_\epsilon\mathbf{u}\cdot\nabla\ln\varrho\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^6 = \int_0^\tau\int_\Omega \Big(\epsilon^2\mathrm{D}\mathbf{u}:(\nabla G_\epsilon\otimes\nabla G_\epsilon) + \frac{\epsilon^2}{2}|\nabla G_\epsilon|^2\mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^7 = -\int_0^\tau\int_\Omega \Big[((\mathbf{H}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon - \frac{1}{2}\nabla|\mathbf{H}_\epsilon|^2)\cdot\mathbf{u} + 2\nu\nabla\mathbf{H}_\epsilon:\nabla\mathbf{H} - \nu|\nabla\mathbf{H}|^2\Big]\mathrm{d}x\mathrm{d}t$$

$$A_\epsilon^8 = -\int_\Omega \Big[\mathbf{H}_\epsilon\cdot\mathbf{H} - \mathbf{H}_{0,\epsilon}\cdot\mathbf{H}_0 - \frac{1}{2}|\mathbf{H}|^2 + \frac{1}{2}|\mathbf{H}_0|^2\Big]\mathrm{d}x.$$

Step II. We introduce a result of the convex function h as follows:

$$h(\varrho_\epsilon) \geq \begin{cases} C(K)(|\varrho_\epsilon-\varrho|^2), & \text{if } \varrho_\epsilon\in K \\ C(K)(1+\varrho_\epsilon^\gamma), & \text{if } \varrho_\epsilon\in(0,\infty)\setminus K \end{cases} \tag{31}$$

for any compact subset $K\subset(0,\infty)$ and some $C(K)>0$. The following notations will be used later:

$$[h]_{\mathrm{ess}} = h1_{\varrho/2<\varrho_\epsilon<2\varrho}, \quad h = [h]_{\mathrm{ess}} + [h]_{\mathrm{res}}.$$

We first compute the residue and essential part which will be used later. Making use of the Holler inequality, we get

$$\begin{aligned} &\Big|\int_\Omega \Big[\frac{\varrho_\epsilon-\varrho}{\varrho}\Big]_{\mathrm{ess}} H(\mathbf{u})\cdot(\mathbf{u}_\epsilon-\mathbf{u})\mathrm{d}x\Big| \\ &\leq \|[\varrho_\epsilon-\varrho]_{\mathrm{ess}}\|_{L^2(\Omega)}\Big\|\frac{H(\mathbf{u})}{\varrho}\Big\|_{L^3(\Omega)}\|\mathbf{u}_\epsilon-\mathbf{u}\|_{L^6(\Omega)} \\ &\leq C(\theta)\mathcal{E}_\epsilon(\tau) + \theta\|\mathbf{u}_\epsilon-\mathbf{u}\|^2_{L^6(\Omega)}, \end{aligned} \tag{32}$$

where $H(\mathbf{u}) = \mu\Delta\mathbf{u} + (\mu+\nu)\nabla\mathrm{div}\mathbf{u} + (\nabla\times\mathbf{H})\times\mathbf{H}$. We also obtain

$$\begin{aligned} &\Big|\int_\Omega \Big[\frac{\varrho_\epsilon-\varrho}{\varrho}\Big]_{\mathrm{res}} H(\mathbf{u})\cdot(\mathbf{u}_\epsilon-\mathbf{u})\mathrm{d}x\Big| \\ &\leq \|[\varrho_\epsilon^{\gamma/2}]_{\mathrm{res}}\|_{L^2(\Omega)}\Big\|\frac{H(\mathbf{u})}{\varrho}\Big\|_{L^3(\Omega)}\|\sqrt{\varrho_\epsilon}(\mathbf{u}_\epsilon-\mathbf{u})\|_{L^2(\Omega)} \\ &\quad + \|[1]_{\mathrm{res}}\|_{L^2(\Omega)}\|H(\mathbf{u})\|_{L^3(\Omega)}\|\mathbf{u}_\epsilon-\mathbf{u}\|_{L^6(\Omega)} \\ &\leq C(\theta)\mathcal{E}_\epsilon(\tau) + \theta\|\mathbf{u}_\epsilon-\mathbf{u}\|^2_{L^6(\Omega)}. \end{aligned} \tag{33}$$

We next control the velocity term in A^1_ϵ and the first term can be written as follows:

$$
\begin{aligned}
&\int_0^\tau\int_\Omega \varrho_\epsilon(\partial_t\mathbf{u}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}+\nabla\ln\varrho)(\mathbf{u}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t\\
&=\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\otimes(\mathbf{u}-\mathbf{u}_\epsilon):\nabla\mathbf{u}\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\ln\varrho)\mathrm{d}x\mathrm{d}t\\
&\le C\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t+\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\ln\varrho)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{34}
$$

Note that

$$(\nabla\times\mathbf{H})\times\mathbf{H}=(\mathbf{H}\cdot\nabla)\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2.$$

Using two facts (32) and (33), we rewrite (34) as

$$
\begin{aligned}
&\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}-\mathbf{u}_\epsilon)\cdot(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\ln\varrho)\mathrm{d}x\mathrm{d}t\\
&=\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega \frac{\varrho_\epsilon-\varrho}{\varrho}(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\mathrm{div}\mathbf{u}+(\nabla\times\mathbf{H})\times\mathbf{H}\Big)\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega (\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\mathrm{div}\mathbf{u}+(\nabla\times\mathbf{H})\times\mathbf{H}\Big)\mathrm{d}x\mathrm{d}t\\
&=\int_0^\tau\int_\Omega \varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega \Big[\frac{\varrho_\epsilon-\varrho}{\varrho}(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\mathrm{div}\mathbf{u}+(\nabla\times\mathbf{H})\times\mathbf{H}\Big)\Big]_{\mathrm{ess}}\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega \Big[\frac{\varrho_\epsilon-\varrho}{\varrho}(\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\mathrm{div}\mathbf{u}+(\nabla\times\mathbf{H})\times\mathbf{H}\Big)\Big]_{\mathrm{res}}\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega (\mathbf{u}-\mathbf{u}_\epsilon)\cdot\Big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla\mathrm{div}\mathbf{u}+(\nabla\times\mathbf{H})\times\mathbf{H}\Big)\mathrm{d}x\mathrm{d}t\\
&\le C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t+\theta\int_0^\tau\|\mathbf{u}_\epsilon-\mathbf{u}\|^2_{L^6(\Omega)}\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega \Big(\mu\nabla(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla\mathbf{u}+(\mu+\nu)\mathrm{div}(\mathbf{u}_\epsilon-\mathbf{u})\mathrm{div}\mathbf{u}+\varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\Big)\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega (\mathbf{u}-\mathbf{u}_\epsilon)\cdot[(\nabla\times\mathbf{H})\times\mathbf{H}]\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where we have here used Holleder's inequality, integration by parts, and the property in (31). Thus, we get

$$
\begin{aligned}
&\int_0^\tau\int_\Omega \varrho_\epsilon(\partial_t\mathbf{u}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{u}+\nabla\ln\varrho)(\mathbf{u}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t\\
&\le \int_0^\tau\int_\Omega\Big(\mu\nabla(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla\mathbf{u}+(\mu+\nu)\mathrm{div}(\mathbf{u}_\epsilon-\mathbf{u})\mathrm{div}\mathbf{u}+\varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\Big)\mathrm{d}x\mathrm{d}t\\
&\quad+\theta\int_0^\tau\|\mathbf{u}_\epsilon-\mathbf{u}\|^2_{L^6(\Omega)}\mathrm{d}t\\
&\quad+\int_0^\tau\int_\Omega(\mathbf{u}-\mathbf{u}_\epsilon)\cdot[(\nabla\times\mathbf{H})\times\mathbf{H}]\mathrm{d}x\mathrm{d}t+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\mathrm{d}t,
\end{aligned}
\tag{35}
$$

which implies that

$$
\begin{aligned}
A^1_\epsilon &\le \int_0^\tau\int_\Omega\Big(\mu\nabla(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla\mathbf{u}+(\mu+\nu)\mathrm{div}(\mathbf{u}_\epsilon-\mathbf{u})\mathrm{div}\mathbf{u}+\varrho_\epsilon(\mathbf{u}_\epsilon-\mathbf{u})\cdot\nabla p'(\varrho)\Big)\mathrm{d}x\mathrm{d}t\\
&\quad+\theta\int_0^\tau\|\nabla(\mathbf{u}_\epsilon-\mathbf{u})\|^2_{L^2(\Omega)}\mathrm{d}t+\int_0^\tau\int_\Omega(\mathbf{u}-\mathbf{u}_\epsilon)\cdot[(\nabla\times\mathbf{H})\times\mathbf{H}]\mathrm{d}x\mathrm{d}t\\
&\quad+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\mathrm{d}t,
\end{aligned}
\tag{36}
$$

where we have here used the Poincare's inequality.

In virtue of (36), we obtain

$$
\begin{aligned}
A^1_\epsilon+A^2_\epsilon+A^3_\epsilon &\le \theta\|\nabla(\mathbf{u}_\epsilon-\mathbf{u})\|^2_{L^2(\Omega;\mathbb{R}^3)}+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\mathrm{div}\mathbf{u}\Big(\varrho_\epsilon^\gamma-\varrho^\gamma-\gamma\varrho^{\gamma-1}(\varrho_\epsilon-\varrho)\Big)\mathrm{d}x\mathrm{d}t\\
&\le \theta\|\nabla(\mathbf{u}_\epsilon-\mathbf{u})\|^2_{L^2(\Omega;\mathbb{R}^3)}+\int_0^\tau\int_\Omega(\mathbf{u}-\mathbf{u}_\epsilon)\cdot[(\nabla\times\mathbf{H})\times\mathbf{H}]\mathrm{d}x\mathrm{d}t\\
&\quad+C(\theta)\int_0^\tau\mathcal{E}_\epsilon(t)\mathrm{d}t
\end{aligned}
\tag{37}
$$

while

$$
\int_0^\tau\int_\Omega(\varrho-\varrho_\epsilon)(\partial_t p'(\varrho)+\nabla p'(\varrho)\cdot\mathbf{u})\mathrm{d}x\mathrm{d}t=-\int_0^\tau\int_\Omega\mathrm{div}\mathbf{u}(\varrho_\epsilon-\varrho)\gamma\varrho^{\gamma-1}\mathrm{d}x\mathrm{d}t.
$$

Step III. In the continuity equation (6), dividing by ϱ gives

$$
\partial_t\ln\varrho+\mathrm{div}\mathbf{u}+\mathbf{u}\cdot\nabla\ln\varrho=0,
$$

which implies that $A^4_\epsilon=0$.

For the terms of $A^5_\epsilon,\ A^6_\epsilon$, we can estimate as follows:

$$
\begin{aligned}
A^5_\epsilon \le C\epsilon\|\epsilon\nabla G_\epsilon\|_{L^\infty(0,T;L^2(\Omega))}\Big(&\|\nabla(\mathbf{u}\cdot\nabla\ln\varrho)\|_{L^\infty(0,T;L^2(\Omega))}\\
&+\|\ln\varrho\|_{W^{1,\infty}(0,T;H^1(\Omega))}\Big)
\end{aligned}
\tag{38}
$$

and

$$A_\epsilon^6 \le C\epsilon\|\epsilon\nabla G_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega))}\|\nabla\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}. \tag{39}$$

Step IV. Finally, it remains to handle A_ϵ^7 and A_ϵ^8. We also use $\mathbf{H}$ as a test function to the magnetic field equation (3) and insert (8), which yields that

$$\begin{aligned}
-\int_\Omega(\mathbf{H}_\epsilon\cdot\mathbf{H})(\tau)\mathrm{d}x = & -\int_\Omega \mathbf{H}_{0,\epsilon}\cdot\mathbf{H}_0\mathrm{d}x + \nu\int_0^\tau\int_\Omega \nabla\mathbf{H}_\epsilon : \nabla\mathbf{H}\mathrm{d}x\mathrm{d}t \\
& + \int_0^\tau\int_\Omega[(\mathrm{div}\mathbf{u}_\epsilon)\mathbf{H}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon - (\mathbf{H}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon]\cdot\mathbf{H}\mathrm{d}x\mathrm{d}t \\
& + \int_0^\tau\int_\Omega \mathbf{H}_\epsilon\cdot[(\mathrm{div}\mathbf{u})\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{H}]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{40}$$

Multiplying $\mathbf{H}$ to the equation (8), we get

$$\begin{aligned}
&\int_\Omega\Big(\frac{1}{2}|\mathbf{H}|^2 - \frac{1}{2}|\mathbf{H}_0|^2\Big)\mathrm{d}x \\
&\quad = -\int_0^\tau\int_\Omega\Big(\mathrm{div}\mathbf{u}|\mathbf{H}|^2 + (\mathbf{u}\cdot\nabla)\mathbf{H}\cdot\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u}\cdot\mathbf{H} + \nu|\nabla\mathbf{H}|^2\Big)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{41}$$

We now insert (40) and (41) into (30) together with using the previous results and so we get

$$\begin{aligned}
&\Big[\mathcal{E}_\epsilon(t)\Big]_0^\tau \\
&\quad + C\int_0^\tau\int_\Omega\Big(\mu|\nabla\mathbf{u}_\epsilon-\nabla\mathbf{u}|^2+(\mu+\lambda)|\mathrm{div}\mathbf{u}_\epsilon-\mathrm{div}\mathbf{u}|^2+\nu|\nabla\mathbf{H}_\epsilon-\nabla\mathbf{H}|^2\Big)\mathrm{d}x\mathrm{d}t \\
&\le -\int_0^\tau\int_\Omega \mathbf{u}_\epsilon\cdot[(\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2]\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau\int_\Omega \mathbf{H}_\epsilon\cdot[(\mathrm{div}\mathbf{u})\mathbf{H} + (\mathbf{u}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u}]\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau\int_\Omega \mathbf{H}\cdot[\mathrm{div}\mathbf{u}_\epsilon\ \mathbf{H}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{H}_\epsilon - (\mathbf{H}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon]\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau\int_\Omega\Big(\mathrm{div}\mathbf{u}|\mathbf{H}|^2 + (\mathbf{u}\cdot\nabla)\mathbf{H}\cdot\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u}\cdot\mathbf{H}\Big)\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C\epsilon \\
&\le \int_0^\tau\int_\Omega\Big([(\mathbf{H}_\epsilon-\mathbf{H})\cdot\nabla]\mathbf{u}\cdot(\mathbf{H}_\epsilon-\mathbf{H})+[(\mathbf{H}_\epsilon-\mathbf{H})\cdot\nabla]\mathbf{H}\cdot(\mathbf{u}_\epsilon-\mathbf{u})\Big)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau\int_\Omega\Big([(\mathbf{H}-\mathbf{H}_\epsilon)\cdot\nabla]\mathbf{H}\cdot(\mathbf{u}_\epsilon-\mathbf{u})+(\mathbf{u}\cdot\nabla)(\mathbf{H}-\mathbf{H}_\epsilon)\cdot(\mathbf{H}-\mathbf{H}_\epsilon)\Big)\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C\epsilon
\end{aligned} \tag{42}$$

$$:= \sum_{j=1}^{4} B_\epsilon^j + C\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C\epsilon,$$

where we have used the integration by parts and using $\mathrm{div}\mathbf{H}_\epsilon = 0,\ \mathrm{div}\mathbf{H} = 0$.

For the term of B_ϵ^1, it is easily to get

$$B_\epsilon^1 \le C\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t, \tag{43}$$

where we have used the regularity in Proposition 2.1.

For the terms of $B_\epsilon^2, B_\epsilon^3$, we can use the Young's inequality and Korn inequality to obtain

$$B_\epsilon^2 + B_\epsilon^3 \le C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + \theta\int_0^\tau\!\!\int_\Omega |\nabla\mathbf{u}_\epsilon - \nabla\mathbf{u}|^2\mathrm{d}x\mathrm{d}t \tag{44}$$

for a suitable $\delta > 0$. Finally, we are also going to apply the Young's inequality to get

$$B_\epsilon^4 \le C(\theta)\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + \theta\int_0^\tau\!\!\int_\Omega |\nabla\mathbf{H}_\epsilon - \nabla\mathbf{H}|^2\mathrm{d}x\mathrm{d}t \tag{45}$$

for a suitable $\theta > 0$. Consequently, by choosing small number $\theta > 0$, the relative entropy in (42) is given by

$$\begin{aligned} &\Big[\mathcal{E}_\epsilon(t)\Big]_{t=0}^{t=\tau} \\ &\quad + C\int_0^\tau\!\!\int_\Omega \Big(\mu|\nabla\mathbf{u}_\epsilon - \nabla\mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}\mathbf{u}_\epsilon - \mathrm{div}\mathbf{u}|^2 + \nu|\nabla\mathbf{H}_\epsilon - \nabla\mathbf{H}|^2\Big)\mathrm{d}x\mathrm{d}t \\ &\le C\int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C\epsilon. \end{aligned} \tag{46}$$

Step V. Complete the proof. Let us apply the Gronwall's inequality to (46) in order to obtain:

$$\mathcal{E}_\epsilon(\tau) \le \exp(TC))\mathcal{E}_\epsilon(0) \tag{47}$$

for all $\tau \in [0, T]$.

For the estimates of the initial data, we use the assumptions (14)-(16) to obtain

$$\begin{aligned} \int_\Omega \varrho_{0,\epsilon}|\mathbf{u}_{0,\epsilon} - \mathbf{u}_0|^2\mathrm{d}x &\le C\int_\Omega |\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \sqrt{\varrho_0}\mathbf{u}_0|^2\mathrm{d}x + C\int_\Omega |\sqrt{\varrho_{0,\epsilon}} - \sqrt{\varrho_0}|^2\mathrm{d}x \\ &\le C\int_\Omega |\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \sqrt{\varrho_0}\mathbf{u}_0|^2\mathrm{d}x + C\int_\Omega |\varrho_{0,\epsilon} - \varrho_0|^2\mathrm{d}x \le C\epsilon, \end{aligned}$$

$$\int_\Omega |\mathbf{H}_{0,\epsilon} - \mathbf{H}_0|^2\mathrm{d}x \le C\epsilon,\ \int_\Omega h(\varrho_{0,\epsilon})\mathrm{d}x \le C\epsilon,$$

and

$$\int_\Omega \Big[\exp V_{0,\epsilon}\ln\Big(\exp V_{0,\epsilon}/\varrho_0\Big) - \exp V_{0,\epsilon} + \varrho_0\Big]\mathrm{d}x \le \int_\Omega (\exp V_{0,\epsilon} - \varrho_0)^2 \le C\epsilon.$$

Thus we get

$$\mathcal{E}_\epsilon(0) \leq C\epsilon$$

and the Gronwall inequality gives the results (17)-(19) in Theorem 2.1. Indeed,

$$\begin{aligned}\int_\Omega |\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon - \sqrt{\varrho}\mathbf{u}|^2 \mathrm{d}x &\leq C\int_\Omega |\sqrt{\varrho_\epsilon}(\mathbf{u}_\epsilon - \mathbf{u})|^2 \mathrm{d}x + C\int_\Omega |\sqrt{\varrho_\epsilon} - \sqrt{\varrho}|^2 \mathrm{d}x \\ &\leq C\epsilon,\end{aligned}$$

$$\int_\Omega \Big([\varrho_\epsilon - \varrho]^2_{\mathrm{ess}} + [\varrho_\epsilon - \varrho]^\gamma_{\mathrm{res}} \Big) \mathrm{d}x \leq \int_\Omega h(\varrho_\epsilon) \mathrm{d}x \leq C\epsilon,$$

$$\int_\Omega |\sqrt{\exp V_\epsilon} - \sqrt{\varrho}|^2 \mathrm{d}x \leq C\epsilon, \quad \int_\Omega |\mathbf{H}_\epsilon - \mathbf{H}|^2 \mathrm{d}x \leq C\epsilon.$$

Hence we complete our proof of Theorem 2.1. □

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