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WEAK NORMAL PROPERTIES OF PARTIAL ISOMETRIES

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Abstract. This paper describes when a partial isometry satisfies several weak normal properties. Topics treated include quasi-normality, subnormality, hyponormality, p-hyponormality ($p > 0$), w-hyponormality, paranormality, normaloidity, spectraloidity, the von Neumann property and Weyl's theorem.

1. Introduction

Throughout this paper, $\mathcal{H}$ will always denote a complex separable infinite dimensional Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$, we denote by $\ker A$ and $\operatorname{ran} A$ the kernel of A and the range of A, respectively.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called a *partial isometry* if $\|Tx\| = \|x\|$ for all $x \in (\ker T)^\perp$. The space $(\ker T)^\perp$ is called the *initial space* of T, and $\operatorname{ran} T$ is called the *final space* of T. Examples of partial isometries include isometries, unitary operators, projections and their direct sums.

Partial isometries appear in the polar decomposition and play a basic role in many aspects of operator theory and operator algebras. Universal C^*-algebras generated by families of partial isometries subject to certain relations provide many important examples of C^*-algebras. For example, the Cuntz algebra is the universal C^*-algebra generated by n isometries satisfying certain relations.

Besides the polar decomposition, general operators on Hilbert spaces can be associated with partial isometries in the following way. Let $A \in \mathcal{B}(\mathcal{H})$ with $\|A\| \le 1$. Denote

$$R(A) = \begin{bmatrix} A & 0 \\ \sqrt{I - A^*A} & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}. \end{matrix}$$

Then it is easy to verify that $R(A)$ is a partial isometry on $\mathcal{H} \oplus \mathcal{H}$. Conversely, if T is a partial isometry, then the compression of T to its initial space is a contraction.

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Previous results show that questions concerning general operators sometimes can be reduced to that concerning partial isometries. By a result of Halmos and McLaughlin [9], if A and B are invertible contractions, then they are unitarily equivalent if and only if $R(A)$ and $R(B)$ are unitarily equivalent. Thus the problem of unitary equivalence for arbitrary operators on $\mathcal{H}$ can be reduced to that for partial isometries. Garcia and Wogen proved in [7] that a contraction A is complex symmetric if and only if so is $R(A)$; the third author of the present paper proved in [16] that A is the norm limit of complex symmetric operators if and only if so is $R(A)$. After observing these interesting results, it is natural to seek more such results, which could reflect connections between A and $R(A)$.

The aim of this paper is to explore connections between contractions A and their extensions $R(A)$ by examining various weak normal properties. Topics treated here include quasi-normality, subnormality, hyponormality, w-hyponormality, p-hyponormality for $p > 0$, paranormality, normaloidity, spectraloidity, the von Neumann property and Weyl's theorem. Now we make a brief introduction to these notions.

The class of normal operators is undoubtedly the best understood among various classes of operators on Hilbert spaces. One of the grand themes in operator theory is to generalize the theory of normal operators. Many generalization of normal operators are hence posed and studied.

Definition 1.1. Let $T \in \mathcal{B}(\mathcal{H})$.

(i) T is said to be *quasi-normal* if $T(T^*T) = (T^*T)T$.

(ii) T is said to be *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N(\mathcal{H}) \subseteq \mathcal{H}$ and $N|_{\mathcal{H}} = T$.

(iii) T is said to be *hyponormal* if $T^*T \geq TT^*$.

(iv) T is said to be *paranormal* if $\|T^2x\| \geq \|Tx\|^2$ for any unit vector $x \in \mathcal{H}$.

(v) T is said to be *p-hyponormal* for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$.

(vi) T is said to be *w-hyponormal* if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$, where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T = U|T|$ is the polar decomposition of T.

Most of these weak normal properties appearing in the preceding definition are defined in terms of operator inequalities, and it is easy to check that normality implies each of them.

We shall show in Section 2 that a partial isometry T satisfies any one of the above mentioned properties if and only if ran $T \subset (\ker T)^{\perp}$ (see Proposition 2.1). As a consequence, this shows for a contraction A that these properties of $R(A)$ imply but are not equivalent to that of A (see Example 2.4).

There are some other notions which can be viewed as generalizations of the notion of normality.

Let $T \in \mathcal{B}(\mathcal{H})$. The *numerical range* of T is defined as $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}$ with $\|x\| = 1\}$, and the *numerical radius* of T is $w(T) = \sup\{|z| : z \in W(T)\}$. We denote by $r(T)$ the *spectral radius* of T, that is, $r(T) = \max\{|z| : z \in \sigma(T)\}$.

If $T \in \mathcal{B}(\mathcal{H})$ is normal, then it is well known that $\|T\| = r(T) = w(T)$. Generalizing the equality, people obtain the following notions.

Definition 1.2. Let $T \in \mathcal{B}(\mathcal{H})$.

(i) T is said to be *normaloid* if $w(T) = \|T\|$.
(ii) T is said to be *spectraloid* if $w(T) = r(T)$.
(iii) T is called a *von Neumann operator* if $f(T)$ is normaloid for each f analytic on some neighborhood of $\sigma(T)$.

Note that an operator T is normaloid if and only if $\|T\| = r(T)$. Also it is known that the normaloid property implies the spectraloid property. However, there exist spectraloid operators that are not normaloid. For example, if the following matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

viewed as an operator on $\mathbb{C}^3$, then $r(T) = w(T) = \frac{1}{2} < 1 = \|T\|$. The reader is referred to [8, Problems 218 & 219] for more details.

The von Neumann property essentially describes the property of normal operators that the functional calculus of a normal operator with respect to analytic functions is isometric. In this sense, the class of von Neumann operators is a generalization of normal operators. Every subnormal operator satisfies the von Neumann property ([4, Proposition II.9.2]).

We shall describe in Section 3 when a partial isometry satisfies the properties appearing in Definition 1.2 (see Theorems 3.1 and 3.8). In particular, as an application, we shall see that these three kinds of properties of $R(A)$ imply but are not equivalent to that of A. Also our result shows that the normaloid property and the spectraloid property are equivalent for partial isometries.

Finally we shall consider another kind of spectral property, which can be also viewed as a weak normal property. For $T \in \mathcal{B}(\mathcal{H})$, the *Weyl spectrum* of T is the set

$$\sigma_w(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \sigma(T + K).$$

Here $\mathcal{K}(\mathcal{H})$ denotes the set of all compact operators on $\mathcal{H}$. If $A \in \mathcal{B}(\mathcal{H})$ is normal, a theorem of H. Weyl [14] states that $\sigma_w(A)$ consists of all spectral points except isolated eigenvalues of finite multiplicity. Coburn [2] proved that Weyl's theorem holds for two classes of nonnormal operators, the hyponormal operators and the Toeplitz operators. Inspired by the results, many work are devoted to the study of Weyl's theorem for more classes of operators (e.g. [1, 5, 6, 10]).

In Section 4, we shall describe when a partial isometry T satisfies Weyl's theorem (see Theorem 4.1). Our result shows that if a contraction A satisfies Weyl's theorem, then so does $R(A)$. However, the converse does not hold (see Example 4.3).

2. Weak normal properties

This section is devoted to describing when a partial isometry satisfies those properties appearing in Definition 1.1.

The main result of this section is the following proposition.

Proposition 2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a partial isometry, then the following statements are equivalent*:

(i) *T is hyponormal*;
(ii) *T is p-hyponormal for $p > 0$*;
(iii) *T is paranormal*;
(iv) *T is quasi-normal*;
(v) *T is subnormal*;
(vi) *T is w-hyponormal*;
(vii) *$\ker T$ reduces T*;
(viii) *$\operatorname{ran} T \subset (\ker T)^{\perp}$*.

Proof. Since $\ker T$ is invariant under T, we may assume that

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T. \end{matrix} \tag{1}$$

If $\ker T = \{0\}$ or $\mathcal{H}$, then either T is an isometry or $T = 0$; hence T clearly satisfies any one of (i)-(viii). So, in the sequel, we directly assume that $\{0\} \subsetneq \ker T \subsetneq \mathcal{H}$.

"(vii)$\Longleftrightarrow$(viii)". From the matrix representation (1) of T, one can check that $\ker T$ reduces T if and only if $B = 0$ if and only if $\operatorname{ran} T \subset (\ker T)^{\perp}$.

"(vii)$\Longrightarrow$ (iv)". From the proof of "(vii)$\Longleftrightarrow$(viii)", one can see that $B = 0$. Since T^*T is the projection onto $(\ker T)^{\perp}$ and

$$T^*T = \begin{bmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T \end{matrix} = \begin{bmatrix} A^*A & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T, \end{matrix}$$

it follows that A^*A is the identity on $(\ker T)^{\perp}$, that is, A is an isometry. So both T and A are quasi-normal.

Likewise, one can prove that (vii) implies any one of (i)-(vi).

"(i)$\Longleftrightarrow$(ii)". Since T is a partial isometry, it follows that both T^*T and TT^* are projections. Then, for any $p > 0$, we have $(T^*T)^p = T^*T$ and $(TT^*)^p = TT^*$. Then the equivalence "(i)$\Longleftrightarrow$(ii)" follows readily.

"(i)$\Longrightarrow$(vii)". Since T is hyponormal, we have $T^*T \geq TT^*$. Since both TT^* and T^*T are projections, it follows $\operatorname{ran} TT^* \subseteq \operatorname{ran} T^*T = (\ker T)^{\perp}$. Note that

$$TT^* = \begin{bmatrix} AA^* & AB^* \\ BA^* & BB^* \end{bmatrix}.$$

So $BB^* = 0$. Thus $B = 0$.

"(iii)$\Longrightarrow$(vii)". For every $x \in (\ker T)^{\perp}$ with $\|x\| = 1$, we have

$$T^2 x = \begin{pmatrix} A^2 x \\ BAx \end{pmatrix} = T(Ax).$$

Note that $x, Ax \in (\ker T)^{\perp}$ and A is a contraction. So

$$1 \geq \|Ax\| = \|T^2x\| \geq \|Tx\|^2 = \|x\|^2 = 1.$$

That is, $\|Ax\| = \|x\|$ for all $x \in (\ker T)^{\perp}$. So A is an isometry, which means that $B = 0$.

"(iv)$\Longrightarrow$(vii) & (v)$\Longrightarrow$(vii)". It is well known that "(iv)$\Longrightarrow$(v)$\Longrightarrow$(i)". Since "(i)$\Longrightarrow$(vii)" has been proved, we are done.

"(vi)$\Longleftrightarrow$(vii)". Direct calculation shows that

$$|T| = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T, \end{matrix} \quad |\widetilde{T}| = \begin{bmatrix} |A| & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T, \end{matrix}$$

where I is the identity operator on $(\ker T)^{\perp}$. Since T is w-hyponormal, it follows that $|A| \geq I$. Note that $\|A\| \leq 1$. So $|A| = I$. Then $A^*A = I$ or equivalently A is an isometry, which implies that $B = 0$. □

Corollary 2.2. *If T is a partial isometry with $\{0\} \subsetneqq \ker T \subsetneqq \mathcal{H}$, then T is hyponormal if and only if $T = S \oplus 0$ for some isometry S.*

Corollary 2.3. *If T is a partial isometry with $\{0\} \subsetneqq \ker T \subsetneqq \mathcal{H}$, then T is normal if and only if* ran $T = (\ker T)^{\perp}$ *if and only if $T = U \oplus 0$ for some unitary operator U.*

We conclude this section with an example, which shows for a contraction A that the weak normal properties of A defined in Definition 1.1 does not imply that of $R(A)$.

Example 2.4. Let $A = I/2$, where I is the identity operator on $\mathcal{H}$. Then A is normal. However $R(A)$ does not satisfy any weak normal properties defined in Definition 1.1.

3. Normaloidity, spectraloidity and the von Neumann property

In this section we shall describe when a partial isometry satisfies those properties appearing in Definition 1.2.

3.1. Normaloidity and spectraloidity

The main result of this subsection is the following theorem which describes normaloid and spectraloid partial isometries.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a nonzero partial isometry. Then the following are equivalent*:

(i) *T is normaloid*;
(ii) *T is spectraloid*;
(iii) *$T|_{(\ker T)^{\perp}}$ is normaloid with norm* 1.

In order to give the proof of Theorem 3.1, we need to make some preparation.

Lemma 3.2. (i) *If* $\{T_n\}_{n=0}^{\infty} \subset \mathcal{B}(\mathcal{H})$ *and* $T_n \to T_0$*, then* $w(T_0) \le \liminf_n w(T_n)$*.*

(ii) *Let* $T \in \mathcal{B}(\mathbb{C}^2)$ *be written as*

$$\begin{bmatrix} \lambda & 0 \\ \mu & 0 \end{bmatrix}$$

relative to some orthonormal basis of $\mathbb{C}^2$*, where* $|\lambda|^2 + |\mu|^2 = 1$*. Then* $w(T) = \frac{1}{2}(1 + |\lambda|)$*.*

Proof. (i) For any unit vector $x \in \mathcal{H}$, we have

$$w(T_n) \ge |\langle T_n x, x\rangle| \to |\langle T_0 x, x\rangle|.$$

Thus $\liminf_n w(T_n) \ge |\langle T_0 x, x\rangle|$. Since x can be any unit vector, the desired result follows readily.

(ii) This is a corollary of [13, Theorem 2.14]. □

Lemma 3.3. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a nonzero partial isometry and*

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{matrix} (\ker T)^{\perp} \\ \ker T. \end{matrix}$$

Then $\sigma(T) \cup \{0\} = \sigma(A) \cup \{0\}$ *and* $r(T) = r(A)$*.*

Proof. Obviously we may directly assume that $\ker T \ne \{0\}$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Straightforward calculation shows that

$$\ker(T - \lambda) = \left\{ \begin{pmatrix} x \\ \frac{Bx}{\lambda} \end{pmatrix} : x \in \ker(A - \lambda) \right\}.$$

Thus

$$\dim \ker(\lambda - A) = \dim \ker(\lambda - T). \tag{2}$$

As a consequence, $\lambda - A$ is injective if and only if $\lambda - T$ is injective. Likewise one can prove that $\dim \ker(\lambda - A)^* = \dim \ker(\lambda - T)^*$.

Since $\lambda \ne 0$, it follows that

$$\text{ran } (\lambda - T) = \text{ran } (\lambda - A) \oplus \ker T. \tag{3}$$

Thus $\lambda - A$ is surjective if and only if $\lambda - T$ is surjective. This combining (2) implies that $\lambda \in \sigma(T)$ if and only if $\lambda \in \sigma(A)$. Hence $\sigma(T) \cup \{0\} = \sigma(A) \cup \{0\}$. Moreover, we obtain $r(T) = r(A)$. □

Let $T \in \mathcal{B}(\mathcal{H})$. Recall that T is called a *semi-Fredholm* operator, if ran T is closed and either $\dim \ker T$ or $\dim \ker T^*$ is finite; in this case, ind $T := \dim \ker T - \dim \ker T^*$ is called the *index* of T. In particular, if T is semi-Fredholm of finite index, then T is called a *Fredholm operator*. The *Wolf spectrum* of T is defined by

$$\sigma_{lre}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}.$$

Given a subset σ of $\mathbb{C}$, we let iso σ, int σ and $\partial\sigma$ denote respectively the set of all isolated points of σ, the set of all interior points of σ and the boundary of σ.

Lemma 3.4 ([3, Theorem XI.6.8]). *If $S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \partial\sigma(S)$, then either $\lambda \in \text{iso } \sigma(S)$ or $\lambda \in \sigma_{lre}(S)$.*

Lemma 3.5 ([3, Proposition XI.6.9]). *If $S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \text{iso } \sigma(S)$, then the following statements are equivalent.*
(i) $\lambda \notin \sigma_{lre}(S)$.
(ii) *The Riesz idempotent corresponding to the singleton λ has finite rank.*
(iii) $S - \lambda$ *is Fredholm and* ind $(S - \lambda) = 0$.

Lemma 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \partial\sigma(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that*

$$T - K = \begin{bmatrix} \lambda & * \\ 0 & A \end{bmatrix} \begin{array}{l} \mathbb{C}e \\ \mathcal{H} \ominus \mathbb{C}e, \end{array}$$

where $e \in \mathcal{H}$ with $\|e\| = 1$.

Proof. If $\lambda \in \sigma_{lre}(T)$, then the result follows from [12, Lemma 3.2.6]. If $\lambda \notin \sigma_{lre}(T)$, then, by Lemmas 3.5 and 3.4, λ is an eigenvalue of T; in this case, we choose a unit vector $e \in \ker(T - \lambda)$. The proof is complete. □

Proof of Theorem 3.1. If $\ker T = \{0\}$, then T is an isometry and the conclusion is clear. In the sequel, we assume that $\{0\} \subsetneq \ker T \subsetneq \mathcal{H}$.

"(i)$\Longrightarrow$(ii)". By [8, Problem 218], the implication is clear.

"(iii)$\Longrightarrow$(i)". Denote $A = T|_{(\ker T)^\perp}$. Then A is a contraction and, by Lemma 3.3, $r(T) = r(A) = \|A\| = 1 = \|T\|$.

"(ii)$\Longrightarrow$(iii)". Denote $\mathcal{H}_1 = (\ker T)^\perp$ and $\mathcal{H}_2 = \ker T$. Then T can be written as

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{array}{l} \mathcal{H}_1 \\ \mathcal{H}_2. \end{array}$$

It suffices to prove that $r(A) = 1$.

For a proof by contradiction, we assume that $r(A) < 1$. Then there exists $\lambda \in \partial\sigma(A)$ such that $|\lambda| = r(A)$. By Lemma 3.6, for any $n \geq 1$, there exist unit vector $e_n \in \mathcal{H}$ and compact $K_n \in \mathcal{B}(\mathcal{H}_1)$ with $\|K_n\| < 1/n$ such that

$$A - K_n = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix} \begin{array}{l} \mathbb{C}e_n \\ \mathcal{H}_1 \ominus \mathbb{C}e_n. \end{array}$$

Denote $E = A - K_n$. Then

$$T = \begin{bmatrix} E + K_n & 0 \\ B & 0 \end{bmatrix} \begin{array}{l} \mathcal{H}_1 \\ \mathcal{H}_2. \end{array}$$

Set $f_n = Be_n$ and $\mu_n = \|f_n\|$. Note that $K_n \to 0$, $|\lambda| < 1$ and

$$1 = \|Te_n\|^2 = \|Ee_n + K_n e_n\|^2 + \|Be_n\|^2 = \|\lambda e_n + K_n e_n\|^2 + \|Be_n\|^2.$$

Thus $\mu_n \neq 0$ for n large enough and $\mu_n \to \sqrt{1 - |\lambda|^2}$. We directly assume that $f_n \neq 0$ for all n. Denote $g_n = f_n/\|f_n\|$ and $a_n = \langle K_n e_n, e_n \rangle$. Then T can be written as

$$T = \left[\begin{array}{cc|cc} \lambda + a_n & * & 0 & 0 \\ * & * & 0 & 0 \\ \hline \mu_n & * & 0 & 0 \\ 0 & * & 0 & 0 \end{array}\right] \begin{array}{c} \mathbb{C}e_n \\ \mathcal{H}_1 \ominus \mathbb{C}e_n \\ \mathbb{C}g_n \\ \mathcal{H}_2 \ominus \mathbb{C}g_n. \end{array}$$

Denote

$$X_n = \begin{bmatrix} \lambda + a_n & 0 \\ \mu_n & 0 \end{bmatrix}, \quad n \geq 1.$$

Then $W(X_n) \subset W(T)$ and

$$X_n \longrightarrow \begin{bmatrix} \lambda & 0 \\ \sqrt{1-|\lambda|^2} & 0 \end{bmatrix} \triangleq X.$$

In view of Lemma 3.2, we have

$$w(T) \geq \liminf_n w(X_n) \geq w(X) = \frac{1}{2}(1+|\lambda|) > |\lambda| = r(A) = r(T).$$

That is, T is not spectraloid, a contradiction. □

Example 3.7. Let $A = I/2$, where I is the identity operator on $\mathcal{H}$. Then A is normal and

$$R(A) = \begin{bmatrix} I/2 & 0 \\ \sqrt{3}I/2 & 0 \end{bmatrix}.$$

It is obvious that $\|R(A)\| = 1$, $r(R(A)) = 1/2$ and, by Lemma 3.2, $w(R(A)) = 3/4$. So

$$\|R(A)\| > w(R(A)) > r(R(A)).$$

Thus $R(A)$ is neither normaloid nor spectraloid.

3.2. The von Neumann property

The aim of this subsection is to prove the following theorem which gives a necessary and sufficient condition for partial isometries to be von Neumann operators.

Theorem 3.8. *A partial isometry T is a von Neumann operator if and only if either T is normal or $\sigma(T) = \overline{\mathbb{D}}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

We list some results that we shall use in the proof of Theorem 3.8.

Lemma 3.9 ([4, page 176, Cor. 3.7])**.** *If $T \in \mathcal{B}(\mathcal{H})$ is a von Neumann operator and Area $(\sigma(T)) = 0$, then T is normal. Here* Area *denotes area measure.*

Lemma 3.10 ([15, Theorem 1.4])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be a von Neumann operator. If $\lambda \in \partial\sigma(T)$, then $T \cong_a \lambda \oplus C$ for some operator C, where λ acts on some Hilbert space of dimension* 1 *and $\cong_a$ denotes approximate unitary equivalence. That is, there exist unitary operators U_n such that $U_n T U_n^* \to \lambda \oplus C$.*

Lemma 3.11 ([15, Corollary 4.7])**.** *If $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T) = \{z \in \mathbb{C} : |z| \leq \delta\}$ for some $\delta \geq 0$, then T is a von Neumann operator if and only if T is normaloid.*

Proof of Theorem 3.8. "$\Longleftarrow$". A normal operator is always a von Neumann operator. So we assume that $\sigma(T) = \overline{\mathbb{D}}$. Then $r(T) = 1 = \|T\|$. So, by Lemma 3.11, T is a von Neumann operator.

"$\Longrightarrow$". Now assume that T is a von Neumann operator. If $\ker T = \mathcal{H}$, then $T = 0$, a normal operator. If $\ker T = \{0\}$, then T is an isometry and T is the direct sum of a unitary operator and a unilateral shift (either may be absent). So either T is normal or $\sigma(T) = \overline{\mathbb{D}}$. Now it remains to treat the case that $\{0\} \subsetneq \ker T \subsetneq \mathcal{H}$. Clearly we have $r(T) = \|T\| = 1$, since $T \neq 0$ and T is normaloid.

Case 1. $(\mathbb{D}\backslash\{0\}) \cap \sigma(T) = \emptyset$.

It follows that $\sigma(T) \subset \{0\} \cup \partial\mathbb{D}$. Then $\mathrm{Area}(\sigma(T)) = 0$, and by Lemma 3.9, T is a normal operator.

Case 2. $(\mathbb{D}\backslash\{0\}) \cap \sigma(T) \neq \emptyset$.

In this case we shall show that $\sigma(T) = \overline{\mathbb{D}}$. For a proof by contradiction, we assume that $\mathbb{D} \nsubseteq \sigma(T)$. Note that $0 \in \sigma(T)$ and $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$. Then one can deduce that $\partial\sigma(T) \cap (\mathbb{D}\backslash\{0\}) \neq \emptyset$. Choose $\lambda \in \partial\sigma(T) \cap (\mathbb{D}\backslash\{0\})$. Then, by Lemma 3.10, $T \cong_a \lambda \oplus C$ for some operator C, where $\cong_a$ denotes approximate unitary equivalence. That is, there exist unitary operators U_n such that $U_n T U_n^* \to \lambda \oplus C$. Since partial isometries constitute a norm closed subset of $\mathcal{B}(\mathcal{H})$, it follows that $\lambda \oplus C$ is also a partial isometry. That is, $|\lambda|^2 \oplus C^*C$ is a projection. This is absurd since $0 < |\lambda| < 1$. This completes the proof. $\square$

4. Weyl's theorem

For $S \in \mathcal{B}(\mathcal{H})$, we denote $\pi_{00}(S) := \{\lambda \in \mathrm{iso}\,\sigma(S) : 0 < \dim\ker(S-\lambda) < \infty\}$. Also, by [3, Theorem XI.6.12], we have

$$\sigma_w(S) = \sigma_{lre}(S) \cup \{\lambda \in \mathbb{C} \setminus \sigma_{lre}(S) : \mathrm{ind}\ (\lambda - S) \neq 0\}.$$

Then Weyl's theorem holds for S if and only if $\sigma(S)\backslash\sigma_w(S) = \pi_{00}(S)$.

The main result of this section is the following theorem.

Theorem 4.1. *Let T be a nonzero partial isometry and let A be the compression of T to $(\ker T)^\perp$. Then Weyl's theorem holds for T if and only if either* (a) *Weyl's theorem holds for A, or* (b) $\dim\ker T = \infty$ *and* $[\sigma(A) \setminus \sigma_w(A)] \setminus \{0\} = \pi_{00}(A) \setminus \{0\}$.

Thus, if a contraction A satisfies Weyl's theorem, then so does $R(A)$. Later we shall provide an example to show that the converse does not hold (see Example 4.3).

Still, to give the proof of Theorem 4.1, we first make some preparation.

Lemma 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a partial isometry with $\{0\} \subsetneq \ker T \subsetneq \mathcal{H}$ and*

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{array}{c} (\ker T)^\perp \\ \ker T. \end{array}$$

Then $\pi_{00}(T) \cup \{0\} = \pi_{00}(A) \cup \{0\}$ and $\sigma_w(T) \cup \{0\} = \sigma_w(A) \cup \{0\}$.

Proof. Let $\lambda \in \mathbb{C}\setminus\{0\}$. By Lemma 3.3, $\lambda \in \operatorname{iso}\sigma(T)$ if and only if $\lambda \in \operatorname{iso}\sigma(A)$. By (2) in the proof of Lemma 3.3, we obtain $\pi_{00}(T)\cup\{0\} = \pi_{00}(A)\cup\{0\}$.

Again, using (3) in the proof of Lemma 3.3, ran $(\lambda - T)$ is closed if and only if ran $(\lambda - A)$ is closed. From the proof of Lemma 3.3, one can see $\dim\ker(\lambda - A)^* = \dim\ker(\lambda - T)^*$. Then $\lambda - A$ is Fredholm if and only if $\lambda - T$ is Fredholm; in this case, ind $(\lambda - T) =$ ind $(\lambda - A)$. Hence we conclude that $\sigma_w(T)\cup\{0\} = \sigma_w(A)\cup\{0\}$. ☐

Now we are going to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Denote $\mathcal{H}_1 = (\ker T)^\perp$ and $\mathcal{H}_2 = \ker T$. Then, relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, T can be written as

$$T = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}.$$

If $\dim\mathcal{H}_1 < \infty$, then T is of finite rank; if $\mathcal{H}_2 = \{0\}$, then T is an isometry. In both cases the result is clear. So we directly assume that $\dim\mathcal{H}_1 = \infty$ and $\mathcal{H}_2 \neq \{0\}$. Thus neither A nor B is absent.

For convenience, given an operator S, we write $S \in$ (W) to denote that Weyl's theorem holds for S.

"$\Longleftarrow$". We first prove that statement (a) implies $T \in$ (W). The proof is divided into three cases.

Case 1. $0 \notin \sigma(A)$.

By [11, Cor. 3.22], T is similar to $A \oplus 0$.

If $\dim\mathcal{H}_2 < \infty$, then we immediately have $\sigma(T) = \sigma(A)\cup\{0\}$, $\sigma_w(T) = \sigma_w(A)$ and $\pi_{00}(T) = \pi_{00}(A)\cup\{0\}$. Noting that $0 \notin \sigma(A)$, we obtain $0 \notin \sigma_w(A)\cup\pi_{00}(A)$. Hence

$$\begin{aligned}\pi_{00}(T) &= \pi_{00}(A)\cup\{0\} \\ &= \big(\sigma(A)\backslash\sigma_w(A)\big)\cup\{0\} \\ &= \big(\sigma(A)\cup\{0\}\big)\backslash\sigma_w(A) \\ &= \sigma(T)\backslash\sigma_w(T).\end{aligned}$$

That is, $T \in$ (W).

On the other hand, if $\dim\mathcal{H}_2 = \infty$, then $0 \notin \pi_{00}(T)$ and $0 \in \sigma_w(T)$, which leads to $\pi_{00}(T) = \pi_{00}(A)$ and $\sigma_w(T) = \sigma_w(A)\cup\{0\}$. So

$$\begin{aligned}\pi_{00}(T) = \pi_{00}(A) &= \sigma(A)\backslash\sigma_w(A) \\ &= (\sigma(A)\cup\{0\})\backslash(\sigma_w(A)\cup\{0\}) \\ &= \sigma(T)\backslash\sigma_w(T).\end{aligned}$$

That is, $T \in$ (W).

Case 2. $0 \in \sigma(A)$ and $0 \notin \sigma_w(A)$.

Since $A \in$ (W), that is, $\pi_{00}(A) = \sigma(A)\backslash\sigma_w(A)$, it follows that $0 \in \pi_{00}(A)$. So $0 \in \operatorname{iso}\sigma(T)$.

If $\dim \mathcal{H}_2 < \infty$, then one has $\sigma_w(A) = \sigma_w(T)$ and $0 \in \pi_{00}(T)$. By Lemma 4.2, the latter implies $\pi_{00}(T) = \pi_{00}(A)$. Since $A \in$ (W) and $\sigma(A) = \sigma(T)$, it follows immediately that $T \in$ (W).

If $\dim \mathcal{H}_2 = \infty$, then $0 \notin \pi_{00}(T)$ and $0 \in \sigma_w(T)$. It follows that $\pi_{00}(A) = \pi_{00}(T) \cup \{0\}$ and $\sigma_w(T) = \sigma_w(A) \cup \{0\}$. Hence

$$\begin{aligned}\pi_{00}(T) &= \pi_{00}(A) \setminus \{0\} = [\sigma(A)\backslash\sigma_w(A)] \setminus \{0\}\\ &= [\sigma(T)\backslash\sigma_w(A)] \setminus \{0\}\\ &= \sigma(T)\backslash\sigma_w(T),\end{aligned}$$

that is, $T \in$ (W).

Case 3. $0 \in \sigma_w(A)$.

Since $A \in$ (W), it follows that $0 \notin \pi_{00}(A)$. We also note that $\sigma(A) = \sigma(T)$.

We first consider the case that $\dim \mathcal{H}_2 < \infty$. Then $\sigma_w(T) = \sigma_w(A)$, which implies $0 \in \sigma_w(T)$. Note that B is finite-rank and $A^*A + B^*B = I$, where I is the identity operator on $\mathcal{H}_1$. Thus A is a semi-Fredholm operator. We claim that $0 \notin \pi_{00}(T)$. In fact, if not, then $0 \in \operatorname{iso} \sigma(T)$, which implies that $0 \in \operatorname{iso} \sigma(A)$. By Lemma 3.5, we obtain $0 \in \pi_{00}(A)$, a contraction. Then, in view of Lemma 4.2, we obtain

$$\sigma(T) \setminus \sigma_w(T) = \sigma(A) \setminus \sigma_w(A) = \pi_{00}(A) = \pi_{00}(T),$$

that is, $T \in$ (W).

On the other hand, when $\dim \mathcal{H}_2 = \infty$, one gets $0 \notin \pi_{00}(T)$ and $0 \in \sigma_w(T)$. Using Lemma 4.2 again, we obtain $T \in$ (W).

Now we shall prove that statement (b) implies $T \in$ (W).

Since $\dim \mathcal{H}_2 = \infty$, it follows that $0 \in \sigma_w(T)$ and $0 \notin \pi_{00}(T)$.

By Lemma 4.2, we have

$$\begin{aligned}\sigma(T) \setminus \sigma_w(T) &= [\sigma(A) \setminus \sigma_w(A)] \setminus \{0\}\\ &= \pi_{00}(A) \setminus \{0\} = \pi_{00}(T).\end{aligned}$$

Thus $T \in$ (W).

"$\Longrightarrow$". Now we assume that $T \in$ (W). We shall prove that either (a) or (b) holds. The proof is divided into four cases.

Case 1. $0 \notin \sigma(A)$.

By [11, Cor. 3.22], T is similar to $A \oplus 0$.

If $\dim \mathcal{H}_2 < \infty$, then we immediately have $\sigma(T) = \sigma(A) \cup \{0\}$, $\sigma_w(T) = \sigma_w(A)$ and $\pi_{00}(T) = \pi_{00}(A) \cup \{0\}$. Since $T \in$ (W), we have

$$\begin{aligned}\pi_{00}(A) \cup \{0\} &= \pi_{00}(T)\\ &= \sigma(T)\backslash\sigma_w(T)\\ &= \big(\sigma(A) \cup \{0\}\big)\backslash\sigma_w(A)\\ &= \big(\sigma(A)\backslash\sigma_w(A)\big) \cup \{0\}.\end{aligned}$$

Noting that $0 \notin \sigma(A)$, we obtain $\pi_{00}(A) = \sigma(A)\backslash\sigma_w(A)$. That is, $A \in$ (W).

On the other hand, if $\dim \mathcal{H}_2 = \infty$, then $0 \notin \pi_{00}(T)$ and $0 \in \sigma_w(T)$, which leads to $\pi_{00}(T) = \pi_{00}(A)$ and $\sigma_w(T) = \sigma_w(A) \cup \{0\}$. So

$$\begin{aligned}\pi_{00}(A) = \pi_{00}(T) &= \sigma(T)\backslash\sigma_w(T)\\ &= \big(\sigma(A) \cup \{0\}\big)\backslash\big(\sigma_w(A) \cup \{0\}\big)\\ &= \sigma(A)\backslash\sigma_w(A).\end{aligned}$$

That is, $A \in$ (W).

Case 2. $0 \in \sigma(A)$ and $0 \notin \sigma_w(T)$.

Since $T \in$ (W), it follows that $0 \in \pi_{00}(T)$ and $\dim \mathcal{H}_2 = \dim \ker T < \infty$. Then $\sigma_w(T) = \sigma_w(A)$ and $0 \notin \sigma_w(A)$. On the other hand, it is clear that $\sigma(T) = \sigma(A)$. Thus $0 \in \operatorname{iso} \sigma(A)$. By Lemma 3.5, we obtain $0 \in \pi_{00}(A)$. In view of Lemma 4.2, one can see that $\sigma(A)\backslash\sigma_w(A) = \pi_{00}(A)$.

Case 3. $0 \in \pi_{00}(A)$ and $0 \in \sigma_w(T)$.

$0 \in \pi_{00}(A)$ implies that $0 \in \operatorname{iso} \sigma(A) = \operatorname{iso} \sigma(T)$. Since $T \in$ (W), it follows that $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$ and $0 \notin \pi_{00}(T)$. Hence $\dim \mathcal{H}_2 = \dim \ker T = \infty$ and

$$[\sigma(T) \setminus \sigma_w(T)] \setminus \{0\} = \pi_{00}(T) \setminus \{0\}.$$

By Lemma 4.2, one can see that statement (b) holds.

Case 4. $0 \in [\sigma(A) \setminus \pi_{00}(A)]$ and $0 \in \sigma_w(T)$.

Since $T \in$ (W) and $0 \in \sigma_w(T)$, it follows that $0 \notin \pi_{00}(T)$. In view of Lemma 4.2, we have $\pi_{00}(T) = \pi_{00}(A)$. Note that $\sigma(T) = \sigma(A)$. It suffices to prove that $\sigma_w(T) = \sigma_w(A)$. In view of Lemma 4.2, we need only prove $0 \in \sigma_w(A)$.

For a proof by contradiction, we assume that $0 \notin \sigma_w(A)$. Then $\operatorname{ind} A = 0$ and $0 < \dim \ker A < \infty$.

If $0 \in \partial\sigma(A)$, then, by Lemma 3.4, $0 \in \operatorname{iso} \sigma(A)$. This means that $0 \in \pi_{00}(A)$, a contradiction.

If $0 \in \operatorname{int} \sigma(A)$, then there exists $\delta > 0$ such that $B(0, \delta) \subset \sigma(A)$ and $\operatorname{ind} (A - z) = 0$ for $z \in B(0, \delta) \setminus \{0\}$. Thus $B(0, \delta) \setminus \{0\} \subset [\sigma(A) \setminus \sigma_w(A)]$. In view of Lemma 4.2, we deduce that $B(0, \delta) \setminus \{0\} \subset [\sigma(T) \setminus \sigma_w(T)]$. This is absurd, since $T \in$ (W) implies that $\sigma(T) \setminus \sigma_w(T)$ is at most countable. This completes the proof. $\square$

Example 4.3. The classical Volterra integration operator on $L^2[0, 1]$ is defined by

$$(Vf)(t) = \int_0^t f(s)\mathrm{d}s, \quad \forall t \in [0, 1],$$

where $f \in L^2[0, 1]$. It is well known that $\sigma(V) = \{0\}$, $\|V\| \leq 1$ and $\dim \ker V = 0$. Denote $A = V \oplus 0$ acting on $L^2[0, 1] \oplus \mathbb{C}$. Thus $\sigma(A) = \sigma_w(A) = \pi_{00}(A) = \{0\}$. Hence Weyl's theorem does not hold for A. Note that $\sigma(R(A)) = \sigma_w(R(A)) = \{0\}$ and $\pi_{00}(R(A)) = \emptyset$. Then $R(A) \in$ (W).

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