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STRONG PRESERVERS OF SYMMETRIC ARCTIC RANK OF NONNEGATIVE REAL MATRICES

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Abstract. A rank 1 matrix has a factorization as $\mathbf{u}\mathbf{v}^t$ for vectors $\mathbf{u}$ and $\mathbf{v}$ of some orders. The arctic rank of a rank 1 matrix is the half number of nonzero entries in $\mathbf{u}$ and $\mathbf{v}$. A matrix of rank k can be expressed as the sum of k rank 1 matrices, a rank 1 decomposition. The arctic rank of a matrix A of rank k is the minimum of the sums of arctic ranks of the rank 1 matrices over all rank 1 decomposition of A. In this paper we obtain characterizations of the linear operators that strongly preserve the symmetric arctic ranks of symmetric matrices over nonnegative reals.

1. Introduction

Lately there have been many articles on the theory of quantum mechanics. In the quantum theory, operators are defined by completely positive operators with additional properties. As an example, a *quantum channel* is a completely positive mapping which preserves trace acting on spaces of operators. So the study on operators that preserve properties of completely positive matrices is interesting and important. The definition of completely positive mapping was originated in the study of quadratic forms. A quadratic form $P = \sum_{i,j=1}^{n} a_{i,j} x_i x_j$ is completely positive if and only if P is a sum of squares of linear forms $q_i = \sum_{k=1}^{n} c_{(i,k)} x_k$, which is equivalent that $P = q_1^2 + q_2^2 + \cdots + q_r^2$ for some r. In [7] it was shown that the operator P is completely positive if and only if there is some $n \times r$ matrix Q with nonnegative entries such that $P = (p_{i,j}) = QQ^t$. The symmetric arctic rank of a completely positive matrix P is the minimum number of nonzero entries in Q for a completely positive factorization of $P = QQ^t$. In [9] the linear operator that preserves the completely positive rank of a completely positive matrix was determined.

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In this paper, we investigate the symmetric arctic rank and characterize the strong linear preservers of sets of completely positive matrices defined by symmetric arctic rank of nonnegative matrices.

In Section 2 we shall give definitions, some basic properties of matrices with various symmetric arctic ranks. In Section 3 we characterize strong linear preservers of sets of nonnegative symmetric matrices defined by symmetric arctic ranks.

2. Preliminaries

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. Then $(\mathbb{R}_+, +, \cdot)$ is a semidomain, which is a commutative semiring with a multiplicative identity 1 different from 0 and without zero divisors.

Let $\mathcal{M}_{m,n}(\mathbb{R}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{R}_+$. If $m = n$, we use the notation $\mathcal{M}_n(\mathbb{R}_+)$ instead of $\mathcal{M}_{n,n}(\mathbb{R}_+)$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix, and we write J_n for $J_{n,n}$ and O_n for $O_{n,n}$. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. For matrices A and B, $A \oplus B$ is the direct sum of A and B so that $A \oplus B = \left[\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right]$.

Consider a subset $\mathcal{N}$ of matrices in $\mathcal{M}_{m,n}(\mathbb{R}_+)$. If $\mathcal{N}$ is closed under addition and scalar multiplication, then it is a semimodule and hence has a basis $\mathcal{B}$, a set of matrices such that any member $A \in \mathcal{N}$ is a linear combination of elements of $\mathcal{B}$ and no member of $\mathcal{B}$ is a linear combination of the remaining members of $\mathcal{B}$. Since $\mathbb{R}_+$ is antinegative, that is only 0 has an additive inverse, any member of $\mathcal{N}$ is a unique linear combination of elements of any specified basis. The elements of a basis are called *base elements* ([2]).

A matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$ is called a *cell* ([4]) if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the $(i, j)^{th}$ position by $E_{i,j}$. We also call $wE_{i,j}$ a *weighted cell* ([4]) for any nonzero $w \in \mathbb{R}_+$. Let $\mathcal{E} = \{E_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Then, $\mathcal{E}$ is a basis for $\mathcal{M}_{m,n}(\mathbb{R}_+)$.

We let $\mathcal{S}_n(\mathbb{R}_+)$ denote the set of all $n \times n$ *symmetric* matrices with entries in $\mathbb{R}_+$. For $1 \leq i < j \leq n$ let $D_{i,j} = E_{i,j} + E_{j,i}$. The matrix $D_{i,j}$ is called a *digon* ([5]). In $\mathcal{M}_{m,n}(\mathbb{R}_+)$ a basis consists of all the cells in $\mathcal{E}$, but in $\mathcal{S}_n(\mathbb{R}_+)$ a basis consists of all digons and diagonal cells. We occasionally use $D_{i,i}$ to represent $E_{i,i}$.

For $A,\ B \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, A *dominates* B, written $A \sqsupseteq B$, if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$ for all i and j. If $A \sqsupseteq B$, then $A \setminus B = C$ is the matrix such that $c_{i,j} = a_{i,j}$ if $b_{i,j} = 0$ and is 0 otherwise.

We now let $\mathcal{B} = \{D_{i,j} \mid 1 \leq i \leq j \leq n\}$ be a basis for $\mathcal{S}_n(\mathbb{R}_+)$ and the term "*base element*" shall refer to members of $\mathcal{B}$.

For $X \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, let $|X|$ denote the number of nonzero entries in X. That is $|\cdot| : \mathcal{M}_{m,n} \to \mathbb{Z}_+$ is the function such that $|X|$ is the number of nonzero entries in X, where $\mathbb{Z}_+$ is the set of nonnegative integers.

For $X \in \mathcal{S}_n(\mathbb{R}_+)$, we let $\#(X)$ denote the number of base elements of $\mathcal{S}_n(\mathbb{R}_+)$ that X dominates. That is, $\# : \mathcal{S}_n(\mathbb{R}_+) \to \mathbb{Z}_+$ is the function such that $\#(X)$ is the number of base elements of $\mathcal{S}_n(\mathbb{R}_+)$ that X dominates. Note that if X is not symmetric, then $\#(X)$ is undefined.

Example 2.1. For a digon $D_{1,2} \in \mathcal{S}_n(\mathbb{R}_+)$, $D_{1,2} = E_{1,2} + E_{2,1}$. Thus we have that $|D_{1,2}| = 2$ but $\#(D_{1,2}) = 1$ since $D_{1,2}$ is a base element of $\mathcal{S}_n(\mathbb{R}_+)$.

For $A = (a_{i,j})$, $B = (b_{i,j}) \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ the *Hadamard* or *Schur product* ([8]) of A and B is the matrix $C = (c_{i,j})$ if the $(i,j)^{th}$ entry $c_{i,j}$ of C is $a_{i,j}b_{i,j}$, and we write $A \circ B = C$.

The *rank* ([10]), $r(A)$ of a nonzero $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ is defined to be the smallest integer h for which there exist $B \in \mathcal{M}_{m,h}(\mathbb{R}_+)$ and $C \in \mathcal{M}_{h,n}(\mathbb{R}_+)$ such that $A = BC$. The rank of the zero matrix is zero.

The rank of $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ is 1 if and only if there exist nonzero vectors $\mathbf{b} \in \mathbb{R}_+^m = \mathcal{M}_{m,1}(\mathbb{R}_+)$ and $\mathbf{c} \in \mathbb{R}_+^n = \mathcal{M}_{n,1}(\mathbb{R}_+)$ such that $A = \mathbf{b}\mathbf{c}^t$. It is clear that these vectors $\mathbf{b}$ and $\mathbf{c}$ are uniquely determined by A up to scalar multiples. That is, if $A = \mathbf{b}\mathbf{c}^t = \mathbf{d}\mathbf{e}^t$, then $\mathbf{d} = x\mathbf{b}$ and $\mathbf{e} = y\mathbf{c}$ with $xy = 1$. We have from [3] that $r(A)$ is the least h such that A is the sum of h matrices of rank 1, which is called a *rank*-1 *decomposition*. It follows that $0 < r(A) \leq m$ for all nonzero $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$.

The *binary Boolean algebra* ([2]) consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$. We also use the Boolean arithmetic extended in a usual way to vector and matrix arguments. We write $\mathcal{M}_{m,n}(\mathbb{B})$ for the set of all $m \times n$ *Boolean matrices* with entries in $\mathbb{B}$. The *Boolean rank*, $r_{\mathbb{B}}(A)$, of a nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is the minimal number of rank-1 matrices needed to obtain A as the Boolean sum. Thus for $A \in \mathcal{M}_{m,n}(\mathbb{B})$, $r_{\mathbb{B}}(A) = h$ if and only if there is the least integer h for which there exist $C \in \mathcal{M}_{m,h}(\mathbb{B})$ and $D \in \mathcal{M}_{h,n}(\mathbb{B})$ such that $A = CD$. The Boolean rank of the zero matrix is zero.

The *arctic rank* of a Boolean rank-1 matrix $A = \mathbf{b}\mathbf{c}^t \in \mathcal{M}_{m,n}(\mathbb{B})$, $arct(A)$, is $\frac{1}{2}(|\mathbf{b}| + |\mathbf{c}|)$. Beasley et al. ([1]) defined the arctic rank of any Boolean matrix and proved that this artic rank gives an upper bound for many other ranks including the Boolean rank (aka. Kapronov rank or Boolean factor rank) and term rank of a given Boolean matrix. The tropical rank (see [6]) gives a lower bound on many ranks of matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ and thus, the name “arctic rank” is an appropriate name.

For a rank-r matrix $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ with a rank-1 decomposition $A = A_1 + A_2 + \cdots + A_r$, the arctic rank of this rank-1 decomposition is the sum of the arctic rank of the rank-1 summands. The *arctic rank* of matrix $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$ of rank r is defined to be the minimum arctic rank of a rank-1 decomposition over all rank-1 decompositions of A ([5]).

For $A \in \mathcal{M}_{m,n}(\mathbb{R}_+)$, let $\mathcal{F}(A)$ be the set of ordered pairs of matrices that factor A. That is,

$$\mathcal{F}(A) = \{(B, C) \mid B \in \mathcal{M}_{m,h},\ C \in \mathcal{M}_{h,n} \text{ for some } h \text{ such that } A = BC\}.$$

Then, $arct(A) = \min_{(B,C)\in\mathcal{F}(A)} \frac{1}{2}\{|B| + |C|\}$. Then it is clear that every matrix in $\mathcal{M}_{m,n}(\mathbb{R}_+)$ whose arctic rank is 1 is a weighted cell.

For $A = (a_{i,j}) \in \mathcal{S}_n(\mathbb{R}_+)$, we say that A has a *symmetric factorization* if there exist $r \in \mathbb{Z}_+$ and $B \in \mathcal{M}_{n,r}$ such that $A = BB^t$. Let $\mathcal{F}_{sym}$ denote the matrices in $\mathcal{S}_n(\mathbb{R}_+)$ that have a symmetric factorization, that is

$$\mathcal{F}_{sym} = \{A \in \mathcal{S}_n(\mathbb{R}_+) \mid A = BB^t \text{ for some } B \in \mathcal{M}_{n,r}(\mathbb{R}_+),\ r \in \mathbb{Z}_+\}.$$

In this case, $A = \mathbf{b}_1\mathbf{b}_1^t + \mathbf{b}_2\mathbf{b}_2^t + \cdots + \mathbf{b}_r\mathbf{b}_r^t$ where $\mathbf{b}_j$ is the j^{th} column of B. Then, the *symmetric arctic rank* of A, $s\cdot arct(A)$, is the minimum number, $|B|$, over all symmetric factorizations of $A = BB^t$ ([5]).

We can check that not all members of $\mathcal{S}_n(\mathbb{R}_+)$ have a symmetric factorization. For example, $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ has rank two, but the sum of any two symmetric rank-1 matrices cannot be $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. If A has a nonzero entry, then AA^t has a nonzero entry on the main diagonal. Thus the product of any 2×2 symmetric matrix and its transpose cannot equal $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. If B does not have symmetric factorization, then we put $s\cdot arct(B) = \infty$.

Let $\Sigma_h = \{A \in \mathcal{F}_{sym} \mid s\cdot arct(A) = h$ or $X \sqsupseteq A \sqsupseteq X$ for some X with $s\cdot arct(X) = h\}$ and

$$\Sigma_\infty = \{A \in \mathcal{S}_n(\mathbb{R}_+) \mid A \text{ does not have symmetric factorization}\}.$$

That is, if $A \in \Sigma_h$, then $s\cdot arct(A) = h$ or $s\cdot arct(A \circ X) = h$ for some $X = (x_{i,j}) \in \mathcal{S}_n(\mathbb{R}_+)$ with all nonzero entries.

Example 2.2. Consider $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 8 \end{smallmatrix}\right]$. Then we have

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}^t = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}^t + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}^t.$$

Thus A has a symmetric factorization and has rank 2 over $\mathbb{R}_+$. Moreover $s\cdot arct(A) = 3$.

Let $A \in \mathcal{S}_n(\mathbb{R}_+)$. If $s\cdot arct(A) = 1$, then A is a weighted diagonal cell. That is, Σ_1 is the set of all weighted diagonal cells;

If $s\cdot arct(A) = 2$, then up to permutational similarity, there are nonzero $a, b \in \mathbb{R}_+$ such that $A = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^t \oplus O_{n-2}$ or $A = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]^t \oplus O_{n-2}$; and

If $s\cdot arct(A) = 3$, then up to permutational similarity, there are nonzero $a, b, c \in \mathbb{R}_+$ such that

$$A_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}^t \oplus O_{n-3},$$

$$A_2 = \begin{bmatrix} a & 0 \\ b & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \\ 0 & c \end{bmatrix}^t \oplus O_{n-3},$$

$$A_3 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^t \oplus O_{n-3}, \text{ or}$$

$$A_4 = \begin{bmatrix} a & b \\ 0 & c \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \\ 0 & 0 \end{bmatrix}^t \oplus O_{n-3} = \begin{bmatrix} a^2+b^2 & bc & 0 \\ bc & c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus O_{n-3}.$$

It is clear that when $s \cdot arct(A) = 3$, we have $\#(A) = 3, 4$ or 6 only.

A mapping $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ is said a *linear operator* if $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all $X, Y \in \mathcal{S}_n(\mathbb{R}_+)$. A linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ is said a (P, P^t, B)-*operator* if there exist a permutation matrix P and a matrix $B = (b_{i,j}) \in \mathcal{S}_n(\mathbb{R}_+)$ with all $b_{i,j} \neq 0$ such that $T(A) = P(A \circ B)P^t$ for all $A \in \mathcal{S}_n(\mathbb{R}_+)$.

A linear operator T is said to *preserve* Σ_h if $X \in \Sigma_h$ implies $T(X) \in \Sigma_h$. Also, T *strongly preserves* Σ_h if $X \in \Sigma_h$ if and only if $T(X) \in \Sigma_h$.

There have been many papers on linear operators that preserve some special subsets of matrices ([2–5], [8–10]). For an excellent survey, see [8]. In [5], the bijective linear operators that preserve sets of matrices defined by symmetric arctic ranks over semirings were characterized. In this article we investigate the linear operators that strongly preserve symmetric arctic rank of symmetric matrices over the nonnegative reals, $\mathbb{R}_+$.

3. Strong preservers of symmetric arctic ranks over nonnegative matrices

In this section we shall classify linear operators on $\mathcal{S}_n(\mathbb{R}_+)$ that strongly preserve the set of matrices of symmetric arctic rank $k \geq 3$.

If a linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ preserves Σ_1, then it is obvious that the image of a diagonal cell is a weighted diagonal cell.

From now on, the set of distinct weighted base elements means a set of weighted base elements, no two of which are of the same base element.

We begin this section with an example.

Example 3.1. Consider a map $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ which is defined by $T(E_{j,j}) = E_{j,j}$, $T(D_{j,k}) = E_{j,j} + E_{k,k}$ and extend linearly. Then T strongly preserves Σ_1, T preserves Σ_2, but T does not preserve any Σ_h, $3 \leq h \leq n$, since, if we consider $B = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 4 \end{smallmatrix}\right] \oplus I_{h-3} \oplus O_{n-h+1} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right] \oplus I_{h-3} \oplus O_{n-h+1}$, then $s \cdot arct(B) = h$ while $s \cdot arct(T(B)) = h - 1$, since $T(B) = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 6 \end{smallmatrix}\right] \oplus I_{h-3} \oplus O_{n-h+1}$.

Lemma 3.2. *If a linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ strongly preserves Σ_1, then T maps the set of diagonal cells bijectively onto a set of n distinct weighted diagonal cells. That is T is a bijective linear mapping from Σ_1 onto Σ_1.*

Proof. Since T strongly preserves Σ_1, the image of a diagonal cell is a weighted diagonal cell. Suppose that the images of two distinct diagonal cells are the weighted diagonal cells of the same diagonal cell. We may assume without loss of generality that $T(E_{1,1}) = w_1E_{q,q}$ and $T(E_{2,2}) = w_2E_{q,q}$. Then $T(E_{1,1} + E_{2,2}) = (w_1 + w_2)E_{q,q} \in \Sigma_1$ while $E_{1,1} + E_{2,2} \in \Sigma_2$, which is a contradiction, since T strongly preserves Σ_1.

Let $\Delta = \{(k, k) : k = 1, \ldots, n\}$ and define a map $g : \Delta \to \Delta$ by $g(k, k) = (j, j)$ if and only if $T(E_{k,k}) = wE_{j,j}$ for some $w \in \mathbb{R}_+$. Then we have that g is injective by above paragraph and moreover g is bijective since Δ is finite. Thus, T is a bijective linear mapping from Σ_1 onto Σ_1. □

Lemma 3.3. (1) *Let* $X = \{(J_{h-1} \oplus O_1) + (O_1 \oplus J_{h-1})\} \oplus O_{n-h}$ *for some* $3 \leq h \leq n$. *Then* $s \cdot arct(X) = 2h - 2$ *and hence* $X \notin \Sigma_h$.

(2) *Let* $D_{i,j} \sqsubseteq (J_h \oplus O_{n-h})$ *for some* $3 \leq h \leq n$. *Then* $Y = (J_h \oplus O_{n-h}) \setminus D_{i,j} + \mu D_{r,s} \notin \Sigma_h$ *for any* $\mu \in \mathbb{R}_+$.

Proof. (1) We have $s \cdot arct(X) = s \cdot arct[\{(J_{h-1} \oplus O_1) + (O_1 \oplus J_{h-1})\} \oplus O_{n-h}] = s \cdot arct[([1\ 1\ \cdots\ 1]^t[1\ 1\ \cdots\ 1] \oplus O_1) + (O_1 \oplus [1\ 1\ \cdots\ 1]^t[1\ 1\ \cdots\ 1]) \oplus O_{n-h}] = (h - 1) + (h - 1) = 2h - 2 \neq h$ for $h \geq 3$. Thus by definition of Σ_h, $X \notin \Sigma_h$.

(2) If $D_{i,j}$ is a diagonal cell or $D_{r,s}$ is a digon, then $Y = (J_h \oplus O_{n-h}) \setminus D_{i,j} + \mu D_{r,s}$ cannot have a symmetric factorization. If $D_{i,j}$ is a digon and $D_{r,s}$ is a diagonal cell, then we may assume that $D_{i,j} = D_{1,h}$ and $D_{r,s} = E_{i,i}$ without loss of generality. Then we choose $X = [\{(J_{h-1} \oplus O_1) + (O_1 \oplus J_{h-1})\} \oplus O_{n-h}] + E_{i,i}$. Then $X \sqsubseteq (J_h \oplus O_{n-h}) \setminus D_{1,h} + \mu E_{i,i} \sqsubseteq X$. But $s \cdot arct(X) = 2(h-1) + 1 = 2h - 1 \neq h$ for $h \geq 2$. Therefore $(J_h \oplus O_{n-h}) \setminus D_{1,h} + \mu E_{i,i} \notin \Sigma_h$ by definition of Σ_h. □

Lemma 3.4. *If a linear operator* $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ *strongly preserves* Σ_h *for some* $3 \leq h \leq n$, *then we have* $(J_h \oplus O_{n-h}) \setminus D_{i,j} \notin \Sigma_h$ *for any* $1 \leq i \leq j \leq h$.

Proof. If $i = j$, then $(J_h \oplus O_{n-h}) \setminus D_{i,i}$ has a 0 in (i, i)-entry but the other entries in the ith row and column are not zero. This implies that it cannot have a symmetric factorization. Hence $s \cdot arct((J_h \oplus O_{n-h}) \setminus D_{i,i}) = \infty$ and $(J_h \oplus O_{n-h}) \setminus D_{i,i} \notin \Sigma_h$.

If $i < j$, then we may assume that $i = 1$, $j = k$ by permuting. That is, $P^t(J_h \oplus O_{n-h}) \setminus D_{i,j})P = (J_h \oplus O_{n-h}) \setminus D_{1,h})$. Then we choose $X = \{(J_{h-1} \oplus O_1) + (O_1 \oplus J_{h-1})\} \oplus O_{n-h}$ with $X \sqsubseteq (J_h \oplus O_{n-h}) \setminus D_{1,h}) \sqsubseteq X$. But $s \cdot arct(X) = 2h - 2 \neq h$ for $h \geq 3$ by Lemma 3.3(1). Therefore $(J_h \oplus O_{n-h}) \setminus D_{1,h} \notin \Sigma_h$ by definition of Σ_h. □

Lemma 3.5. *For a linear operator* $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$, *if* T *strongly preserves* Σ_h *for some* $3 \leq h \leq n$, *then* T *maps the set of base elements bijectively onto a set of weighted base elements.*

Proof. Suppose that $T(D_{i,j}) = O$ for some $i \leq j$. We may assume without loss of generality that $1 \leq i, j \leq 2$. Then, $T((J_h \oplus O_{n-h}) \setminus D_{i,j}) = T(J_h \oplus O_n - h)$, which contradicts that T strongly preserves Σ_h since $(J_h \oplus O_{n-h}) \in \Sigma_h$ but $(J_h \oplus O_{n-h}) \setminus D_{i,j} \notin \Sigma_h$ by Lemma 3.4. Therefore T is nonsingular.

Suppose that $\#(T(D_{i,j})) \geq 2$ for some $i \leq j$. Without loss of generality, assume that $1 \leq i \leq j \leq h$. Then, there is a base element $D_{r,s}(\neq D_{i,j})$ such that $T(J_h \oplus O_{n-h}) \sqsupseteq T((J_h \oplus O_{n-h}) \setminus D_{r,s}) \sqsupseteq T(J_h \oplus O_{n-h})$. Since $(J_h \oplus O_{n-h}) \in \Sigma_h$, we have $T(J_h \oplus O_{n-h}) \in \Sigma_h$ since T strongly preserves Σ_h. By definition of Σ_h, $T((J_h \oplus O_{n-h}) \setminus D_{r,s}) \in \Sigma_h$ but $((J_h \oplus O_{n-h}) \setminus D_{r,s}) \notin \Sigma_h$ by Lemma 3.4. So we have a contradiction, since T strongly preserves Σ_h. Therefore T maps each base element to a weighted base elements.

Suppose that $D_{a,b}$ and $D_{c,d}$ are distinct base elements and $T(D_{a,b}) = \mu T(D_{c,d})$. If for some permutation matrix P, $(D_{a,b} + D_{c,d}) \sqsubseteq P^t(J_h \oplus O_{n-h})P$, then

$$\begin{aligned} T(P^t(J_h \oplus O_{n-h})P) &= T((P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) + D_{a,b}) \\ &= T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) + T(D_{a,b}) \\ &= T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) + \mu T(D_{c,d}) \\ &= T((P^t(J_h \oplus O_{n-h})P \setminus D_{a,b} + \mu D_{c,d}) \\ &\sqsubseteq T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) \\ &\sqsubseteq T(P^t(J_h \oplus O_{n-h})P), \end{aligned}$$

which is a contradiction since $T(P^t(J_h \oplus O_{n-h})P) \in \Sigma_h$ and hence $T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) \in \Sigma_h$ by definition of Σ_h but $P^t(J_h \oplus O_{n-h})P \setminus D_{a,b} \notin \Sigma_h$ by Lemma 3.4.

Note that if $h > 3$, then we can always find such a permutation matrix P.

Hence if we cannot find a permutation matrix P such that $P(D_{a,b} + D_{c,d})P^t \sqsubseteq (J_h \oplus O_{n-h})$, then $h < 4$. But, there is a permutation P such that $D_{a,b} \sqsubseteq P^t(J_h \oplus O_{n-h})P$. In this case, as above we have that

$$\begin{aligned} T(P^t(J_h \oplus O_{n-h})P) &= T((P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) + D_{a,b}) \\ &= T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) + T(D_{a,b}) \\ &= T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b}) + \mu T(D_{c,d}) \\ &= T(P^t(J_h \oplus O_{n-h})P \setminus D_{a,b} + \mu D_{c,d}), \end{aligned}$$

which is a contradiction since $P^t(J_h \oplus O_{n-h})P \in \Sigma_h$ while $P^t(J_h \oplus O_{n-h})P \setminus D_{a,b} + \mu D_{c,d} \notin \Sigma_h$ by Lemma 3.3(2).

Thus we have that T maps the set of base elements injectively (and hence bijectively) onto a set of weighted base elements. □

Corollary 3.6. *For a linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$, if T strongly preserves Σ_h for some $3 \leq h \leq n$, then T is bijective on the set of weighted base elements and hence on $\mathcal{S}_n(\mathbb{R}_+)$.*

Proof. By Lemma 3.5, T maps the set of base elements bijectively onto a set of weighted base elements. Hence the preimage of each base element under T is also a weighted base element. That is, T is bijective on the set of weighted base elements. Since every member of $\mathcal{S}_n(\mathbb{R}_+)$ is a unique sum of weighted base element, T is bijective on $\mathcal{S}_n(\mathbb{R}_+)$. □

We define the *pattern* of a matrix $X \in \mathcal{S}_n(\mathbb{R}_+)$, X^*, to be the $(0,1)$-matrix in $\mathcal{S}_n(\mathbb{B})$ such that the (j,k) entry of X^* is 1 if and only if $x_{j,k} \neq 0$. Then we have $X \sqsubseteq X^* \sqsubseteq X$, $(X+Y)^* = X^* + Y^*$ and $(XY)^* = X^*Y^*$. We write $s \cdot arct_{\mathbb{B}}(X^*)$ for the symmetric arctic rank of $X^* \in \mathcal{S}_n(\mathbb{B})$ over $\mathbb{B}$.

Lemma 3.7. *Let $B \in \mathcal{F}_{sym}$ with $s \cdot arct(B) = h$ for some $3 \le h \le n$. If $\#(B) = \frac{h^2+h}{2}$, then there exists a permutation matrix P such that $B^* = P^t(J_h \oplus O_{n-h})P$.*

Proof. Let P be a permutation matrix such that $P(B \circ I)^* P^t = I_\ell \oplus O_{n-\ell}$ and let $B = B_1 + B_2 + \cdots + B_h$ be a rank-1 decomposition of B such that $s \cdot arct(B) = \sum_{i=1}^h s \cdot arct(B_i)$. Then $|B_i \circ I| = s \cdot arct(B_i)$ and $\ell = |B \circ I| = |(\sum_{i=1}^h B_i) \circ I| \le \sum_{i=1}^h |B_i \circ I| = \sum_{i=1}^h s \cdot arct(B_i) = s \cdot arct(B) = h$. Since $B \in \mathcal{F}_{sym}$, we have $B \sqsubseteq P^t(J_h \oplus O_{n-h})P$. But $\#(J_h \oplus O_{n-h}) = \frac{h^2+h}{2} = \#(B)$ so we must have that $B^* = P^t(J_h \oplus O_{n-h})P$. □

For $X \in \mathcal{S}_n(\mathbb{R}_+)$, we use $X[1,\ldots,h|1,\ldots,h]$ to denote the submatrix of X consisting of the intersection of the first h rows and the first h columns. We use $X(1,\ldots,h|1,\ldots,h)$ to denote the submatrix of X consisting of the intersection of the last $n-h$ rows and the last $n-h$ columns.

Lemma 3.8. *For a linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$, if T strongly preserves Σ_h for some $3 \le h \le n$, then T preserves Σ_1.*

Proof. Suppose that T strongly preserves Σ_h $(3 \le h \le n)$. By Corollary 3.6, T is bijective on the set of weighted base elements.

Suppose that $T(E_{i,i}) = bD_{r,s}$ for some $r \neq s$ and $b \neq 0$. By permuting, we may assume that $i = 1$. Consider $T(J_h \oplus O_{n-h})$. Since T is bijective on the set of weighted base elements, $\#(T(J_h \oplus O_{n-h})) = \#(J_h \oplus O_{n-h}) = \frac{h^2+h}{2}$ and $(T(J_h \oplus O_{n-h})) \in \Sigma_h$. By Lemma 3.7 there is some permutation matrix P such that $T(J_h \oplus O_{n-h})^* = P^t(J_h \oplus O_{n-h})P$. Without loss of generality, by permuting, we assume that $(J_h \oplus O_{n-h}) \sqsupseteq T(J_h \oplus O_{n-h}) \sqsupseteq (J_h \oplus O_{n-h})$.

Similarly as above, there is some permutation matrix Q such that $Q^tT(O_1 \oplus J_h \oplus O_{n-h-1})Q \sqsupseteq J_h \oplus O_{n-h}$ so that we have that $T(O_1 \oplus J_h \oplus O_{n-h-1})$ is a rank-1 matrix since T strongly preserves Σ_h.

Let $X = T(O_1 \oplus J_h \oplus O_{n-h-1})$ and let R be a permutation matrix such that $R(X[1,\ldots,h|1,\ldots,h])^* R^t = O_{h-\ell} \oplus J_\ell$. Let S be a permutation matrix such that $S(X(1,\ldots,h|1,\ldots,h)^* S^t = J_{h-\ell} \oplus O_{n-2h+\ell}$. Then ,

$$(R \oplus S)X^*(R \oplus S)^t = \begin{bmatrix} O_{h-\ell} & O & A_1 & A_2 \\ O & J_\ell & A_3 & A_4 \\ A_1^t & A_3^t & J_{h-\ell} & O \\ A_2^t & A_4^t & O & O_{n-2h+\ell} \end{bmatrix}.$$

We use the fact that if $Y \in \mathcal{F}_{sym}$ and the i^{th} row or column of Y has a nonzero entry, then $y_{i,i} \neq 0$. Thus, A_1, A_2, and A_4 are all zero matrices. Since X is rank-1, we must have that $A_3 = J_{\ell.h-\ell}$. Hence we have that $(R \oplus S)X^*(R \oplus S)^t = O_{h-\ell} \oplus J_h \oplus O_{n-2h+\ell}$. And $(R \oplus S)(J_h \oplus O_{n-h})(R \oplus S)^t = RJ_hR^t \oplus SO_{n-h}S^t = J_h \oplus O_{n-h}$ so that $(R \oplus S)T(J_h \oplus O_{n-h})^*(R \oplus S)^t = J_h \oplus O_{n-h}$.

Define $f : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$ by $f(X) = (R \oplus S)T(X)(R \oplus S)^t$. Then, f is bijective on the set of weighted base elements, $f(E_{1,1}) = bD_{p,q}$ for some nonzero b and $p \neq q$, $f(J_h \oplus O_{n-h})^* = J_h \oplus O_{n-h}$, and $f(O_1 \oplus J_h \oplus O_{n-h-1})^* = O_{h-\ell} \oplus J_h \oplus O_{n-2h+\ell}$. Since $O_1 \oplus J_{h-1} \oplus O_{n-h} \sqsubseteq J_h \oplus O_{n-h}$ and $O_1 \oplus J_{h-1} \oplus O_{n-h} \sqsubseteq O_1 \oplus J_h \oplus O_{n-h-1}$, we must have that $f(O_1 \oplus J_{h-1} \oplus O_{n-h})^* \sqsubseteq f(J_h \oplus O_{n-h})^* = J_h \oplus O_{n-h}$ and $L(O_1 \oplus J_{h-1} \oplus O_{n-h})^* \sqsubseteq f(O_1 \oplus J_h \oplus O_{n-h-1})^* = O_{h-\ell} \oplus J_h \oplus O_{n-2h+\ell}$. Since $\#(O_1 \oplus J_{h-1} \oplus O_{n-h}) = \frac{(h-1)^2+(h-1)}{2}$ and f is bijective on the weighted base elements, we must have that $\ell = h-1$. Hence $f(O_1 \oplus J_{h-1} \oplus O_{n-h})^* = (O_1 \oplus J_{h-1} \oplus O_{n-h})$.

Now we have that $f(E_{1,1}+(O_1 \oplus J_{h-1} \oplus O_{n-h}))^* = D_{p,q}+(O_1 \oplus J_{h-1} \oplus O_{n-h})$, and $bD_{p,q} = f(E_{1,1}) \not\sqsubseteq f(O_1 \oplus J_{h-1} \oplus O_{n-h}) \sqsubseteq O_1 \oplus J_{h-1} \oplus O_{n-h}$ since f is bijective on the set of weighted base elements. We have a contradiction since $s \cdot arct(E_{1,1} + (O_1 \oplus J_{h-1} \oplus O_{n-h})) = h$ and $s \cdot arct(Y) = \infty$ for any Y such that $Y^* = (D_{p,q} + (O_1 \oplus J_{h-1} \oplus O_{n-h})) = f(E_{1,1} + (O_1 \oplus J_{h-1} \oplus O_{n-h}))^*$. This contradiction implies that $f(E_{1,1}) = bD_{p,q} = bE_{p,p}$ for some p and $b \neq 0$. Thus f, and hence T, preserves Σ_1. □

Lemma 3.9. *For a linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$, if T strongly preserves Σ_h for some $3 \leq h \leq n$, then T maps the set of diagonal cells bijectively onto a set of n distinct weighted diagonal cells. That is, T is a bijective mapping from Σ_1 into Σ_1.*

Proof. Assume that T strongly preserves Σ_h. Then T preserves Σ_1 by Lemma 3.8. Hence the image of a diagonal cell is a weighted diagonal cell. Suppose that the images of two diagonal cells are weighted cells of the same diagonal cell. Without loss of generality, we assume that $T(E_{1,1}) = bE_{r,r}$ and $T(E_{2,2}) = dE_{r,r}$. Then $T(I_h \oplus O_{n-h})$ is the sum of at most $h-1$ weighted diagonal cells. Thus $s \cdot arct(T(I_h \oplus O_{n-h})) \leq h-1$, while $s \cdot arct(I_h \oplus O_{n-h}) = h$, which is a contradiction since T preserves Σ_h. Hence T maps the set of diagonal cells bijectively onto a set of n distinct weighted diagonal cells. That is, T is a bijective mapping from Σ_1 into Σ_1. □

Theorem 3.10. *For a linear operator $T : \mathcal{S}_n(\mathbb{R}_+) \to \mathcal{S}_n(\mathbb{R}_+)$, we have that T strongly preserves Σ_h for some $4 \leq h \leq n-1$, if and only if there exist a matrix $B \in \mathcal{S}_n(\mathbb{R}_+)$ with all nonzero entries, scalars $c_{i,j}$, $d_{i,j} \in \mathcal{S}_n(\mathbb{R}_+)$ for $1 \leq i < j \leq n$, and a permutation P such that*

$$T(Y) = P\left(Y \circ B + \sum_{1 \leq i < j \leq n} y_{i,j}(c_{i,j}E_{i,i} + d_{i,j}E_{j,j})\right)P^t$$

for all $Y \in \mathcal{S}_n(\mathbb{R}_+)$.

Proof. Assume that T strongly preserves Σ_h for some $4 \leq h \leq n-1$. Then by Lemma 3.9, T maps the set of diagonal cells bijectively onto a set of n distinct weighted diagonal cells. Permute by P so that $f(Y) = P^tT(Y)P$ for all Y and such that for some $b_i \neq 0$, $T(E_{i,i}) = b_iE_{i,i}$ for all i. Since permutational similarity preserves all symmetric arctic ranks, it follows that f strongly preserves Σ_h.

Step 1) Suppose that $f(D^+_{i,j})$ dominates at least three diagonal cells, where $D^+_{i,j} = E_{i,i} + E_{j,j} + D_{i,j}$ for all $1 \leq i,\ j \leq n$. Let Δ_1 be the sum of $h-2$ diagonal cells such that $s \cdot arct(D^+_{i,j} + \Delta_1) = h$ and $f(D^+_{i,j} + \Delta_1)$ dominates $h+1$ diagonal cells. This is always possible since $f(E_{i,i}) = b_iE_{i,i}$. Then, $s \cdot arct(f(D^+_{i,j}+\Delta_1)) \geq h+1$, which is a contradiction since $s \cdot arct(D^+_{i,j}+\Delta_1) = h$ and f strongly preserves Σ_h.

Step 2) Suppose that $f(D_{i,j})$ dominates two digons. Then there is some k such that $f(D^+_{i,j})$ dominates $D_{k,\ell}$ for some ℓ, but does not dominate $E_{k,k}$. Let Δ_1 be the sum of $h-2$ diagonal cells whose image does not dominate $E_{k,k}$ and such that $s \cdot arct(D^+_{i,j} + \Delta_1) = h$. Then we have a contradiction since $s \cdot arct(f(D^+_{i,j} + \Delta_1)) = \infty$ from $f(D^+_{i,j} + \Delta_1) \notin \mathcal{F}_{sym}$. Thus, for each (i,j) there is some (p,q) such that $f(D^+_{i,j}) \sqsubseteq D^+_{p.q}$.

Step 3) Suppose that $f(D^+_{i,j}) \sqsubseteq D^+_{p,q}$ and $f(D^+_{h,\ell}) \sqsubseteq D^+_{p.q}$ for $(i,j) \neq (h,\ell)$. Let Δ_2 be the sum of either $h-3$ or $h-4$ diagonal cells such that $s \cdot arct(D^+_{i,j} + D^+_{h,\ell} + \Delta_2) = h$. But, $f(D^+_{i,j} + D^+_{h,\ell} + \Delta_2) \sqsubseteq D^+_{p,q} + f(\Delta_2)$ has symmetric arctic rank at most h-1, which is a contradiction since f strongly preserves Σ_h.

By the above steps, we have that $f(D^+_{i,j}) \sqsubseteq D^+_{i,j}$ for all (i,j). Suppose that $f(D^+_{i,j}) \sqsubseteq E_{i,i}+E_{j,j}$. Without loss of generality, we may assume that $f(D^+_{1,2}) = cE_{1,1} + dE_{2,2}$. Then, $s \cdot arct(D^+_{1,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{h-1,h-1}) = h$ and hence $D^+_{1,2}+D^+_{1,3}+E_{4,4}+\cdots+E_{h-1,h-1} \in \Sigma_h$. But $E_{2,2}+D^+_{1,3}+E_{4,4}+\cdots+E_{h-1,h-1} \sqsubseteq f(D^+_{1,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{h-1,h-1}) \sqsubseteq E_{2,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{h-1,h-1}$ and $s \cdot arct(E_{2,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{k-1,k-1}) = h-1$, we have $f(D^+_{1,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{h-1,h-1}) \in \Sigma_{h-1}$. Thus we have a contradiction since f strongly preserves Σ_h. Hence, for all (i,j), $T(D^+_{i,j}) = bD_{i,j} + cE_{i,i} + dE_{j,j}$ for some $b, c, d \in \mathbb{R}_+$.

Since $f(D_{i,j}) \sqsubseteq D^+_{i,j}$, we can define that $f(D_{i,j}) = b_{i,j}D_{i,j} + c_{i,j}E_{i,i} + d_{i,j}E_{j,j}$. Let us denote $B = (b_{i,j})$. Since $f(Y) = P^tT(Y)P$, we have that $T(Y) = P\left(Y \circ B + \sum_{1\leq i<j\leq n} y_{i,j}(c_{i,j}E_{i,i} + d_{i,j}E_{j,j})\right)P^t$ for all $Y \in \mathcal{S}_n(\mathbb{R}_+)$.

Now we claim that B has rank 1. For, if not, we have a 2×2 principal submatrix B_1 of B with rank 2. By permuting, we may assume that $B_1 = \begin{bmatrix} b_{1.1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$ has rank 2. Since $b_{i,j} \neq 0$ for all $1 \leq i, j \leq 2$, we have $s \cdot arct(B_1) = 3$. Then $f(J_2 \oplus I_{h-2} \oplus O_{n-h}) = \begin{bmatrix} b_{1.1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \oplus G_{h-2} \oplus O_{n-h}$ where

G_{h-2} is a $(h-2)\times(h-2)$ diagonal matrix with all nonzero entries in the main diagonal. We have a contradiction from $s\cdot arct(J_2\oplus I_{h-2}\oplus O_{n-h})=h$ and $s\cdot arct(\left[\begin{smallmatrix} b_{1,1} & b_{1,2}\\ b_{2,1} & b_{2,2}\end{smallmatrix}\right]\oplus G_{h-2}\oplus O_{n-h})=h+1$ since T strongly preserves Σ_h. This contradiction implies that B has rank 1.

For the converse implication, assume that B has rank 1, P is a permutation, and $f(Y)=P\left(Y\circ B+\sum_{1\le i<j\le n} y_{i,j}(c_{i,j}E_{i,i}+d_{i,j}E_{j,j})\right)P^t$ for all $Y\in\mathcal{S}_n(\mathbb{R}_+)$, then f strongly preserves all symmetric arctic ranks except $s\cdot arct(Y)=\infty$. □

Corollary 3.11. *For a linear operator $T:\mathcal{S}_n(\mathbb{R}_+)\to\mathcal{S}_n(\mathbb{R}_+)$, we have that T strongly preserves Σ_h for some $4\le h\le n-1$ if and only if T is a (P,P^t,B)-operator for some rank-1 matrix B with all nonzero entries.*

Proof. Suppose T strongly preserves Σ_h for some $4\le h\le n-1$. By Lemma 3.8, T preserves Σ_1. And by Theorem 3.10, there exist a matrix $B\in\mathcal{S}_n(\mathbb{R}_+)$ with all nonzero entries, scalars $c_{i,j},d_{i,j}\in\mathcal{S}_n(\mathbb{R}_+)$ for $1\le i<j\le n$, and a permutation P such that

$$T(Y)=P\left(Y\circ B+\sum_{1\le i<j\le n} y_{i,j}(c_{i,j}E_{i,i}+d_{i,j}E_{j,j})\right)P^t$$

for all $Y\in\mathcal{S}_n(\mathbb{R}_+)$.

Suppose that $c_{i,j}\neq 0$. Since T is bijective on the set of weighted base elements by Lemma 3.9, we can find matrices $U,V,W\in\mathcal{S}_n(\mathbb{R}_+)$ such that $T(U)=D_{i,j},T(V)=E_{i,i}$ and $T(W)=E_{j,j}$. Then

$$\begin{aligned}T(D_{i,j})&=b_{i,j}D_{i,j}+c_{i,j}E_{i,i}+d_{i,j}E_{j,j}\\&=b_{i,j}T(U)+c_{i,j}T(V)+d_{i,j}T(W)\\&=T(b_{i,j}U+c_{i,j}V+d_{i,j}W).\end{aligned}$$

Since T is bijective, $D_{i,j}=b_{i,j}U+c_{i,j}V+d_{i,j}W$. Since $\mathbb{R}_+$ is antinegative semiring, we have nonzero $x,y,z\in\mathbb{R}_+$ such that $U=xD_{i,j},V=yD_{i,j}$ and $W=zD_{i,j}$. Then $T(xyD_{i,j})=xT(yD_{i,j})=xT(V)=xE_{i,i}$, and $T(xyD_{i,j})=yT(xD_{i,j})=yT(U)=yD_{i,j}$. Since T is bijective, we have $xE_{i,i}=yD_{i,j}$ and hence $x=y=0$ and $U=V=0$, which is impossible from $T(U)=D_{i,j},T(V)=E_{i,i}$. Thus $c_{i,j}=0$. Similarly $d_{i,j}=0$. Thus we have $T(Y)=P(Y\circ B)P^t$ for all $Y\in\mathcal{S}_n(\mathbb{R}_+)$. The fact that B is rank-1 with all nonzero entries follows from the proof of Theorem 3.10.

The converse is obvious. □

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