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EXISTENCE OF WEAK SOLUTIONS TO A CLASS OF SCHRÖDINGER TYPE EQUATIONS INVOLVING THE FRACTIONAL p-LAPLACIAN IN $\mathbb{R}^N$

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Abstract. We are concerned with the following elliptic equations:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda g(x,u) \quad \text{in} \quad \mathbb{R}^N,$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0 < s < 1 < p < +\infty$, $sp < N$, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is a continuous potential function, and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. We show the existence of at least one weak solution for the problem above without the Ambrosetti and Rabinowitz condition. Moreover, we give a positive interval of the parameter λ for which the problem admits at least one nontrivial weak solution when the nonlinearity g has the subcritical growth condition.

1. Introduction

A great attention has been drawn to the study of nonlocal type operators in view of the mathematical theory to concrete some phenomena: social sciences, quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process and Levy process [6, 8, 14, 21, 24, 32, 33] and the references therein. In particular, the fractional Schrödinger equation which is initially introduced by Laskin [24] has received considerable attention in recent years (see e.g. [20, 40, 47]).

Motivated by huge interest in the current literature, exploiting variational methods, we investigate the existence of nontrivial weak solutions for the fractional p-Laplacian problems. To be more precise, we consider the existence results of a nontrivial weak solution for the following nonlinear elliptic equations of the fractional p-Laplace type:

$$(P_\lambda) \qquad (-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda g(x,u) \quad \text{in} \quad \mathbb{R}^N,$$

where λ is a real parameter, $0 < s < 1 < p < +\infty$, $sp < N$, the potential function $V : \mathbb{R}^N \to (0,\infty)$ is continuous, $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory

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function, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon^N(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon^N(x) := \{y \in \mathbb{R}^N : |y-x| \le \varepsilon\}$. Many researchers have been extensively studied the fractional p-Laplacian type problems in various way; see [4, 5, 8, 13, 14, 19, 22, 26, 36, 38, 44, 47] and the references therein.

Since the seminal work of Ambrosetti and Rabinowitz in [3], critical point theory has become one of the most effective analytic tools for establishing the existence of solutions to elliptic equations of variational type. Afterward, many results for the existence of nontrivial solutions to nonlinear elliptic problems involving the fractional p-Laplacian type have been obtained; see for example [4, 5, 7, 15, 20, 22, 26, 38, 41, 42, 46]. Especially, the existence and multiplicity results for the fractional p-Laplacian type problems have been studied by Iannizzotto et al. [22] and Servadei [38] for a bounded domain in $\mathbb{R}^N$. The key ingredient for obtaining these results is the Ambrosetti and Rabinowitz condition ((AR)-condition for short) in [3];

(AR) There exist positive constants C_0 and η such that $\eta > p$ and

$$0 < \eta G(x,t) \le g(x,t)t \quad \text{for } x \in \Omega \text{ and } |t| \ge C_0,$$

where $G(x,t) = \int_0^t g(x,s)\, ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

It is well known that the (AR)-condition is essential to verify the compactness condition of the Euler-Lagrange functional which plays a central role in applying critical point theory. However this condition is very restrictive and eliminates many nonlinearities. Miyagaki and Souto [34] established the existence of a nontrivial solution for the superlinear problems without assuming the (AR)-condition. Inspired by this paper, the existence of at least one solution and infinitely many solutions for the p-Laplacian problem in a bounded domain $\Omega \subset \mathbb{R}^N$ was presented by Liu-Li [31] under the following assumption:

(LL) There exists $C_* > 0$ such that

$$\mathcal{G}(x,t) \le \mathcal{G}(x,\tau) + C_*$$

for each $x \in \Omega$, $0 < t < \tau$ or $\tau < t < 0$, where $\mathcal{G}(x,t) = g(x,t)t - pG(x,t)$.

See also [50] for $p = 2$. In this direction, Wei-Su in [42] showed that the fractional Laplacian problem possesses infinitely many weak solutions. On the other hand, the existence and multiplicity of weak solutions for the p-Laplacian equation in case of the whole space $\mathbb{R}^N$ were obtained by Liu [30] under the following assumption:

(Je) There exists $\eta \ge 1$ such that

$$\eta \mathcal{G}(x,t) \ge \mathcal{G}(x,\tau t)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $\tau \in [0,1]$, where $\mathcal{G}(x,t) = g(x,t)t - pG(x,t)$ and $G(x,t) = \int_0^t g(x,s)ds$.

Recently, under this condition, Torres in [41] obtained the existence result for the fractional p-Laplacian problem by using the mountain pass theorem. In fact, the condition above is originally due to Jeanjean [23] in the case of $p = 2$. Following in [31], the condition (Je) is weaker than the condition that for each $x \in \mathbb{R}^N$,

$$\frac{g(x,t)}{|t|^{p-1}} \quad \text{is an increasing function of } t \in \mathbb{R}^N \setminus \{0\}.$$

In the last few decades, there were extensive studies dealing with the p-Laplacian problems by the assumption (Je); see [27–29] for the p-Laplacian and [2,39, 45] for the $p(x)$-Laplacian. In this respect, authors in [7, 20, 46] extended the existence of infinitely many weak solutions to the fractional Laplacian problems.

The purpose of this study is twofold. First, by using the mountain pass theorem under the Cerami condition that is slightly weaker than the well known Palais-Smale condition, we present the existence of a nontrivial weak solution for our problem when the condition on g has mild and different assumptions from the condition (Je) based on the arguments in [28, 44, 48]. Second, we concretely provides an estimate of the positive interval of the parameters λ for which the problem (P_λ) admits at least one nontrivial weak solution when the nonlinearity g has the subcritical growth condition (but may not always be p-superlinear). To do this, we give an abstract result that is based on the work of Bonanno [9]. It is worth noticing that we obtain the existence of the nontrivial weak solution for our problem without the facts that the energy functional associated with (P_λ) satisfies the Cerami condition and the mountain pass geometry that is crucial to apply the mountain pass theorem.

This paper is structured as follows. In Section 2, we recall briefly some basic results for the fractional Sobolev spaces. And under various conditions on g, we obtain several existence results of nontrivial weak solutions for problem (P_λ) by utilizing the variational principle. Also we obtain the existence of at least one nontrivial weak solution whenever the parameter λ belongs to a positive interval.

2. Preliminaries and existence of a nontrivial weak solution

In this section, we briefly recall some definitions and basic properties of the fractional Sobolev spaces. We refer the reader to [1,21,35] for further references.

Let $s \in (0,1)$ and $p \in (1, +\infty)$. We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dxdy < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W^{s,p}(\mathbb{R}^N)} \right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |u|^p \, dx \quad \text{and} \quad |u|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dxdy.$$

Let $s \in (0,1)$ and $1 < p < +\infty$. Then $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space. Also, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, that is $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ (see e.g. [1, 35]).

Lemma 2.1 ([17]). *Let $\Omega \subset \mathbb{R}^N$ a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $p \in (1, +\infty)$. Then we have the following continuous embeddings*:

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } \ q \in [1, p_s^*], && \text{if } \ sp < N; \\ &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for every } \ q \in [1, \infty), && \text{if } \ sp = N; \\ &W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) && \text{for all } \ \lambda < s - N/p, && \text{if } \ sp > N, \end{aligned}$$

where p_s^* is the fractional critical Sobolev exponent, that is

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

In particular, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [p, p_s^*)$.

Lemma 2.2 ([35, 37]). *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Then there exists a positive constant $C = C(N, p, s)$ such that for all $u \in W^{s,p}(\mathbb{R}^N)$,*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leq C \, |u|_{W^{s,p}(\mathbb{R}^N)}.$$

Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$. In particular, we denote the best constant $S_{s,p}$ in the fractional Sobolev inequality by

$$S_{s,p} = \inf_{u \in L^{p_s^*}(\mathbb{R}^N) \setminus \{0\}, |u|_{W^{s,p}(\mathbb{R}^N)} < \infty} \frac{|u|^p_{W^{s,p}(\mathbb{R}^N)}}{\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)}}.$$

For our analysis, we assume that

(V) $V \in C(\mathbb{R}^N)$, $\inf_{x \in \mathbb{R}^N} V(x) > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$ for all $M \in \mathbb{R}$.

When V satisfies (V), the basic space

$$X_s(\mathbb{R}^N) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : V|u|^p \in L^1(\mathbb{R}^N) \right\}$$

denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X_s(\mathbb{R}^N)} := \left(|u|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}} u\|^p_{L^p(\mathbb{R}^N)} \right)^{\frac{1}{p}}.$$

Lemma 2.3 ([41]). *Let $0 < s < 1 < p < +\infty$ with $ps < N$ and suppose that the assumption* (V) *holds. Then there is a compact embedding $X_s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*)$.*

2.1. Existence of a weak solution via the mountain pass theorem

In this subsection, we deal with the existence of a nontrivial weak solution for the problem (P_λ) under suitable assumptions.

Definition 2.4. Let $0 < s < 1 < p < +\infty$. We say that $u \in X_s(\mathbb{R}^N)$ is a weak solution of the problem (P_λ) if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|u(x)|^{p-2}\,uv\,dx = \lambda\int_{\mathbb{R}^N} g(x,u)v\,dx$$

for all $v \in X_s(\mathbb{R}^N)$.

Let us define a functional $\Phi_{s,p} : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi_{s,p}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|u(x)|^p\,dx.$$

So then from Lemma 3.2 of [41], the functional $\Phi_{s,p}$ is well defined on $X_s(\mathbb{R}^N)$, $\Phi_{s,p} \in C^1(X_s(\mathbb{R}^N), \mathbb{R})$ and its Fréchet derivative is given by for any $v \in X_s(\mathbb{R}^N)$,

$$\langle \Phi'_{s,p}(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|u(x)|^{p-2}\,uv\,dx.$$

Lemma 2.5 ([36]). *Let $0 < s < 1 < p < +\infty$ and let the assumption* (V) *hold. Then the functional $\Phi'_{s,p}$ is of type (S_+), i.e., if $u_n \rightharpoonup u$ in $X_s(\mathbb{R}^N)$ and $\limsup_{n\to\infty} \langle \Phi'_{s,p}(u_n) - \Phi'_{s,p}(u), u_n - u\rangle \le 0$, then $u_n \to u$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$.*

Denoting $G(x,t) = \int_0^t g(x,s)\,ds$ and we suppose that for $1 < p < q < p_s^*$ and $x \in \mathbb{R}^N$,

(G1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

(G2) There exist nonnegative functions $a \in L^{q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N)$ such that

$$|g(x,t)| \le a(x) + b(x)\,|t|^{q-1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $1/q + 1/q' = 1$.

(G3) There exists $\delta > 0$ such that

$$G(x,t) \le 0 \quad \text{for } x \in \mathbb{R}^N,\ |t| < \delta.$$

(G4) $\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^p} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.

(G5) There exist $c_0 \geq 0$, $r_0 \geq 0$, and $\tau > \frac{N}{ps}$ such that

$$|G(x,t)|^\tau \leq c_0 \, |t|^{\tau p} \, \mathfrak{G}(x,t)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $|t| \geq r_0$, where $\mathfrak{G}(x,t) = (1/p)g(x,t)t - G(x,t) \geq 0$.

(G6) There exist $\mu > p$ and $\varrho > 0$ such that

$$\mu G(x,t) \leq t g(x,t) + \varrho t^p$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Under the assumptions (G1) and (G2), we define the functional $\Psi : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} G(x,u)\, dx.$$

Then it follows from the same arguments as those of Proposition 1.12 in [43] that $\Psi \in C^1(X_s(\mathbb{R}^N), \mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(u), v\rangle \; = \int_{\mathbb{R}^N} g(x,u)v \, dx$$

for any $u, v \in X_s(\mathbb{R}^N)$. Next we define a functional $\mathcal{I}_\lambda : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) = \Phi_{s,p}(u) - \lambda\Psi(u).$$

Then we know that the functional $\mathcal{I}_\lambda \in C^1(X_s(\mathbb{R}^N), \mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle \mathcal{I}'_\lambda(u), v\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy \\ &+ \int_{\mathbb{R}^N} V(x)\,|u(x)|^{p-2}\,uv\, dx - \lambda \int_{\mathbb{R}^N} g(x,u)v\, dx\end{aligned}$$

for any $u, v \in X_s(\mathbb{R}^N)$.

Remark 2.6. As seen before, there were many existence results of solution for elliptic problems under the assumption (Je): see [27–29, 39, 45]. The authors in [28], however gave some examples which satisfy the assumptions (G5), (G6) not (Je). In that sense, our analysis is motivated by this counterexample. For example,

$$g(x,t) = a(x)\,|t|^{p-2}\,t(4\,|t|^3 + 2t\sin t - 4\cos t),$$

where $0 < \inf_{\mathbb{R}^N} a \leq \sup_{\mathbb{R}^N} a < \infty$.

First of all, in this setting, we need the following lemma.

Lemma 2.7. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V) *and* (G1)–(G2) *hold. Then Ψ and Ψ' are weakly strongly continuous in $X_s(\mathbb{R}^N)$.*

Proof. Let $\{u_n\}$ be a sequence in $X_s(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$. Then $\{u_n\}$ is bounded in $X_s(\mathbb{R}^N)$ and Lemma 2.3 guarantees that the embeddings $X_s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ and $X_s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ are compact for $p < q < p_s^*$. So we know that

$$u_n \to u \text{ in } L^p(\mathbb{R}^N) \text{ and } u_n \to u \text{ in } L^q(\mathbb{R}^N) \text{ as } n \to \infty. \tag{2.1}$$

First we prove that Ψ is weakly strongly continuous in $X_s(\mathbb{R}^N)$. Let $u_n \to u$ in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ as $n \to \infty$. By the convergence principle, there exist a subsequence $\{u_{n_k}\}$ and a function $v \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ such that $u_{n_k}(x) \to u(x)$ as $k \to \infty$ for almost all $x \in \mathbb{R}^N$ and $|u_{n_k}(x)| \le v(x)$ for all $k \in \mathbb{N}$ and for almost all $x \in \mathbb{R}^N$. Therefore taking (G2) into account, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^N} |G(x, u_{n_k}) - G(x, u)|\, dx \\
&\le \int_{\mathbb{R}^N} |G(x, u_{n_k})| + |G(x, u)|\, dx \\
&\le \int_{\mathbb{R}^N} a(x)|u_{n_k}(x)| + b(x)|u_{n_k}(x)|^q + a(x)|u(x)| + b(x)|u(x)|^q\, dx \\
&\le \int_{\mathbb{R}^N} a(x)|v(x)| + b(x)|v(x)|^q + a(x)|u(x)| + b(x)|u(x)|^q\, dx
\end{aligned}$$

and thus the integral at the left-hand side is dominated by an integrable function. Since g is the Carathéodory function, we have that $G(x, u_{n_k}) \to G(x, u)$ as $k \to \infty$ for almost all $x \in \mathbb{R}^N$ by (G1). Therefore, the Lebesgue convergence theorem tells us that

$$\int_{\mathbb{R}^N} G(x, u_{n_k})\, dx \to \int_{\mathbb{R}^N} G(x, u)\, dx$$

as $k \to \infty$. This implies that Ψ is weakly strongly continuous in $X_s(\mathbb{R}^N)$.

Next, we show that Ψ' is weakly strongly continuous in $X_s(\mathbb{R}^N)$. Using (G2) and Hölder's inequality, we assert that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |g(x, u_n) - g(x, u)|^{q'}\, dx \\
&\le C_1 \int_{\mathbb{R}^N} |g(x, u_n)|^{q'} + |g(x, u)|^{q'}\, dx \\
&\le C_2 \int_{\mathbb{R}^N} |a(x)|^{q'} + \|b\|^{q'}_{L^\infty(\mathbb{R}^N)}(|u_n(x)|^q + |u(x)|^q)\, dx
\end{aligned} \tag{2.2}$$

for some positive constants C_1, C_2. Since $u_n \to u$ as $n \to \infty$ in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ and almost all in $\mathbb{R}^N$, it follows from (2.2) and the convergence principle that

$$|g(x, u_n) - g(x, u)|^{q'} \le h(x) \text{ for almost all } x \in \mathbb{R}^N \text{ and for some } h \in L^1(\mathbb{R}^N),$$

and thus $g(x,u_n) \to g(x,u)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$. This together with the Lebesgue convergence Theorem yields that

$$\begin{aligned}\|\Psi'(u_n) - \Psi'(u)\|_{X_s^*(\mathbb{R}^N)} &= \sup_{\|\varphi\|_{X_s(\mathbb{R}^N)} \le 1} |\langle \Psi'(u_n) - \Psi'(u), \varphi\rangle| \\ &= \sup_{\|\varphi\|_{X_s(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|\,|\varphi(x)|\,dx \\ &\le \left(\int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|^{q'}\,dx\right)^{\frac{1}{q'}} \to 0 \text{ as } n \to \infty.\end{aligned}$$

This completes the proof. □

The following assertion is to show that the energy functional $\mathcal{I}_\lambda$ satisfies the mountain pass geometry.

Lemma 2.8. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V) *and* (G1)–(G4) *hold. Then the geometric conditions in the mountain pass theorem are satisfied, i.e.,*

(1) *$u = 0$ is a strict local minimum for $\mathcal{I}_\lambda$,*
(2) *$\mathcal{I}_\lambda$ is unbounded from below in $X_s(\mathbb{R}^N)$.*

Proof. By the condition (G3), it is trivial that $u = 0$ is a strict local minimum for $\mathcal{I}_\lambda$. Next we prove the condition (2). By the condition (G4), for any positive constant $\mathcal{M}$, we can choose a constant $\delta > 0$ such that

$$G(x,t) \ge \mathcal{M}\,|t|^p \tag{2.3}$$

for $|t| > \delta$ and for almost all $x \in \mathbb{R}^N$. Take $v \in X_s(\mathbb{R}^N) \setminus \{0\}$. Then the relation (2.3) implies that

$$\mathcal{I}_\lambda(tv) = \Phi_{s,p}(tv) - \lambda\Psi(tv) \le t^p \left(\frac{1}{p}\|v\|^p_{X_s(\mathbb{R}^N)} - \lambda\mathcal{M}\int_{\mathbb{R}^N} |v(x)|^p\,dx\right)$$

for sufficiently large $t > 1$. If $\mathcal{M}$ is large enough, then we conclude that $\mathcal{I}_\lambda(tv) \to -\infty$ as $t \to \infty$ and therefore the functional $\mathcal{I}_\lambda$ is unbounded from below. This completes the proof. □

With the help of Lemmas 2.5 and 2.7, we show that the energy functional $\mathcal{I}_\lambda$ satisfies the Cerami condition ($(C)_c$-condition for brevity), *i.e.*, for $c \in \mathbb{R}$, any sequence $\{u_n\} \subset X_s(\mathbb{R}^N)$ such that

$$\mathcal{I}_\lambda(u_n) \to c \ \text{ and } \ \|\mathcal{I}'_\lambda(u_n)\|_{X_s^*(\mathbb{R}^N)}(1 + \|u_n\|_{X_s(\mathbb{R}^N)}) \to 0 \ \text{ as } n \to \infty$$

has a convergent subsequence. The basic idea of the proofs for the following Lemmas 2.9 and 2.11 comes from the paper [28]. These play a decisive role in showing the existence of a nontrivial weak solution for problem (P_λ).

Lemma 2.9. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V) *and* (G1)–(G5) *hold. For any $\lambda > 0$, the functional $\mathcal{I}_\lambda$ satisfies the $(C)_c$-condition.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(C)_c$-sequence in $X_s(\mathbb{R}^N)$, that is,

$$\mathcal{I}_\lambda(u_n) \to c \text{ and } \|\mathcal{I}'_\lambda(u_n)\|_{X_s^*(\mathbb{R}^N)}(1 + \|u_n\|_{X_s(\mathbb{R}^N)}) \to 0 \text{ as } n \to \infty, \tag{2.4}$$

which show that

$$c = \mathcal{I}_\lambda(u_n) + o(1) \quad \text{and} \quad \langle \mathcal{I}'_\lambda(u_n), u_n\rangle = o(1), \tag{2.5}$$

where $o(1) \to 0$ as $n \to \infty$. It follows from Lemmas 2.5 and 2.7 that $\Phi'_{s,p}$ and Ψ' are mappings of type (S_+). Since $\mathcal{I}'_\lambda$ is of type (S_+) and $X_s(\mathbb{R}^N)$ is reflexive, it suffices to verify that the sequence $\{u_n\}$ is bounded in $X_s(\mathbb{R}^N)$. Suppose to the contrary that the sequence $\{u_n\}$ is unbounded in $X_s(\mathbb{R}^N)$. So then we may suppose that

$$\|u_n\|_{X_s(\mathbb{R}^N)} > 1 \ \text{ and } \ \|u_n\|_{X_s(\mathbb{R}^N)} \to \infty \ \text{ as } \ n \to \infty.$$

Define a sequence $\{w_n\}$ by $w_n = u_n/\|u_n\|_{X_s(\mathbb{R}^N)}$. Then it is obvious that $\{w_n\} \subset X_s(\mathbb{R}^N)$ and $\|w_n\|_{X_s(\mathbb{R}^N)} = 1$. Hence, up to a subsequence, still denoted by $\{w_n\}$, we obtain $w_n \rightharpoonup w$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$ and by Lemma 2.3, we have

$$w_n \to w \ \text{ a.e. in } \ \mathbb{R}^N \ \text{ and } \ w_n \to w \ \text{ in } \ L^{p_0}(\mathbb{R}^N) \ \text{ as } \ n \to \infty \tag{2.6}$$

for $p \le p_0 < p_s^*$. Set $\Omega = \big\{x \in \mathbb{R}^N : w(x) \neq 0\big\}$. Due to the condition (2.5), we have that

$$\begin{aligned} c = \mathcal{I}_\lambda(u_n) + o(1) &= \Phi_{s,p}(u_n) - \lambda\Psi(u_n) + o(1) \\ &= \frac{1}{p}\|u_n\|^p_{X_s(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} G(x, u_n)\, dx + o(1). \end{aligned} \tag{2.7}$$

Since $\|u_n\|_{X_s(\mathbb{R}^N)} \to \infty$ as $n \to \infty$, we assert that

$$\int_{\mathbb{R}^N} G(x, u_n)\, dx = \frac{1}{\lambda p}\|u_n\|^p_{X_s(\mathbb{R}^N)} - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty \ \text{ as } \ n \to \infty. \tag{2.8}$$

From the assumptions (G1) and (G2), we have that there exists a positive constant M such that $|G(x,t)| \le M$ for all $(x,t) \in \mathbb{R}^N \times [-t_0, t_0]$. This together with (G4) yields that there is a real number M_0 such that $G(x,t) \ge M_0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{G(x, u_n) - M_0}{\|u_n\|^p_{X_s(\mathbb{R}^N)}} \ge 0 \tag{2.9}$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. By the convergence (2.6), we know that $|u_n| = |w_n|\,\|u_n\|_{X_s(\mathbb{R}^N)} \to \infty$ as $n \to \infty$ for all $x \in \Omega$. Furthermore, owing to the condition (G4), we have

$$\lim_{n\to\infty} \frac{G(x, u_n)}{\|u_n\|^p_{X_s(\mathbb{R}^N)}} = \lim_{n\to\infty} \frac{G(x, u_n)}{|u_n|^p}\, |w_n|^p = \infty \tag{2.10}$$

for all $x \in \Omega$. Hence we get that $\operatorname{meas}(\Omega) = 0$. Indeed, if $\operatorname{meas}(\Omega) \neq 0$, then according to (2.7)–(2.10), and the Fatou lemma, we deduce that

$$\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(x,u_n)\,dx}{\lambda \int_{\mathbb{R}^N} G(x,u_n)\,dx + c - o(1)} \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{G(x,u_n)}{\frac{1}{p}\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \\
&\geq \liminf_{n\to\infty} \int_\Omega \frac{pG(x,u_n)}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx - \limsup_{n\to\infty} \int_\Omega \frac{pM_0}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \\
&= \liminf_{n\to\infty} \int_\Omega \frac{p(G(x,u_n)-M_0)}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \qquad (2.11)\\
&\geq \int_\Omega \liminf_{n\to\infty} \frac{p(G(x,u_n)-M_0)}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \\
&= \int_\Omega \liminf_{n\to\infty} \frac{pG(x,u_n)}{|u_n|^p_{X_s(\mathbb{R}^N)}}\,|w_n|^p_{X_s(\mathbb{R}^N)}\,dx - \int_\Omega \limsup_{n\to\infty} \frac{pM_0}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \\
&= \infty,
\end{aligned}$$

which is a contradiction. Thus $w(x) = 0$ for almost all $x \in \mathbb{R}^N$.

Observe that

$$\begin{aligned}
c+1 &\geq \mathcal{I}_\lambda(u_n) - \frac{1}{p}\,\langle \mathcal{I}'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{p}\|u_n\|^p_{X_s(\mathbb{R}^N)} - \lambda \int_{\mathbb{R}^N} G(x,u_n)\,dx - \frac{1}{p}\|u_n\|^p_{X_s(\mathbb{R}^N)} \qquad (2.12)\\
&\quad + \frac{\lambda}{p}\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx \\
&\geq \lambda \int_{\mathbb{R}^N} \mathfrak{G}(x,u_n)\,dx
\end{aligned}$$

for n large enough and $\mathfrak{G}$ is defined in (G5). Let us define $\Omega_n(a,b) := \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}$ for $a \geq 0$. By the convergence (2.6), we note that

$$w_n \to 0 \ \text{ in } \ L^{p_0}(\mathbb{R}^N) \ \text{ and } \ w_n \to 0 \ \text{ a.e. in } \ \mathbb{R}^N \ \text{ as } \ n \to \infty \qquad (2.13)$$

for $p \leq p_0 < p^*_s$. Hence from the relation (2.8) we get

$$0 < \frac{1}{\lambda p} \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{|G(x,u_n)|}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx. \qquad (2.14)$$

On the other hand, we can choose a positive constant K_{p_0} such that $\|w_n\|_{L^{p_0}(\mathbb{R}^N)} \leq K_{p_0}\|w_n\|_{X_s(\mathbb{R}^N)}$ for $p \leq p_0 < p^*_s$, because $\|w_n\|_{X_s(\mathbb{R}^N)} = 1$. From the assumption (G2) and (2.13), we have

$$\int_{\Omega_n(0,d)} \frac{G(x,u_n)}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \leq \int_{\Omega_n(0,d)} \frac{a(x)\,|u_n(x)| + \frac{b(x)}{q}\,|u_n(x)|^q}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx$$

$$
\begin{aligned}
&\le \frac{1}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\|a\|_{L^{q'}(\mathbb{R}^N)}\|u_n\|_{L^q(\mathbb{R}^N)} \\
&\quad + \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\int_{\Omega_n(0,d)} |u_n(x)|^{q-p}\,|w_n(x)|^p\,dx \\
&\le \frac{C_3\|a\|_{L^{q'}(\mathbb{R}^N)}}{\|u_n\|^{p-1}_{X_s(\mathbb{R}^N)}}\|w_n\|_{L^q(\mathbb{R}^N)} \\
&\quad + \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}d^{q-p}\int_{\mathbb{R}^N} |w_n(x)|^p\,dx \\
&\le \frac{K_qC_4}{\|u_n\|^{p-1}_{X_s(\mathbb{R}^N)}} + \frac{\|b\|_{L^\infty(\mathbb{R}^N)}}{q}d^{q-p}\int_{\mathbb{R}^N} |w_n(x)|^p\,dx \to 0 \\
&\quad \text{as } n\to\infty,
\end{aligned} \tag{2.15}
$$

where C_3 and C_4 are positive constants. Set $\tau' = \tau/(\tau-1)$. Since $\tau > N/ps$, we see that $p < \tau' p < p^*_s$. Hence, it follows from (G5), (2.12), and (2.13) that

$$
\begin{aligned}
&\int_{\Omega_n(d,\infty)} \frac{|G(x,u_n)|}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \\
&= \int_{\Omega_n(d,\infty)} \frac{|G(x,u_n)|}{|u_n(x)|^p}\,|w_n(x)|^p\,dx \\
&\le \left\{\int_{\Omega_n(d,\infty)} \left(\frac{|G(x,u_n)|}{|u_n(x)|^p}\right)^\tau dx\right\}^{\frac{1}{\tau}} \left\{\int_{\Omega_n(d,\infty)} |w_n(x)|^{\tau' p}\right\}^{\frac{1}{\tau'}} \\
&\le c_0^{\frac{1}{\tau}}\left\{\int_{\Omega_n(d,\infty)} \mathfrak{G}(x,u_n)\,dx\right\}^{\frac{1}{\tau}} \left\{\int_{\mathbb{R}^N} |w_n(x)|^{\tau' p}\right\}^{\frac{1}{\tau'}} \\
&\le c_0^{\frac{1}{\tau}}\left(\frac{c+1}{\lambda}\right)^{\frac{1}{\tau}} \left\{\int_{\mathbb{R}^N} |w_n(x)|^{\tau' p}\right\}^{\frac{1}{\tau'}} \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{2.16}
$$

Combining (2.15) with (2.16), we have

$$
\int_{\mathbb{R}^N} \frac{|G(x,u_n)|}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx = \int_{\Omega_n(0,d)} \frac{|G(x,u_n)|}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx + \int_{\Omega_n(d,\infty)} \frac{|G(x,u_n)|}{\|u_n\|^p_{X_s(\mathbb{R}^N)}}\,dx \to 0
$$

as $n\to\infty$, which contradicts (2.14). This completes the proof. □

Theorem 2.10. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V) *and* (G1)–(G5) *hold. Then the problem (P_λ) has a nontrivial weak solution for all $\lambda > 0$.*

Proof. Note that $\mathcal{I}_\lambda(0) = 0$. By Lemmas 2.8 and 2.9, we verify that all conditions of the mountain pass theorem are satisfied. Consequently, the problem (P_λ) has a nontrivial weak solution for all $\lambda > 0$. □

Lemma 2.11. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V), (G1)–(G4), *and* (G6) *hold. Then the functional $\mathcal{I}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda > 0$.*

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence in $X_s(\mathbb{R}^N)$ satisfying (2.4). Then the relation (2.5) is fulfilled. As in the proof of Lemma 2.9, we only show that $\{u_n\}$ is bounded in $X_s(\mathbb{R}^N)$. To this end, arguing by contradiction, suppose that $\|u_n\|_{X_s(\mathbb{R}^N)} \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|_{X_s(\mathbb{R}^N)}$. Then $\|v_n\|_{X_s(\mathbb{R}^N)} = 1$ and for $p \le p_0 < p_s^*$, $\|v_n\|_{L^{p_0}(\mathbb{R}^N)} \le K_{p_0}\|v_n\|_{X_s(\mathbb{R}^N)} = K_{p_0}$, where K_{p_0} is the Sobolev constant. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$, then by Lemma 2.3,

$$v_n \to v \text{ in } L^{p_0}(\mathbb{R}^N) \text{ and } v_n(x) \to v(x) \text{ almost all } x \in \mathbb{R}^N \text{ as } n \to \infty.$$

By the assumption (G6), one has

$$\begin{aligned} c + 1 &\ge \mathcal{I}_\lambda(u_n) - \frac{1}{\mu}\langle \mathcal{I}'_\lambda(u_n), u_n\rangle \\ &= \frac{1}{p}\|u_n\|^p_{X_s(\mathbb{R}^N)} - \lambda\int_{\mathbb{R}^N} G(x,u_n)\,dx - \frac{1}{\mu}\|u_n\|^p_{X_s(\mathbb{R}^N)} \\ &\quad + \frac{\lambda}{\mu}\int_{\mathbb{R}^N} g(x,u_n)u_n\,dx \\ &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|^p_{X_s(\mathbb{R}^N)} - \frac{\lambda\varrho}{\mu}\int_{\mathbb{R}^N} |u_n(x)|^p\,dx \end{aligned}$$

which implies

$$1 \le \frac{\lambda\varrho p}{\mu - p}\limsup_{n\to\infty}\|v_n\|^p_{L^p(\mathbb{R}^N)} = \frac{\lambda\varrho p}{\mu - p}\|v\|^p_{L^p(\mathbb{R}^N)}. \tag{2.17}$$

Hence, it follows from (2.17) that $v \neq 0$. If we follow the same argument as in Lemma 2.9, we can check the relations (2.8), (2.9), and (2.10) and hence yield the relation (2.11). Therefore we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in $X_s(\mathbb{R}^N)$. □

Remark 2.12. Although we replace (G5) with (G6) in the assumption of Theorem 2.10, we assert that the problem (P_λ) possesses a nontrivial weak solution for all $\lambda > 0$ via Lemma 2.11.

2.2. Another approach for the existence of a nontrivial weak solution

In this subsection, we also give the existence of a nontrivial solution for our problem without the assumptions (G5) and (G6) which play a decisive role in obtaining the fact that the energy functional $\mathcal{I}_\lambda$ satisfies the $(C)_c$-condition. To show this result, for the time being, we need the following additional assumptions for g:

(H) $m(x) > 0$ for all $x \in \mathbb{R}^N$ and $m \in L^{\frac{\gamma}{\gamma - q}}(\mathbb{R}^N)$ with some γ satisfying $q < \gamma < p_s^*$.

(G7) There exist nonnegative functions

$$a_1 \in L^{p'}(\mathbb{R}^N) \cap L^{r'}(\mathbb{R}^N),\ b_1 \in L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$$

with $1 < r < p_s^*$ such that

$$|g(x,t)| \le a_1(x) + b_1(x)\,|t|^{r-1}$$

holds for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(G8) $\limsup_{s\to 0} \frac{|g(x,s)|}{m(x)|s|^{\xi-1}} < +\infty$ uniformly for almost all $x \in \mathbb{R}^N$ with $q < \xi < p_s^*$.

(G9) $\limsup_{|s|\to\infty} \left(\operatorname{ess\,sup}_{x\in\mathbb{R}^N} \frac{|g(x,s)|}{m(x)|s|^{q-1}} \right) < +\infty$, where $p < q < p_s^*$.

We recall the functional $\mathcal{I}_\lambda : X_s(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) = \Phi_{s,p}(u) - \lambda\Psi(u). \tag{2.18}$$

Here, Ψ can be defined as in Section 2.1 under the assumptions (G1) and (G7).

We observe that the growth of g allowed by both (G2) and (G7) is subcritical. The assumptions (G1) and (G8) imply that $g(x,0) = 0$ for almost all $x \in \mathbb{R}^N$. Furthermore, $\limsup_{s\to 0} \frac{|G(x,s)|}{m(x)|s|^{\xi}} < +\infty$ uniformly almost everywhere in $\mathbb{R}^N$, by L'Hôpital's rule. From the analogous argument as in [16], define the crucial value

$$\mathcal{C}_g = \operatorname*{ess\,sup}_{s\neq 0, x\in\mathbb{R}^N} \frac{|g(x,s)|}{m(x)\,|s|^{q-1}}, \tag{2.19}$$

where $q < \xi$. So then $\mathcal{C}_g$ is a positive constant. The assumptions (G8) and (G9) yield $\mathcal{C}_g < \infty$, and furthermore the following relation

$$\operatorname*{ess\,sup}_{s\neq 0, x\in\mathbb{R}^N} \frac{|G(x,s)|}{m(x)\,|s|^{q}} = \frac{\mathcal{C}_g}{q}$$

holds.

Under the circumstance for the functional $\mathcal{I}_\lambda$ in (2.18), we need the following lemma. However, since the growth condition for g in (G7) is different from that in (G2), we cannot follow the lines of the proof as in Lemma 2.7 to get the following assertion directly. Hence we give the proof.

Lemma 2.13. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V), (G1), *and* (G7) *hold. Then Ψ and Ψ' are weakly strongly continuous in $X_s(\mathbb{R}^N)$.*

Proof. In the same way to that of Proposition 1.12 in [43], it is obvious that the functional Ψ is Gâteaux differentiable in $X_s(\mathbb{R}^N)$. So, we only need to prove that the functionals Ψ and Ψ' are weakly strongly continuous in $X_s(\mathbb{R}^N)$. Let us assume that $u_n \rightharpoonup u$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$. First we prove that Ψ is weakly strongly continuous on $X_s(\mathbb{R}^N)$. Observe that

$$|\Psi(u_n) - \Psi(u)| \le \int_{\mathbb{R}^N} |G(x,u_n) - G(x,u)|\,dx$$

$$\begin{aligned} &\leq \int_{\mathbb{R}^N \setminus B_K^N(0)} |G(x,u_n) - G(x,u)|\,dx \\ &\quad + \int_{B_K^N(0)} |G(x,u_n) - G(x,u)|\,dx, \end{aligned} \tag{2.20}$$

where $B_K^N(0) := \{x \in \mathbb{R}^N : |x| \leq K\}$ for $K \in \mathbb{N}$. For the first term in the right side of the inequality (2.20), the assumption (G7) and the Hölder inequality imply that

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus B_K^N(0)} |G(x,u_n) - G(x,u)|\,dx \\ &\leq \int_{\mathbb{R}^N \setminus B_K^N(0)} |G(x,u_n)| + |G(x,u)|\,dx \\ &\leq \int_{\mathbb{R}^N \setminus B_K^N(0)} a_1(x)(|u_n(x)| + |u(x)|) + \frac{b_1(x)}{r}(|u_n(x)|^r + |u(x)|^r)\,dx \\ &\leq C_5 \|a_1\|_{L^{p'}(\mathbb{R}^N \setminus B_K^N(0))} \|u_n + u\|_{L^p(\mathbb{R}^N \setminus B_K^N(0))} \\ &\quad + C_6 \|b_1\|_{L^{\frac{p_s^*}{p_s^* - r}}(\mathbb{R}^N \setminus B_K^N(0))} \|u_n + u\|^r_{L^{p_s^*}(\mathbb{R}^N \setminus B_K^N(0))} \end{aligned}$$

for some positive constants C_5 and C_6. Note that $a_1 \in L^{p'}(\mathbb{R}^N)$ and $b_1 \in L^{\frac{p_s^*}{p_s^* - r}}(\mathbb{R}^N)$. Then, for any $\varepsilon > 0$, there exists $N(K) \in \mathbb{N}$ such that $K > N(K)$ implies

$$\int_{\mathbb{R}^N \setminus B_K^N(0)} |a_1(x)|^{p'}\,dx < \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_K^N(0)} |b_1(x)|^{\frac{p_s^*}{p_s^* - r}}\,dx < \varepsilon.$$

Since $X_s(\mathbb{R}^N)$ is reflexive, $\{u_n\}$ is bounded in $X_s(\mathbb{R}^N)$ and $\{u_n + u\}$ is also bounded in $X_s(\mathbb{R}^N)$. By Lemma 2.2, $\{u_n\}$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$ and so $\{u_n + u\}$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$. Consequently, we assert that

$$\int_{\mathbb{R}^N \setminus B_K^N(0)} |G(x,u_n) - G(x,u)|\,dx < 2C_7\varepsilon \tag{2.21}$$

for some positive constant C_7.

Next, we consider the second term in the right side of the inequality (2.20). Since $u_n \rightharpoonup u$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$, it is easy to check that $u_n \rightharpoonup u$ in $X_s(B_K^N(0))$ and $u_n \to u$ in $L^r(B_K^N(0))$ by Lemma 2.1. By the convergence principle, there exist a subsequence, still denoted by $\{u_n\}$, in $X_s(B_K^N(0))$ and a function $v \in L^p(B_K^N(0)) \cap L^{\frac{p_s^*}{r}}(B_K^N(0))$ such that $u_n(x) \to u(x)$ for almost all $x \in B_K^N(0)$ as $n \to \infty$ and $|u_n(x)| \leq v(x)$ for all $n \in \mathbb{N}$ and for almost all $x \in B_K^N(0)$. Therefore from (G7), we get

$$\int_{B_K^N(0)} |G(x,u_n)|\,dx \leq \int_{B_K^N(0)} a_1(x)|u_n(x)| + b_1(x)|u_n(x)|^r\,dx$$

$$\leq \int_{B_K^N(0)} a_1(x)|v(x)| + b_1(x)|v(x)|^r \, dx$$

and thus $\{G(x,u_n)\}$ is uniformly integrable on $B_K^N(0)$. By the assumptions (G1) and (G7), we deduce that $G(x,u_n) \to G(x,u)$ as $n \to \infty$ and

$$|G(x,u)| \leq \|a_1\|_{L^\infty(\mathbb{R}^N)}|u(x)| + \|b_1\|_{L^\infty(\mathbb{R}^N)}|u(x)|^r < \infty$$

for almost all $x \in B_K^N(0)$, respectively. By the Vitali convergence theorem,

$$\int_{B_K^N(0)} G(x,u_n)\, dx \to \int_{B_K^N(0)} G(x,u)\, dx$$

as $n \to \infty$. Hence for above $\varepsilon > 0$, there exists $N(K) \in \mathbb{N}$ such that

$$\int_{B_K^N(0)} |G(x,u_n) - G(x,u)|\, dx < \varepsilon \tag{2.22}$$

for $K > N(K)$. From (2.21) and (2.22), we have

$$\int_{\mathbb{R}^N} G(x,u_n)\, dx \to \int_{\mathbb{R}^N} G(x,u)\, dx$$

as $n \to \infty$. This implies that Ψ is weakly strongly continuous on $X_s(\mathbb{R}^N)$.

Next we prove that Ψ' is weakly strongly continuous on $X_s(\mathbb{R}^N)$. Note that

$$\begin{aligned}
&\sup_{\|\varphi\|_{X_s(\mathbb{R}^N)}\leq 1} \left|\langle \Psi'(u_n) - \Psi'(u), \varphi\rangle\right| \\
&= \sup_{\|\varphi\|_{X_s(\mathbb{R}^N)}\leq 1} \left|\int_{\mathbb{R}^N} (g(x,u_n) - g(x,u))\varphi\, dx\right| \\
&\leq \sup_{\|\varphi\|_{X_s(\mathbb{R}^N)}\leq 1} \left|\int_{B_K^N(0)} (g(x,u_n) - g(x,u))\varphi\, dx\right| \qquad (2.23) \\
&\quad + \sup_{\|\varphi\|_{X_s(\mathbb{R}^N)}\leq 1} \left|\int_{\mathbb{R}^N\setminus B_K^N(0)} (g(x,u_n) - g(x,u))\varphi\, dx\right|
\end{aligned}$$

for some constant K and for any $\varphi \in X_s(\mathbb{R}^N)$. Since $1 < p < p_s^*$, the compact embedding

$$W^{s,p}(B_K^N(0)) \hookrightarrow L^p(B_K^N(0)) \text{ implies } u_n \to u \text{ in } L^p(B_K^N(0)) \text{ as } n \to \infty.$$

This together with the continuity of the Nemytskii operator with g and acting from $L^p(B_K^N(0))$ into $L^{r'}(B_K^N(0))$ yields that it is easy to see that the first term in the right side of the inequality (2.23) tends to 0 as $n \to \infty$. For the second term in (2.23), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N\setminus B_K^N(0)} (g(x,u_n) - g(x,u))\varphi dx\right| \\
&\leq \int_{\mathbb{R}^N\setminus B_K^N(0)} (a_1(x) + b_1(x)\,|u_n(x)|^{r-1})\,|\varphi|\ dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N\setminus B_K^N(0)}\left(a_1(x)+b_1(x)\,|u(x)|^{r-1}\right)|\varphi|\,dx\\
&\le 2\|a_1\|_{L^{p'}(\mathbb{R}^N\setminus B_K^N(0))}\|\varphi\|_{L^p(\mathbb{R}^N)}\\
&\quad+\|b_1\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N\setminus B_K^N(0))}\left(\|u_n\|^{r-1}_{L^{p_s^*}(\mathbb{R}^N)}+\|u\|^{r-1}_{L^{p_s^*}(\mathbb{R}^N)}\right)\|\varphi\|_{L^{p_s^*}(\mathbb{R}^N)}.
\end{aligned}
$$

According to the assumption (G7), for above $\varepsilon>0$, there exists $N(K)\in\mathbb{N}$ such that

$$
\|a_1\|_{L^{p'}(\mathbb{R}^N\setminus B_K^N(0))}<\varepsilon \quad\text{and}\quad \|b_1\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N\setminus B_K^N(0))}<\varepsilon
$$

for $K>N(K)$. As the sequence $\{u_n\}$ is bounded in $X_s(\mathbb{R}^N)$, invoking Lemma 2.2, $\{u_n\}$ is bounded in $L^p(\mathbb{R}^N)$ and also in $L^{p_s^*}(\mathbb{R}^N)$. Thus, it is immediate that

$$
\left|\int_{\mathbb{R}^N\setminus B_K^N(0)}(g(x,u_n)-g(x,u))\varphi\,dx\right|\le C_8\varepsilon \tag{2.24}
$$

for a positive constant C_8. Due to (2.24), we can deduce that

$$
\int_{\mathbb{R}^N}(g(x,u_n)-g(x,u))\varphi\,dx\to 0 \ \text{ as } \ n\to\infty.
$$

This implies that Ψ' is weakly strongly continuous in $X_s(\mathbb{R}^N)$. Therefore, the proof is completed. □

As mentioned earlier, we give an abstract critical point theory based on [9] to get our main result of this subsection. To do this, we briefly introduce the following definitions and some properties for locally Lipschitz continuous functionals.

Let X be a real Banach space. A functional $I:X\to\mathbb{R}$ is called locally Lipschitz continuous when, for every $u\in X$, there exist a neighborhood U of u and a constant $\mathcal{L}\ge 0$ such that

$$
|I(v)-I(w)|\le\mathcal{L}\|v-w\|_X \quad\text{for all}\ \ v,w\in U.
$$

Let $u,v\in X$. The symbol $I^\circ(u;v)$ indicates the generalized directional derivative of I at point u along direction v, namely

$$
I^\circ(u;v):=\limsup_{w\to u,t\to 0^+}\frac{I(w+tv)-I(w)}{t}.
$$

The generalized gradient of the function I at u, denoted by $\partial I(u)$, is the set

$$
\partial I(u):=\left\{u^*\in X:\langle u^*,v\rangle\le I^\circ(u;v)\quad\text{for all } v\in X\right\}.
$$

Definition 2.14. A functional $I:X\to\mathbb{R}$ is called Gâteaux differentiable at $u\in X$ if there is $\psi\in X^*$(denoted by $I'(u)$) such that

$$
\lim_{t\to 0^+}\frac{I(u+tv)-I(u)}{t}=I'(u)(v)
$$

for all $v \in X$. It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \to I'(u)$ is a continuous map from X to its dual X^*.

We recall that if I is continuously Gâteaux differentiable, then it is locally Lipschitz continuous and one has $I^\circ(u; v) = I'(u)(v)$ for all $u, v \in X$.

For a real Banach space $(X, \|\cdot\|_X)$, we say that a functional $I(u) = \Phi(u) - \Psi(u)$ satisfies the $(C)^{[\mu]}$-condition cut off upper at μ if any Cerami sequence $\{x_n\} \subset X$ for I with $0 < \Phi(x_n) < \mu$ has a convergent subsequence of $\{x_n\}$.

We recall the key lemma in [49] (see also [25]) to prove the abstract critical points result under the $(C)^{[\mu]}$-condition (Theorem 2.16).

Lemma 2.15. *Let X be a real Banach space and let $I : X \to \mathbb{R}$ be a locally Lipschitz continuous function with bounded from below. Then, for all minimizing sequence of I, $\{u_n\}_{n\in\mathbb{N}} \subseteq X$, there exists a minimizing sequence of I, $\{v_n\}_{n\in\mathbb{N}} \subseteq X$, such that for any $n \in \mathbb{N}$,*

$$I(v_n) \le I(u_n) \quad \text{and} \quad I^\circ(v_n; h) \ge \frac{-\varepsilon_n \|h\|_X}{1 + \|v_n\|_X}$$

for all $h \in X$, and $n \in \mathbb{N}$, where $\varepsilon_n \to 0^+$.

Let us introduce two functions

$$\varphi_1(\mu) = \inf_{v\in\Phi^{-1}((0,\mu))} \frac{\sup_{u\in\Phi^{-1}((0,\mu))} \Psi(u) - \Psi(v)}{\mu - \Phi(v)}$$

and

$$\varphi_2(\mu) = \sup_{v\in\Phi^{-1}((0,\mu))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,0])} \Psi(u)}{\Phi(v)}$$

for all $\mu \in \mathbb{R}$.

Using Lemma 2.15, we obtain the following result; see [12] for the case of the Palais-Smale condition. The proof of this theorem proceeds in the analogous way to those of Theorems 2.3 and 2.4 in [12]. For the sake of convenience, we give the proof.

Theorem 2.16. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functionals. Suppose that*

$$\text{there exists } \mu \in \mathbb{R} \ \text{ such that } \ \varphi_1(\mu) < \varphi_2(\mu).$$

Moreover, assume that for each $\lambda \in \Lambda := \left(\frac{1}{\varphi_2(\mu)}, \frac{1}{\varphi_1(\mu)}\right)$ the functional $\mathcal{I}_\lambda := \Phi - \lambda\Psi$ satisfies $(C)^{[\mu]}$-condition. Then, for each $\lambda \in \Lambda$, the functional $\mathcal{I}_\lambda$ has a nontrivial point $u_{0,\lambda}$ in $\Phi^{-1}((0,\mu))$ such that $\mathcal{I}_\lambda(u_{0,\lambda}) \le \mathcal{I}_\lambda(u)$ for all u in $\Phi^{-1}((0,\mu))$ with $u_{0,\lambda}$ being a critical point of $\mathcal{I}_\lambda$.

Proof. For each $\lambda \in \left(\frac{1}{\varphi_2(\mu)}, \frac{1}{\varphi_1(\mu)}\right)$, we have $\varphi_1(\mu) < 1/\lambda < \varphi_2(\mu)$ which implies the existence of $v_1, v_2 \in \Phi^{-1}((0,\mu))$ such that

$$\begin{aligned} \frac{1}{\lambda} &> \frac{\sup_{u\in\Phi^{-1}((0,\mu))}\Psi(u) - \Psi(v_1)}{\mu - \Phi(v_1)} \quad \text{and} \\ \frac{1}{\lambda} &< \frac{\Psi(v_2) - \sup_{u\in\Phi^{-1}((-\infty,0])}\Psi(u)}{\Phi(v_2)}. \end{aligned} \tag{2.25}$$

Now, let $x_0 \in \Phi^{-1}((0,\mu))$ be such that

$$\Phi(x_0) - \lambda\Psi(x_0) = \min\{\Phi(v_1) - \lambda\Psi(v_1), \Phi(v_2) - \lambda\Psi(v_2)\}.$$

From (2.25), we have

$$\sup_{u\in\Phi^{-1}((0,\mu))} \lambda\Psi(u) \le \mu - \Phi(x_0) + \lambda\Psi(x_0), \tag{2.26}$$

and

$$\sup_{u\in\Phi^{-1}((-\infty,0])} \lambda\Psi(u) \le -\Phi(x_0) + \lambda\Psi(x_0). \tag{2.27}$$

Set

$$\mathcal{K} = \mu - \Phi(x_0) + \lambda\Psi(x_0). \tag{2.28}$$

Define

$$\Phi_0(u) = \max\{\Phi(u), 0\}, \qquad \lambda\Psi_{\mathcal{K}}(u) = \min\{\lambda\Psi(u), \mathcal{K}\}, \tag{2.29}$$

and

$$J_\lambda = \Phi_0 - \lambda\Psi_{\mathcal{K}}.$$

Clearly, J_λ is locally Lipschitz continuous and bounded from below. Given a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} J_\lambda(u_n) = \inf_X J_\lambda$, it follows from Lemma 2.15 that we choose a sequence $\{v_n\}$ in X such that

$$\lim_{n\to\infty} J_\lambda(v_n) = \inf_X J_\lambda \quad \text{and} \quad J_\lambda^\circ(v_n; h) \ge -\frac{\varepsilon_n \|h\|_X}{1 + \|v_n\|_X} \tag{2.30}$$

for all $h \in X$ and for all $n \in \mathbb{N}$, where $\varepsilon_n \to 0^+$.

Assume that $J_\lambda(x_0) = \inf_X J_\lambda$. Due to the relation (2.26) and the definition of $\mathcal{K}$, we get

$$\lambda\Psi(u) \le \mathcal{K} \quad \text{for all} \quad u \in \Phi^{-1}((0,\mu)).$$

This together with the relation (2.29) yields that $J_\lambda(u) = \mathcal{I}_\lambda(u)$ and

$$\mathcal{I}_\lambda(x_0) = J_\lambda(x_0) \le J_\lambda(u) = \mathcal{I}_\lambda(u) \quad \text{for all } u \in \Phi^{-1}((0,\mu)).$$

On the other hands, if $\inf_X J_\lambda < J_\lambda(x_0)$, then there exists a positive integer n_0 such that $J_\lambda(v_n) < J_\lambda(x_0)$ for all $n > n_0$. From (2.29) and the choice of x_0, we get

$$\Phi(v_n) - \lambda\Psi_{\mathcal{K}}(v_n) < \Phi_0(v_n) - \lambda\Psi_{\mathcal{K}}(v_n) = J_\lambda(v_n) < J_\lambda(x_0) = \Phi(x_0) - \lambda\Psi(x_0)$$

for all $n > n_0$. Thus, we have

$$\Phi(v_n) < \lambda\Psi_{\mathcal{K}}(v_n) + \Phi(x_0) - \lambda\Psi(x_0) \leq \mathcal{K} + \Phi(x_0) - \lambda\Psi(x_0) = \mu.$$

This implies $\Phi(v_n) < \mu$ for all $n > n_0$. Next, we prove that $\Phi(v_n) > 0$ for all $n > n_0$. Suppose to the contrary that $\Phi(v_n) \leq 0$ for all $n > n_0$. This yields

$$-\lambda\Psi(v_n) = \Phi_0(v_n) - \lambda\Psi(v_n) \leq \Phi(x_0) - \lambda\Psi(x_0)$$

or equivalently

$$-\Phi(x_0) + \lambda\Psi(x_0) < \lambda\Psi(v_n).$$

Due to (2.27), we have $\Phi(v_n) > 0$, which is a contradiction. Consequently, we obtain

$$0 < \Phi(v_n) < \mu \quad \text{for all } n > n_0. \tag{2.31}$$

Then, from (2.26) and (2.31) we obtain

$$J_\lambda(v_n) = \mathcal{I}_\lambda(v_n) \quad \text{and} \quad J_\lambda^\circ(v_n; h) = \mathcal{I}_\lambda^\circ(v_n; h)$$

for all $n > n_0$ and for all $h \in X$. Therefore it follows from (2.30) that

$$\lim_{n\to\infty} \mathcal{I}_\lambda(v_n) = \lim_{n\to\infty} J_\lambda(v_n) = \inf_X J_\lambda$$

and

$$\mathcal{I}_\lambda^\circ(v_n; h) \geq -\frac{\varepsilon_n \|h\|_X}{1 + \|v_n\|_X}$$

for all $h \in X$. Since $\mathcal{I}_\lambda$ satisfies $(C)^{[\mu]}$-condition, the sequence $\{v_n\}$ admits a subsequence strongly converging to v^* in X as $n \to \infty$. Thus,

$$\mathcal{I}_\lambda(v^*) = \inf_X J_\lambda \leq J_\lambda(u) = \mathcal{I}_\lambda(u)$$

for all $u \in \Phi^{-1}((0, \mu))$. To put it shortly, we obtain

$$\mathcal{I}_\lambda(v^*) \leq \mathcal{I}_\lambda(u) \quad \text{for all } u \in \Phi^{-1}((0, \mu)). \tag{2.32}$$

From (2.31) and the continuity of Φ, we have $v^* \in \Phi^{-1}([0, \mu])$. To complete our proof, we consider the following three cases:

Case 1: If $v^* \in \Phi^{-1}((0, \mu))$, by (2.32) the conclusion holds.

Case 2: If $\Phi(v^*) = 0$, then (2.27) implies

$$\mathcal{I}_\lambda(v^*) = -\lambda\Psi(v^*) \geq \Phi(x_0) - \lambda\Psi(x_0) = \mathcal{I}_\lambda(x_0).$$

This combined with (2.32) gives $\mathcal{I}_\lambda(x_0) \leq \mathcal{I}_\lambda(u)$ for all $u \in \Phi^{-1}((0, \mu))$, as claimed.

Case 3: If $\Phi(v^*) = \mu$, we have $\lambda\Psi(v^*) = \lambda\Psi_{\mathcal{K}}(v^*) \leq \mathcal{K}$ since $\mathcal{I}_\lambda(v^*) = J_\lambda(v^*)$. Next, we prove that $\mathcal{I}_\lambda(v^*) = \mathcal{I}_\lambda(x_0)$. In fact, if we suppose that $\mathcal{I}_\lambda(v^*) < \mathcal{I}_\lambda(x_0)$, then by (2.28), we have

$$\mathcal{I}_\lambda(v^*) = \mu - \lambda\Psi(v^*) \geq \mu - \mathcal{K} = \Phi(x_0) - \lambda\Psi(x_0) = \mathcal{I}_\lambda(x_0),$$

that is, $\mathcal{I}_\lambda(v^*) \geq \mathcal{I}_\lambda(x_0)$ which contradicts with the assumption. Hence, from (2.32) we have $\mathcal{I}_\lambda(x_0) \leq \mathcal{I}_\lambda(u)$ for all $u \in \Phi^{-1}((0, \mu))$.

This completes the proof taking into account that each local minimum is also a critical point of $\mathcal{I}_\lambda$. □

The following corollary is an immediate consequence of Theorem 2.16. This is applied to obtain our main result of this subsection.

Corollary 2.17. *Let $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist a positive constant μ and an element $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < \mu$, such that

$$\frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.33}$$

holds and for each $\lambda \in \Lambda_\mu := \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{\mu}{\sup_{\Phi(u)\le\mu}\Psi(u)}\right)$, the functional $\mathcal{I}_\lambda := \Phi - \lambda\Psi$ satisfies the $(C)^{[\mu]}$-condition. Then, for each $\lambda \in \Lambda_\mu$, the functional $\mathcal{I}_\lambda$ has a nontrivial point x_λ in $\Phi^{-1}((0,\mu))$ such that $\mathcal{I}_\lambda(x_\lambda) \le \mathcal{I}_\lambda(x)$ for all x in $\Phi^{-1}((0,\mu))$ and $I'_\lambda(x_\lambda) = 0$.

Proof. From the same argument as Theorem 2.5 in [12], it is easy to check that

$$\left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{\mu}{\sup_{\Phi(u)\le\mu}\Psi(u)}\right) \subseteq \left(\frac{1}{\varphi_2(\mu)}, \frac{1}{\varphi_1(\mu)}\right).$$

Theorem 2.16 implies that the conclusion holds. □

To localize the precise interval of λ for which the problem (P_λ) has at least one weak solution, we consider the following eigenvalue problem

$$(-\Delta)_p^s u + V(x)\,|u|^{p-2}\,u = \lambda m(x)\,|u|^{q-2}\,u \quad \text{in } \mathbb{R}^N. \tag{E}$$

Definition 2.18. Let $0 < s < 1 < p < +\infty$. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the eigenvalue problem (E) if

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)\,|u|^{p-2}\,uv\,dx = \lambda\int_{\mathbb{R}^N} m(x)\,|u|^{q-2}\,uv\,dx$$

holds for any $v \in X_s(\mathbb{R}^N)$ and $p < q < p_s^*$. Then u is called an eigenfunction associated with the eigenvalue λ.

Now we obtain the existence of the positive principal eigenvalue for the problem (E). The basic idea of the proof of the following consequence follows the lines of that of Lemma 3.1 in [18].

Proposition 2.19. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V) *and* (H) *hold. Then the eigenvalue problem* (E) *has a pair (λ_1, u_1) of a principal eigenvalue λ_1 and an eigenfunction u_1 with $\lambda_1 > 0$ and $0 \leq u_1 \in X_s(\mathbb{R}^N)(u_1 \not\equiv 0)$.*

Proof. Let us denote the quantity

$$\lambda_1 = \inf\Big\{ |v|^p_{W^{s,p}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V\,|v|^p\,dx : \int_{\mathbb{R}^N} m(x)|v|^q dx = 1 \Big\}.$$

We shall prove that λ_1 is the least eigenvalue of (E). Obviously $\lambda_1 \geq 0$. Let $\{v_n\}_{n=1}^{\infty}$ be the minimizing sequence for λ_1, *i.e.*,

$$\int_{\mathbb{R}^N} m(x)|v_n|^q\,dx = 1 \text{ and } |v_n|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}} v_n\|^p_{L^p(\mathbb{R}^N)} = \lambda_1 + \delta_n \tag{2.34}$$

with $\delta_n \to 0^+$ for $n \to \infty$. It follows from (2.34) that $\|v_n\|_{X_s(\mathbb{R}^N)} \leq C_9$ for some constant $C_9 > 0$. The reflexivity of $X_s(\mathbb{R}^N)$ yields the weak convergence $v_n \rightharpoonup u_1$ in $X_s(\mathbb{R}^N)$ as $n \to \infty$ for some u_1 (at least for some subsequence of $\{v_n\}$). The compact embedding $X_s(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N)$ implies the strong convergence $v_n \to u_1$ in $L^\gamma(\mathbb{R}^N)$ as $n \to \infty$. It follows from (H), (2.34), the Minkowski and Hölder inequalities that

$$\begin{aligned}
1 =& \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} m(x)\,|v_n(x)|^q\,dx \right)^{\frac{1}{q}} \\
\leq& \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} m(x)\,|v_n(x) - u_1(x)|^q\,dx \right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^N} m(x)\,|u_1(x)|^q\,dx \right)^{\frac{1}{q}} \\
\leq& \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} m(x)^{\frac{\gamma}{\gamma-q}} dx \right)^{\frac{\gamma-q}{q\gamma}} \left(\int_{\mathbb{R}^N} |v_n(x) - u_1(x)|^\gamma\,dx \right)^{\frac{1}{\gamma}} \\
&+ \left(\int_{\mathbb{R}^N} m(x)\,|u_1(x)|^q\,dx \right)^{\frac{1}{q}} \\
=& \left(\int_{\mathbb{R}^N} m(x)\,|u_1(x)|^q\,dx \right)^{\frac{1}{q}}.
\end{aligned} \tag{2.35}$$

On the other hand, we note that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} m(x)\,|u_1(x)|^q\,dx \right)^{\frac{1}{q}} \\
&\leq \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} m(x)^{\frac{\gamma}{\gamma-q}} dx \right)^{\frac{\gamma-q}{q\gamma}} \left(\int_{\mathbb{R}^N} |u_1(x) - v_n(x)|^\gamma\,dx \right)^{\frac{1}{\gamma}} \\
&\quad + \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} m(x)\,|v_n(x)|^q\,dx \right)^{\frac{1}{q}} = 1.
\end{aligned} \tag{2.36}$$

From the inequalities (2.35) and (2.36), we have

$$\int_{\mathbb{R}^N} m(x)\,|u_1(x)|^q\,dx = 1.$$

In particular, $u_1 \neq 0$. The weak lower semicontinuity of the norm in $X_s(\mathbb{R}^N)$ yields

$$\begin{aligned}\lambda_1 &\le |u_1|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}}u_1\|^p_{L^p(\mathbb{R}^N)} = \|u_1\|_{X_s(\mathbb{R}^N)} \le \liminf_{n\to\infty}\|v_n\|_{X_s(\mathbb{R}^N)}\\ &= \liminf_{n\to\infty}\Big\{|v_n|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}}v_n\|^p_{L^p(\mathbb{R}^N)}\Big\} = \liminf_{n\to\infty}(\lambda_1+\delta_n) = \lambda_1,\end{aligned}$$

i.e.,

$$\lambda_1 = |u_1|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}}u_1\|^p_{L^p(\mathbb{R}^N)}. \tag{2.37}$$

It follows from (2.37) that $\lambda_1 > 0$ and it is easy to check that λ_1 is the least eigenvalue of (E) with the corresponding eigenfunction u_1. Moreover, if u is an eigenfunction associated with λ_1, then $|u|$ is also an eigenfunction associated with λ_1. Hence we can suppose that $u_1 \ge 0$ almost everywhere in $\mathbb{R}^N$. □

Theorem 2.20. *Let $0 < s < 1 < p < +\infty$ with $ps < N$. Assume that* (V), (H), (G1), *and* (G7)-(G9) *hold. If furthermore g satisfies the following assumption*:

(G10) *There exist a real number s_0, a positive constant r_0, and an element x_0 in $\mathbb{R}^N$ with*

$$2\,|s_0|^p\,\omega_N^2 r_0^{N-sp}\mathcal{M} < p$$

such that

$$\int_{B^N_{r_0}(x_0)} G(x,|s_0|)\,dx > 0 \quad \text{and} \quad G(x,t) \ge 0$$

for almost all $x \in B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)$ and for all $0 \le t \le |s_0|$. Here $B^N_{r_0}(x_0) = \{x\in\mathbb{R}^N : |x-x_0| \le r_0\}$ and

$$\frac{\mathcal{C}_g}{q\lambda_1} < \frac{r_0^{sp}\,\operatorname{ess\,inf}_{B^N_{r_0/2}(x_0)} G(x,|s_0|)}{2^{N+1}\,|s_0|^p\,\omega_N\mathcal{M}},$$

where ω_N is the volume of $B^N_1(0)$, $\mathcal{C}_g$ is given in (2.19) *and*

$$\begin{aligned}\mathcal{M} := {}& \frac{2^{2p+N-sp-1}}{(p-sp)(N-sp+p)} + \frac{1}{2^{N-sp-1}sp(N+p-sp)}\\ &+ \frac{1}{sp(N-sp)} + \frac{r_0^{sp}\sup_{x\in B^N_{r_0}(x_0)} V(x)}{2N\omega_N}.\end{aligned}$$

Then, for every

$$\lambda \in \tilde{\Lambda} := \left(\frac{2^{N+1}\,|s_0|^p\,\omega_N\mathcal{M}}{p\;r_0^{sp}\operatorname{ess\,inf}_{x\in B^N_{r_0/2}(x_0)} G(x,|s_0|)}, \frac{q\lambda_1}{p\mathcal{C}_g}\right),$$

the problem (P_λ) has at least one nontrivial weak solution.

Proof. The functionals $\Phi_{s,p}, \Psi : X_s(\mathbb{R}^N) \to \mathbb{R}$ are defined as

$$\Phi_{s,p}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|u(x)|^p\,dx,$$

and

$$\Psi(u) = \int_{\mathbb{R}^N} G(x,u)\,dx.$$

First of all, we show that $\mathcal{I}_\lambda$ satisfies the $(C)^{[\mu]}$-condition. Let μ be a fixed positive number and let $\{u_n\}$ be a Cerami sequence in $X_s(\mathbb{R}^N)$ with $\Phi_{s,p}(u_n) < \mu$. It follows from Lemma 2.3, we have

$$\begin{aligned}\mu > \Phi_{s,p}(u_n) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|u_n(x)|^p\,dx\\ &= \frac{1}{p}\|u_n\|^p_{X_s(\mathbb{R}^N)}.\end{aligned}$$

Thus, the sequence $\{u_n\}$ is bounded and we may suppose that $u_n \rightharpoonup u_0$ as $n \to \infty$ for some $u_0 \in X_s(\mathbb{R}^N)$. By Lemma 2.7, we know that $\Psi'(u_n) \to \Psi'(u_0)$ as $n \to \infty$, which implies that

$$\limsup_{n\to\infty}\langle \lambda\Psi'(u_n), u_n - u_0\rangle = 0. \tag{2.38}$$

From the definition of $(C)^{[\mu]}$-condition and the boundedness of $\{u_n\}$, it follows that $\langle \mathcal{I}'_\lambda(u_n), v\rangle \to 0$ as $n \to \infty$ for any v in $X_s(\mathbb{R}^N)$. Combining this with (2.38), we have

$$\limsup_{n\to+\infty}\langle \Phi'_{s,p}(u_n), u_n - u_0\rangle \le \limsup_{n\to+\infty}\langle \lambda\Psi'(u_n), u_n - u_0\rangle = 0.$$

Since $\Phi_{s,p}$ is of type (S_+), we conclude that $u_n \to u_0$ as $n \to \infty$ in $X_s(\mathbb{R}^N)$.

Next, to apply Lemma 2.17 with $\Phi = \Phi_{s,p}$, we will show that there exist a positive constant μ and an element $\tilde{u} \in X$ satisfying $\Phi_{s,p}(\tilde{u}) < \mu$ and the relation (2.33). Define

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B^N_{r_0}(x_0),\\ |s_0| & \text{if } x \in B^N_{r_0/2}(x_0),\\ \frac{2|s_0|}{r_0}(r_0 - |x-x_0|) & \text{if } x \in B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0).\end{cases} \tag{2.39}$$

Then it is clear that $0 \le \tilde{u}(x) \le |s_0|$ for all $x \in \mathbb{R}^N$, and so $\tilde{u} \in X_s(\mathbb{R}^N)$. It follows from (G10) that

$$\begin{aligned}\Phi_{s,p}(\tilde{u}) &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)\,|\tilde{u}(x)|^p\,dx\\ &= \frac{1}{p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{\mathbb{R}^N\setminus B^N_{r_0}(x_0)}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy\\
&+\frac{2}{p}\int_{B^N_{r_0/2}(x_0)}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy\\
&+\frac{2}{p}\int_{\mathbb{R}^N\setminus B^N_{r_0}(x_0)}\int_{B^N_{r_0/2}(x_0)}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy\\
&+\frac{1}{p}\int_{\mathbb{R}^N}V(x)\,|\tilde{u}(x)|^p\,dx\\
&=:\frac{1}{p}(I_1+2I_2+2I_3+2I_4+I_5).
\end{aligned}
$$

Next we estimate I_1–I_5, by the direct calculation, respectively:

• Estimate of I_1: For any positive constant ε small enough,

$$
\begin{aligned}
I_1&=\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy\\
&\le\frac{2^p\,|s_0|^p}{r_0^p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\frac{|x-y|^p}{|x-y|^{N+sp}}\,dxdy\\
&\le\frac{2^p\,|s_0|^p\,\omega_N}{r_0^p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{\varepsilon}^{r_0+|y-x_0|}r^{p-sp-1}\,drdy\\
&\le\frac{2^p\,|s_0|^p\,\omega_N}{r_0^p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\frac{(r_0+|y-x_0|)^{p-sp}}{p-sp}\,dy\\
&=\frac{2^p\,|s_0|^p\,\omega_N^2}{(p-sp)r_0^p}\int_{\frac{3}{2}r_0}^{2r_0}t^{p+N-sp-1}\,dt\\
&=\frac{2^p\,|s_0|^p\,\omega_N^2 r_0^{N-sp}}{(p-sp)(p+N-sp)}\left(2^{p+N-sp}-\left(\frac{3}{2}\right)^{p+N-sp}\right).
\end{aligned}
$$

• Estimate of I_2:

$$
\begin{aligned}
I_2&=\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{\mathbb{R}^N\setminus B^N_{r_0}(x_0)}\frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\,dxdy\\
&\le\frac{2^p\,|s_0|^p}{r_0^p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{\mathbb{R}^N\setminus B^N_{r_0}(x_0)}\frac{|r_0-|y-x_0||^p}{|x-y|^{N+sp}}\,dxdy\\
&=\frac{2^p\,|s_0|^p\,\omega_N}{r_0^p}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}\int_{r_0-|y-x_0|}^{\infty}\frac{|r_0-|y-x_0||^p}{r^{sp+1}}\,drdy\\
&=\frac{2^p\,|s_0|^p\,\omega_N}{r_0^p sp}\int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)}|r_0-|y-x_0|\,|^{p-sp}\,dy
\end{aligned}
$$

$$= \frac{2^p |s_0|^p \omega_N^2}{r_0^p sp} \int_0^{\frac{r_0}{2}} t^{N+p-sp-1}\, dt$$
$$= \frac{|s_0|^p r_0^{N-sp} \omega_N^2}{2^{N-sp} sp(N+p-sp)}.$$

- Estimate of I_3:

$$I_3 = \int_{B^N_{r_0/2}(x_0)} \int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\, dxdy$$
$$= \frac{2^p |s_0|^p}{r_0^p} \int_{B^N_{r_0/2}(x_0)} \int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)} \frac{\left|-\frac{r_0}{2}+|x-x_0|\right|^p}{|x-y|^{N+sp}}\, dxdy$$
$$= \frac{2^p |s_0|^p}{r_0^p} \int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)} \int_{B^N_{r_0/2}(x_0)} \frac{\left|-\frac{r_0}{2}+|x-x_0|\right|^p}{|x-y|^{N+sp}}\, dydx$$
$$= \frac{2^p |s_0|^p \omega_N}{r_0^p} \int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)} \left|-\frac{r_0}{2}+|x-x_0|\right|^p \int_{|x-x_0|-\frac{r_0}{2}}^{|x-x_0|+\frac{r_0}{2}} \frac{1}{r^{sp+1}}\, drdx$$
$$\le \frac{2^p |s_0|^p \omega_N}{r_0^p sp} \int_{B^N_{r_0}(x_0)\setminus B^N_{r_0/2}(x_0)} \left|-\frac{r_0}{2}+|x-x_0|\right|^{p-sp} dx$$
$$= \frac{2^p |s_0|^p \omega_N^2}{r_0^p sp} \int_0^{\frac{r_0}{2}} t^{N+p-sp-1}\, dt$$
$$= \frac{r_0^{N-sp} |s_0|^p \omega_N^2}{2^{N-sp} sp(N+p-sp)}.$$

- Estimate of I_4:

$$I_4 = \int_{B^N_{r_0/2}(x_0)} \int_{\mathbb{R}^N\setminus B^N_{r_0}(x_0)} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}}\, dxdy$$
$$= |s_0|^p \int_{B^N_{r_0/2}(x_0)} \int_{\mathbb{R}^N\setminus B^N_{r_0}(x_0)} \frac{1}{|x-y|^{N+sp}}\, dxdy$$
$$= |s_0|^p \omega_N \int_{B^N_{r_0/2}(x_0)} \int_{r_0-|y-x_0|}^{\infty} r^{-sp-1}\, drdy$$
$$= |s_0|^p \omega_N \int_{B^N_{r_0/2}(x_0)} \frac{1}{sp(r_0-|y-x_0|)^{sp}}\, dy$$
$$= \frac{|s_0|^p \omega_N^2}{sp} \int_{\frac{r_0}{2}}^{r_0} t^{N-sp-1}\, dt$$
$$= \frac{|s_0|^p \omega_N^2 r_0^{N-sp}}{(N-sp)sp} \left(1-\frac{1}{2^{N-sp}}\right).$$

• Estimate of I_5:

$$\begin{aligned} I_5 &= \int_{\mathbb{R}^N} V(x)\,|\tilde{u}|^p\,dx \le \sup_{x\in B^N_{r_0}(x_0)} V(x) \int_{B^N_{r_0}(x_0)} |\tilde{u}|^p\,dx \\ &= \sup_{x\in B^N_{r_0}(x_0)} V(x) \int_{B^N_{\frac{r_0}{2}}(x_0)} |s_0|^p\,dx \\ &\quad + \sup_{x\in B^N_{r_0}(x_0)} V(x) \int_{B^N_{r_0}(x_0)\setminus B^N_{\frac{r_0}{2}}(x_0)} \frac{2^p\,|s_0|^p}{r_0^p}(r_0-|x-x_0|)^p\,dx \\ &\le \frac{\sup_{x\in B^N_{r_0}(x_0)} V(x)\,|s_0|^p\,\omega_N r_0^N}{N}. \end{aligned}$$

Hence

$$\Phi_{s,p}(\tilde{u}) \le \frac{2\,|s_0|^p\,\omega_N^2 r_0^{N-sp}\mathcal{M}}{p} < 1,$$

where $\mathcal{M} = \frac{2^{2p+N-sp-1}}{(p-sp)(N-sp+p)} + \frac{1}{2^{N-sp-1}sp(N+p-sp)} + \frac{1}{sp(N-sp)} + \frac{r_0^{sp}\sup_{x\in B^N_{r_0}(x_0)} V(x)}{2N\omega_N}$. Owing to the assumption (G10), we deduce that

$$\begin{aligned} \Psi(\tilde{u}) &\ge \int_{B^N_{r_0/2}(x_0)} G(x,\tilde{u})\,dx \\ &\ge \operatorname*{ess\,inf}_{x\in B^N_{r_0/2}(x_0)} G(x,|s_0|)\left(\frac{\omega_N r_0^N}{2^N}\right) \end{aligned}$$

and thus

$$\frac{\Psi(\tilde{u})}{\Phi_{s,p}(\tilde{u})} \ge \frac{p\,r_0^{sp}\operatorname{ess\,inf}_{x\in B^N_{r_0/2}(x_0)} G(x,|s_0|)}{2^{N+1}\,|s_0|^p\,\omega_N\mathcal{M}}. \tag{2.40}$$

Also Lemma 2.3, Proposition 2.19 and the definition of $\mathcal{C}_g$ imply that

$$\begin{aligned} \Psi(u) &= \int_{\mathbb{R}^N} G(x,u)\,dx \\ &\le \int_{\mathbb{R}^N} \frac{|G(x,u)|}{m(x)\,|u(x)|^q} m(x)\,|u(x)|^q\,dx \\ &\le \frac{\mathcal{C}_g}{q}\int_{\mathbb{R}^N} m(x)\,|u(x)|^q\,dx \\ &\le \frac{\mathcal{C}_g}{q\lambda_1}\left(|u|^p_{W^{s,p}(\mathbb{R}^N)} + \|V^{\frac{1}{p}}u\|^p_{L^p(\mathbb{R}^N)}\right), \end{aligned}$$

and hence

$$\sup_{u\in\Phi^{-1}_{s,p}((-\infty,1])} \Psi(u) \le \frac{p\,\mathcal{C}_g}{q\lambda_1}.$$

Due to the inequality (2.40) and the assumption (G10), we have

$$\sup_{u\in\Phi_{s,p}^{-1}((-\infty,1])}\Psi(u)<\frac{\Psi(\tilde{u})}{\Phi_{s,p}(\tilde{u})}.$$

Therefore, we deduce $\tilde{\Lambda}\subseteq(\frac{\Phi_{s,p}(\tilde{u})}{\Psi(\tilde{u})},\frac{1}{\sup_{\Phi_{s,p}(u)\leq 1}\Psi(u)})$. By applying Corollary 2.17 with $\mu=1$ and $\Phi=\Phi_{s,p}$, we conclude that the problem (P_λ) has at least one nontrivial weak solution for each $\lambda\in\tilde{\Lambda}$. □

On the other hand, without using the principal eigenvalue λ_1 and the crucial number C_g, we can also prove that the problem (P_λ) admits at least one nontrivial weak solution whenever the parameter λ belongs to an appreciate positive interval. To obtain this result, we only need the minimum prerequisite for the nonlinear term g. The proof for this consequence is similar to that of Theorem 2.20.

Theorem 2.21. *Let $0<s<1<p<+\infty$ with $ps<N$. Assume that* (V) *and* (G1) *hold. If furthermore g satisfies the following assumptions*:

(G11) *There exist nonnegative functions*

$$a_2\in L^{(p_s^*)'}(\mathbb{R}^N)\cap L^{r'}(\mathbb{R}^N),\ b_2\in L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^N)$$

with $1<r<p_s^*$ such that

$$|g(x,t)|\leq a_2(x)+b_2(x)\,|t|^{r-1}$$

holds for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$.

(G12) *There exist a real number s_0, a positive constant r_0, and an element x_0 in $\mathbb{R}^N$ with*

$$2\,|s_0|^p\,\omega_N^2 r_0^{N-sp}\mathcal{M}<p$$

such that

$$\int_{B_{r_0}^N(x_0)}G(x,|s_0|)\,dx>0\quad\text{and}\quad G(x,t)\geq 0$$

for almost all $x\in B_{r_0}^N(x_0)\setminus B_{r_0/2}^N(x_0)$ and for all $0\leq t\leq|s_0|$. Here

$$S_{s,p}^{\frac{-1}{p}}\|a_2\|_{L^{(p_s^*)'}(\mathbb{R}^N)}+S_{s,p}^{\frac{-r}{p}}\|b_2\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)}<\frac{r_0^{sp}\operatorname{ess\,inf}_{B_{r_0/2}^N(x_0)}G(x,|s_0|)}{2^{N+1}\,|s_0|^p\,\omega_N\mathcal{M}},$$

where $S_{s,p}$ and $\mathcal{M}$ are given in Lemma 2.2 and Theorem 2.20, respectively.

Then, for every

$$\lambda\in\tilde{\Lambda}:=\left(\frac{2^{N+1}|s_0|^p\omega_N\mathcal{M}}{p\ r_1^{sp}\operatorname{ess\,inf}_{x\in B_{r_0/2}^N(x_0)}G(x,|s_0|)},\frac{1}{pS_{s,p}^{\frac{-1}{p}}\|a_2\|_{L^{(p_s^*)'}(\mathbb{R}^N)}+pS_{s,p}^{\frac{-r}{p}}\|b_2\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)}}\right),$$

the problem (P_λ) has at least one nontrivial weak solution.

Proof. The functionals $\Phi_{s,p}, \Psi : X_s(\mathbb{R}^N) \to \mathbb{R}$ are defined as in Theorem 2.20. With the same arguments as Lemma 2.13, it easily follows that Ψ and Ψ' are weakly strongly continuous in $X_s(\mathbb{R}^N)$. Repeating the same procedures as in the proof of Theorem 2.20, we know that

$$\Phi_{s,p}(\tilde{u}) \leq \frac{2\,|s_0|^p\,\omega_N^2 r_0^{N-sp}\mathcal{M}}{p} < 1$$

and

$$\frac{\Psi(\tilde{u})}{\Phi_{s,p}(\tilde{u})} \geq \frac{p\,r_0^{sp}\operatorname{ess\,inf}_{x\in B^N_{r_0/2}(x_0)} G(x,|s_0|)}{2^{N+1}\,|s_0|^p\,\omega_N\mathcal{M}}, \tag{2.41}$$

where $\tilde{u}$ is defined as (2.39).

Owing to the assumption (G11), Lemma 2.2, and the Hölder inequality, we deduce that

$$\begin{aligned}
\Psi(u) &= \int_{\mathbb{R}^N} G(x,u)\,dx \\
&\leq \int_{\mathbb{R}^N} a_2(x)\,|u| + b_2(x)\,|u|^r\ dx \\
&\leq \|a_1\|_{L^{(p_s^*)'}(\mathbb{R}^N)}\|u\|_{L^{p_s^*}(\mathbb{R}^N)} + \|b_1\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)}\|u\|^r_{L^{p_s^*}(\mathbb{R}^N)} \\
&\leq S_{s,p}^{\frac{-1}{p}}\|a_1\|_{L^{(p_s^*)'}(\mathbb{R}^N)}\|u\|_{X_s(\mathbb{R}^N)} + S_{s,p}^{\frac{-r}{p}}\|b_1\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)}\|u\|^r_{X_s(\mathbb{R}^N)}
\end{aligned}$$

and hence

$$\sup_{u\in\Phi_{s,p}^{-1}((-\infty,1])}\Psi(u) \leq pS_{s,p}^{\frac{-1}{p}}\|a_1\|_{L^{(p_s^*)'}(\mathbb{R}^N)} + pS_{s,p}^{\frac{-r}{p}}\|b_1\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)}.$$

Due to the inequality (2.41), we have

$$\sup_{u\in\Phi_{s,p}^{-1}((-\infty,1])}\Psi(u) < \frac{\Psi(\tilde{u})}{\Phi_{s,p}(\tilde{u})}.$$

Therefore, we deduce $\tilde{\Lambda} \subseteq (\frac{\Phi_{s,p}(\tilde{u})}{\Psi(\tilde{u})}, \frac{1}{\sup_{\Phi_{s,p}(u)\leq 1}\Psi(u)})$. As proved in Theorem 2.20, $\mathcal{I}_\lambda$ satisfies the $(C)^{[\mu]}$-condition. So, by applying Lemma 2.17 with $\mu = 1$ and $\Phi = \Phi_{s,p}$, we conclude that the problem (P_λ) has at least one nontrivial weak solution for each $\lambda \in \tilde{\Lambda}$. □

3. Conclusion

In summary, when the nonlinearity of g is subcritical and p-superlinear, we demonstrate the existence and multiplicity of weak solutions to a class of Schrödinger type equations involving the fractional p-Laplacian without the Ambrosetti and Rabinowitz condition via variational method. Furthermore, by considering a critical point theorem for an energy functional satisfying the Cerami condition as a variant of Theorems 2.3 and 2.4 in [9], we give an accurate

positive interval of the parameters λ for which our problem admits at least one nontrivial weak solution in the case that the nonlinear term g has the subcritical growth condition (but may not always be p-superlinear). Especially, more complicated analysis than the papers [9–11] has to be carefully carried out when we determine this interval.

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