J. Korean Math. Soc. **56** (2019), No. 6, pp. 1463–1474
https://doi.org/10.4134/JKMS.j180740
pISSN: 0304-9914 / eISSN: 2234-3008

NUMBER OF WEAK GALOIS-WEIERSTRASS POINTS WITH WEIERSTRASS SEMIGROUPS GENERATED BY TWO ELEMENTS

Jiryo Komeda and Takeshi Takahashi

Abstract. Let C be a nonsingular projective curve of genus ≥ 2 over an algebraically closed field of characteristic 0. For a point P in C, the Weierstrass semigroup $H(P)$ is defined as the set of non-negative integers n for which there exists a rational function f on C such that the order of the pole of f at P is equal to n, and f is regular away from P. A point P in C is referred to as a weak Galois-Weierstrass point if P is a Weierstrass point and there exists a Galois morphism $\varphi : C \to \mathbb{P}^1$ such that P is a total ramification point of φ. In this paper, we investigate the number of weak Galois-Weierstrass points of which the Weierstrass semigroups are generated by two positive integers.

1. Introduction and theorem

In our previous paper [7], we investigated a number of relations between weak Galois-Weierstrass points and Galois points. In this paper, to perform a similar study of Galois point theory, we present an investigation of the number of weak Galois-Weierstrass points, of which, the Weierstrass semigroups are generated by two positive integers. First, let us recall the motives and several definitions in brief.

In this paper, a curve refers to a complete nonsingular algebraic curve over an algebraically closed field k of characteristic 0. A plane curve refers to a (complete nonsingular) curve in $\mathbb{P}^2$.

Yoshihara introduced the notion of a Galois point for a plane curve as follows.

Definition 1.1 ([9, 11])**.** Let C be a plane curve of degree $d \geq 3$. For a point $P \in \mathbb{P}^2$, the projection $\pi_P : C \to \mathbb{P}^1$ from P induces an extension of function fields $\pi_P^* : k(\mathbb{P}^1) \hookrightarrow k(C)$. P is referred to as a *Galois point* for C if the extension is Galois. Moreover, when $P \in C$ or when $P \notin C$, the point is said to be an inner or outer Galois point, respectively.

Received November 1, 2018; Revised June 13, 2019; Accepted July 25, 2019.
2010 *Mathematics Subject Classification.* Primary 14H55; Secondly 14H50, 14H30, 20M14.
Key words and phrases. weak Galois-Weierstrass point, Weierstrass semigroup of a point.
This work was supported by JSPS KAKENHI Grant Numbers 15K04830 and 16K05094.

One of the most representative results pertaining to Galois points is as follows.

Theorem 1.1 ([5, 9, 11])**.** *Let C be a plane curve of degree d. Let $(X : Y : Z)$ be a system of homogeneous coordinates of $\mathbb{P}^2$.*

(1) *If $d \geq 5$, then the number of inner Galois points for C equals 0 or 1.*
(2) *If $d = 4$, then the number of inner Galois points for C equals 0, 1 or 4. Moreover, the number of inner Galois points equals 4 if and only if C is projectively equivalent to the curve $XZ^3 + X^4 + Y^4 = 0$.*
(3) *If $d \geq 3$, then the number of outer Galois points for C equals 0, 1 or 3. Moreover, the number of outer Galois points equals 3 if and only if C is projectively equivalent to the curve $X^d + Y^d + Z^d = 0$.*
(4) *Assume that $d \geq 4$. There exist both an inner Galois point and an outer Galois point for C if and only if C is projectively equivalent to the curve $XZ^{d-1} + X^d + Y^d = 0$.*

Remark 1.1. Several researchers have studied various other problems relating to Galois points (see [6]).

The Galois points for a plane curve are characterized as weak Galois-Weierstrass points, which are described as follows. Let $\mathbb{Z}_{\geq 0}$ be the set of all non-negative integers.

Definition 1.2 ([10])**.** Let C be a curve of genus $g \geq 2$. A point $P \in C$ is termed a *Galois-Weierstrass point* (GW point), if $\Phi_{|aP|} : C \to \mathbb{P}^1$ is a Galois covering, where a is the smallest positive integer of the Weierstrass semigroup

$$H(P) := \{n \in \mathbb{Z}_{\geq 0} \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\}.$$

Remark 1.2. A GW point may be a non-Weierstrass point. However, in this study, we are only interested in GW points that are Weierstrass points.

Definition 1.3. Let C be a curve of genus $g \geq 2$. We refer to $P \in C$ as a *weak GW point* if

(1) P is a total ramification point of some Galois covering $f : C \to \mathbb{P}^1$, and
(2) P is a Weierstrass point of C.

Moreover, if P is a weak GW point and not a GW point, we refer to P as a *pseudo-GW point.* For a weak GW point P, we denote

$$\mathrm{degGW}(P) := \{\deg f \mid \text{Galois covering } f : C \to \mathbb{P}^1 \text{ which is totally ramified at } P\}$$

and we refer to it as *the set of degrees of the weak GW point.*

Remark 1.3. (1) If P is a weak GW point and $a \in \mathrm{degGW}(P)$, where a is the least positive integer in $H(P)$, then P is a GW point.

(2) On Definition 1.3, the Galois group of the Galois extension $k(C)/f^*(k(\mathbb{P}^1))$ is isomorphic to the group $\mathrm{Gal}(f) := \{\sigma \in \mathrm{Aut}(C) \mid f \circ \sigma = f\}$, and the group is a cyclic group since $\sigma(P) = P$ for every $\sigma \in \mathrm{Gal}(f)$.

We denote $\langle a, b\rangle$ as the semigroup generated by elements $a, b \in \mathbb{N}$.

Theorem 1.2 (Theorem 2.3 in [7])**.** *If a point $P \in \mathbb{P}^2$ is a Galois point for a plane curve C, then some of the ramification points of π_P are weak GW points with $H(Q) = \langle d-1, d\rangle$. More precisely,*

1. *if P is an inner Galois point, then P is a GW point, and $H(P) = \langle d-1, d\rangle$.*
2. *if P is an outer Galois point, then every ramification point Q of π_P is a weak GW point with $H(Q) = \langle d-1, d\rangle$.*

Conversely, if Q is a weak GW point of a curve C with $H(Q) = \langle d-1, d\rangle$, then C is isomorphic to a plane curve of degree d and Q is a ramification point of the projection from a Galois point.

In the study presented in this paper, as in Theorem 1.1, we investigate the number of weak GW points of which the Weierstrass semigroups are generated by two positive integers.

Our main theorem is as follows.

Theorem 1.3. *Let $a, b \in \mathbb{N}$ satisfy that $\gcd(a, b) = 1$, $2 < a$ and $a+1 < b$. Let C be a curve.*

1. *If $b \equiv a-1 \pmod{a}$, then the number of GW points $P \in C$ with $H(P) = \langle a, b\rangle$ is 0 or $b+1$. If $b \not\equiv a-1 \pmod{a}$, then the number is equal to 0 or 1.*
2. *The number of weak GW points $P \in C$ with $H(P) = \langle a, b\rangle$ and $b \in \mathrm{degGW}(P)$ is equal to 0 or 1.*
3. *There exists a weak GW point $P \in C$ with $H(P) = \langle a, b\rangle$ and $a, b \in \mathrm{degGW}(P)$ if and only if C is birationally equivalent to the singular plane curve $X^b = Y^a Z^{b-a} + Z^b$.*

It must be noted that by Theorem 1.2, for the case of $a+1 = b$, studying weak GW points with $H(P) = \langle a, b\rangle$ is the same as studying the Galois points for plane curves. Thus, we have Theorem 1.1.

On the other hand, Coppens proved the following, recently.

Theorem 1.4 ([4])**.** *Let $a, b \in \mathbb{N}$ satisfy that $\gcd(a, b) = 1$, $2 < a$ and $a+1 < b$. Assume that $b \not\equiv a-1 \pmod{a}$. Let C be a curve. Then, the number of Weierstrass points $P \in C$ with $H(P) = \langle a, b\rangle$ is equal to 0 or 1.*

By Theorem 1.4, when $b \not\equiv a-1 \pmod{a}$, Theorem 1.3(1) and (2) are obvious; this notwithstanding, we present our proof in this paper.

In Section 2, we provide some preliminary results on weak GW points and use these results for the proof of Theorem 1.3. The proof of this theorem appears in Section 3. Section 4 contains some examples of weak GW points.

2. Preliminary

In this section, let a, b be integers such that $1 < a < b$ and $\gcd(a, b) = 1$.

Lemma 2.1. *If there exists a point $P \in C$ with $H(P) = \langle a, b\rangle$, then $g = (a-1)(b-1)/2$ and $K \sim (2g-2)P$.*

Proof. Let $r : \{1, \dots, a-1\} \to \{1, \dots, a-1\}$ be the map given by $r(i) \equiv bi \pmod{a}$. Since $\gcd(a, b) = 1$, the map r is bijective. In particular, $\sum_{i=1}^{a-1} r(i) = \sum_{i=1}^{a-1} i = a(a-1)/2$. For $i = 0, \dots, a-1$, we have that $ib \in H(P)$ and $ib - a \notin H(P)$. Hence, $\mathbb{Z}_{\geq 0} \setminus H(P) = \{ib - ja \mid i = 1, \dots, a-1,\, j \in \mathbb{N},\, ib - ja > 0\}$ and

$$g = \#(\mathbb{Z}_{\geq 0} \setminus H(P)) = \sum_{i=1}^{a-1} (ib - r(i))/a = (a-1)(b-1)/2.$$

According to the Riemann-Roch theorem, $\dim H^0(C, \mathcal{O}_C((2g-1)P)) = g$. Since $2g - 1 = (a-1)b - a \notin H(P)$, we have $\dim H^0(C, \mathcal{O}_C((2g-2)P)) = g$. The Riemann-Roch theorem has that $\dim H^0(C, \mathcal{O}_C(K_C - (2g-2)P)) = 1$. Hence, we see $K \sim (2g-2)P$. □

Based on the proof of Theorem 10.1 and Corollary 6.3 in [2], or Proposition 1 in [4], we have the following.

Theorem 2.1 ([2,4])**.** *Assume that $a + 1 < b$ and there exists a point P on C with $H(P) = \langle a, b\rangle$. Then, the gonality of C is equal to a, and a base point free a-gonal pencil of C is unique, that is $|aP|$.*

The two assertions below are fundamental on study of a weak GW point P of a curve C with $H(P) = \langle a, b\rangle$.

Proposition 2.1 ([7])**.** *If P is a weak GW point of a curve C with $H(a) = \langle a, b\rangle$, then $a \in \mathrm{degGW}(P)$ or $b \in \mathrm{degGW}(P)$.*

Proposition 2.2 ([7])**.** *Let P be a weak GW point of a curve C with $a \in \mathrm{degGW}(P)$ (resp. $b \in \mathrm{degGW}(P)$) and $H(P) = \langle a, b\rangle$. Then, there exist rational functions x and $y \in k(C)$ with $(x)_\infty = aP$ and $(y)_\infty = bP$ such that $k(C) = k(x, y)$ and $y^a = \prod_{i=1}^{b} (x - c_i)$ (resp. $x^b = \prod_{i=1}^{a} (y - c_i)$), where $c_1, c_2, \dots$ are mutually distinct elements in k.*

Remark 2.1. Let $f : C \to \mathbb{P}^1$ be a Galois morphism of degree a or b such that P is a total ramification point. Then, every ramification point of f is a total ramification point.

Some results on the relations between Galois points for a plane curve and weak GW points with certain Weierstrass semigroups through double coverings are described in [7]. We first review one of these results as follows, after which we use it below.

Lemma 2.2 ([7])**.** *Let P be a weak GW point of a curve C with $H(P) = \langle a, 2a-1\rangle$ and $2a - 1 \in \mathrm{degGW}(P)$. Let $\varphi : C \to \mathbb{P}^1$ be a Galois morphism of degree $2a-1$ such that P is a total ramification point. Let $P_1, \dots, P_{2a-1}$ be mutually distinct points such that $P_1 + \cdots + P_{2a-1}$ is a fiber of φ. Then, there exists a nonsingular plane curve $D \subset \mathbb{P}^2$ of degree $2a$ and a double covering*

$f : D \to C$ such that $\mathrm{Branch}(f) = \{P, P_1, \dots, P_{2a-1}\}$ *and $f^{-1}(P)$ is an inner Galois point for D.*

Proof. We construct a double covering by the method in [1, I.17] as follows. Let $\mathcal{L} := \mathcal{O}_C(aP)$, L be its total space, and $p : L \to C$ be the bundle projection. Note that $2aP \sim P + P_1 + \cdots + P_{2a-1}$. Let s be a global section of $\mathcal{L}^{\otimes 2}$ such that $\mathrm{div}(s) = P + P_1 + \cdots + P_{2a-1}$. Let $t \in \Gamma(L, p^*\mathcal{L})$ be a tautological section. Then, the zero divisor of $p^*s - t^2$ defines a nonsingular curve in L, say D. Let $f := p|_D : D \to C$. We have that f is a double covering such that $\mathrm{Branch}(f) = \{P, P_1, \dots, P_{2a-1}\}$. Remark that this double covering depends on only the branch locus and $\mathcal{L}$. To show that D is a nonsingular plane curve and $f^{-1}(P)$ is a Galois point, let us investigate this construction in detail.

By Proposition 2.2, we may assume that C is the compactification of the affine plane curve $x^{2a-1} = \prod_{i=1}^{a}(y - c_i)$. Let C' be the singular plane curve given by $X^{2a-1} = Z^{a-1}\prod_{i=1}^{a}(Y - c_iZ)$, and $\rho : C \to C'$ be the resolution of singularities. Note that C' has a cusp $(0 : 1 : 0)$ and $\rho(P) = (0 : 1 : 0)$. Then, $x = \rho^*((X/Z)|_{C'})$, $y = \rho^*((Y/Z)|_{C'})$, and $\varphi : C \to \mathbb{P}^1$ is given by the projection $\pi_{(0:1:0)} : (x : y : 1) \mapsto (y : 1)$. We may assume that $P_1 + \cdots + P_{2a-1}$ is the fiber $\varphi^{-1}((0 : 1))$, i.e., we may assume that $(y)_0 = P_1 + \cdots + P_{2a-1}$.

Let $U_1 := C \setminus \{P\}$, and U_2 be a sufficiently small neighborhood of P. Then, $\{(U_1, 1), (U_2, 1/x)\}$ is a system of local equations of aP. Hence, $\{g_{11} = 1, g_{12} = x, g_{21} = 1/x, g_{22} = 1\}$ is a system of transition functions of $\mathcal{L}$. The total space L is obtained by the gluing of $U_1 \times \mathbb{A}^1$ and $U_2 \times \mathbb{A}^1$ with the equivalent relation $(Q_1, t_1) \sim (Q_2, t_2) \Leftrightarrow Q_1 = Q_2, t_1 = g_{12}t_2$. The system of transition functions of $\mathcal{L}^{\otimes 2}$ is given by $g_{12}^2 = x^2$. Let s be the section given by the system of local equations $\{(U_1, y), (U_2, y/x^2)\}$. Then, $\mathrm{div}(s) = P + P_1 + \cdots + P_{2a-1}$. Hence, D is given by $\{(U_1 \times \mathbb{A}^1, t_1^2 - y), (U_2 \times \mathbb{A}^1, t_2^2 - y/x^2)\}$.

We see that D is isomorphic to the plane curve defined by $X^{2a-1}Z = \prod_{i=1}^{a}(Y^2 - c_iZ^2)$. Indeed, $D \cap (U_1 \times \mathbb{A}^1)$ is isomorphic to the affine curve defined by $x^{2a-1} - \prod_{i=1}^{a}(y - c_i) = 0$ and $t_1^2 - y = 0$ in $\mathbb{A}^3$, and this is isomorphic to the affine plane curve defined by $x^{2a-1} = \prod_{i=1}^{a}(t_1^2 - c_i)$ in $\mathbb{A}^2$. On the other hand, the plane curve $X^{2a-1}Z = \prod_{i=1}^{a}(Y^2 - c_iZ^2)$ is nonsingular, and contains the affine plane curve $x^{2a-1} = \prod_{i=1}^{a}(t_1^2 - c_i)$ as a Zariski open set.

We treat D as the plane curve $X^{2a-1}Z = \prod_{i=1}^{a}(Y^2 - c_iZ^2)$. Since f is given by $D \ni (X : Y : Z) \mapsto (XZ : Y^2 : Z^2) \in C'$, we have that $f^{-1}(P) = \{(1 : 0 : 0)\}$. By [11, Proposition 5], the point $(1 : 0 : 0)$ is an inner Galois point for D. □

3. Proof of Theorem 1.3

Let $a, b \in \mathbb{N}$ satisfy that $\gcd(a, b) = 1$, $2 < a$ and $a + 1 < b$. We prove Theorem 1.3.

3.1. Proof of Theorem 1.3(1)

We prove that if $b \not\equiv a-1 \pmod a$, then the number of Weierstrass points P with $H(P) = \langle a, b\rangle$ equals 0 or 1. In order to demonstrate the contraposition, let P and Q be two distinct GW points with $H(P) = H(Q) = \langle a, b\rangle$. By Theorem 2.1, we have that $aP \sim aQ$. By Lemma 2.1, we have that $(2g-2)P \sim (2g-2)Q$, where $g = (a-1)(b-1)/2$. Thus, $(b+1)P \sim (b+1)Q$, that is, $b+1 \in H(P) = \langle a, b\rangle$. Hence, $b+1 \equiv 0 \pmod a$.

We assume $b \equiv a-1 \pmod a$ and we prove that the number of GW points P with $H(P) = \langle a, b\rangle$ equals 0 or $b+1$. Let P be a GW point with $H(P) = \langle a, b\rangle$. By Theorem 2.2, there exist $x, y \in k(C)$ such that $k(x, y) = k(C)$, $(x)_\infty = aP$, $(y)_\infty = bP$ and $y^a = \prod_{i=1}^{b}(x - c_i)$, where $c_1, \ldots, c_b \in k$ are mutually distinct. Then, we have that the morphism $\Phi_{|aP|}$ is given by the function x, and the ramification point Q_i holds for $x(Q_i) = c_i$ and $y(Q_i) = 0$. The number of these points $P, Q_1, Q_2, \ldots, Q_b$ equals $b+1$, and $aP \sim aQ_i$ for every i. By Theorem 2.1, if a point Q holds $H(Q) = \langle a, b\rangle$, Q must be P or Q_i. Let us prove that $H(Q_i) = \langle a, b\rangle$. Let m be an integer such that $b = (m+1)a - 1$. Let $\tilde{x}_i := 1/(x - c_i)$ and $\tilde{y}_i := \tilde{x}_i^{m+1} y$. Because $\mathrm{div}(x - c_i) = aP - aQ_i$ and $\mathrm{div}(y) = Q_1 + \cdots + Q_b - bP$, we have that $\mathrm{div}(\tilde{x}_i) = aP - aQ_i$ and $\mathrm{div}(\tilde{y}_i) = (m+1)(aP - aQ_i) + Q_1 + \cdots + Q_b - bP = (Q_1 + \cdots + Q_b - Q_i) + P - bQ_i$. Hence, $a, b \in H(Q_i)$; thus, $\langle a, b\rangle \subset H(Q_i)$. Since the number of elements of $\mathbb{Z}_{\geq 0} \setminus H(Q_i) = g = \mathbb{Z}_{\geq 0} \setminus \langle a, b\rangle$, we conclude that $H(Q_i) = \langle a, b\rangle$.

Remark 3.1. Let P be a GW point with $H(P) = \langle a, b\rangle$. In the case that $b \not\equiv a-1 \pmod a$, we can also calculate the Weierstrass semigroup $H(Q_i)$, where Q_i is a ramification point of $\Phi_{|aP|}$. See [8].

3.2. Proof of Theorem 1.3(2)

We prove the theorem by contradiction by assuming that there exist two weak GW points P and Q on C with $H(P) = H(Q) = \langle a, b\rangle$ and $b \in \mathrm{degGW}(P)$, $b \in \mathrm{degGW}(Q)$.

Claim 3.1. $b \equiv a-1 \pmod a$.

By Theorem 2.1 and Lemma 2.1, we have $aP \sim aQ$ and $(2g-2)P \sim (2g-2)Q$. Since $g = (a-1)(b-1)/2$, we have $(b+1)P \sim (b+1)Q$. That is, $b+1$ is a non-gap value of P. Since $H(P) = \langle a, b\rangle$, we have $b+1 \equiv 0 \pmod a$, so, $b \equiv a-1 \pmod a$.

Claim 3.2. The number of weak GW points R with $H(R) = \langle a, b\rangle$ and $b \in \mathrm{degGW}(R)$ equals $b+1$.

By Theorem 2.1, for a point R with $H(R) = \langle a, b\rangle$, we have $aP \sim aR$. By the Riemann-Hurwitz formula, the number of total ramification points of $\Phi_{|aP|} : C \to \mathbb{P}^1$ is at most $b+1$. Hence, the number of points R with $H(R) = \langle a, b\rangle$ is at most $b+1$.

From Proposition 2.2, we see that for a Galois covering $\varphi : C \to \mathbb{P}^1$ of degree b such that P is a total ramification point, every ramification point is a total ramification point. Let σ be a generator of the Galois group belonging to φ, which is a cyclic group. Then, the set of ramification points of φ coincides with the set of fixed points of σ. We have $\sigma(Q) \neq Q$. Indeed, if $\sigma(Q) = Q$, then $bP \sim bQ$. Since $aP \sim aQ$ and $\gcd(a, b) = 1$, we have $P \sim Q$, and this is a contradiction. Since every ramification point of φ is a total ramification point, the number of elements of $\{\sigma^i(Q) \mid i = 0, \dots, b-1\}$ equals b. Let $P_\infty := P$, $P_i := \sigma^{i-1}(Q)$ $(i = 1, \dots, b)$. Every P_i is a weak GW point and $H(P_i) = \langle a, b\rangle$ and $b \in \mathrm{degGW}(P_i)$. Hence, we conclude Claim 3.2.

For P_i $(i = 1, \dots, b, \infty)$ above, let $\varphi_i : C \to \mathbb{P}^1$ be a Galois covering of degree b such that P_i is a total ramification point, and let σ_i be a generator of the Galois group $\mathrm{Gal}(\varphi_i)$ belonging to φ_i. Let $G := \langle \sigma_\infty, \sigma_1, \dots, \sigma_b\rangle$.

Claim 3.3. $\#G \geq b^2$.

Since $G \supset \{1\} \cup \{\sigma_i^l \mid i = \infty, 1, \dots, b,\ l = 1, \dots, b-1\}$ and these b^2 elements are mutually distinct, we conclude Claim 3.3.

Let $\mathrm{Branch}(\pi) = \{y_1, \dots, y_n\}$ be the branch locus of the natural map $\pi : C \to C/G$.

Claim 3.4. $\pi^{-1}(\pi(P_\infty)) = \{P_\infty, P_1, \dots, P_b\}$.

By the definition of P_i, $\{\sigma_\infty^l(P_1) \mid l = 0, \dots, b-1\} = \{P_1, P_2, \dots, P_b\}$. Since the number of weak GW points P with $H(P) = \langle a, b\rangle$ and $b \in \mathrm{degGW}(P)$ equals $b+1$, we have $\{\sigma_1^l(P_\infty) \mid l = 0, \dots, b-1\} = \{P_\infty, P_2, \dots, P_b\}$. Hence, $\{P_\infty, P_1, \dots, P_b\} \subset \pi^{-1}(\pi(P_\infty))$.

For a point $R \in \pi^{-1}(\pi(P_\infty))$, there exists $\sigma \in G$ such that $\sigma(P_\infty) = R$. The point R is a weak GW point with $H(R) = \langle a, b\rangle$ and $b \in \mathrm{degGW}(R)$. Hence, $R \in \{P_\infty, P_1, \dots, P_b\}$. Namely, $\{P_\infty, P_1, \dots, P_b\} \supset \pi^{-1}(\pi(P_\infty))$. We conclude Claim 3.4.

Let $y_1 = \pi(P_\infty)$. Let Q_∞ be a ramification point of φ_∞ such that $Q_\infty \neq P_\infty$. Since $Q_\infty \notin \{P_\infty, P_1, \dots, P_b\}$, we have $\pi(Q_\infty) \neq y_1$. Since $\sigma_\infty(Q_\infty) = Q_\infty$, we have $\pi(Q_\infty) \in \mathrm{Branch}(\pi)$. Let $y_2 = \pi(Q_\infty)$. For $y_i \in \mathrm{Branch}(\pi)$ $(i = 1, \dots, n)$, let r_i be the ramification number of π at a point in $\pi^{-1}(y_i)$. (Remark that the ramification number at every point in $\pi^{-1}(y_i)$ equals r_i, since π is Galois.) We may assume $r_3 \leq r_4 \leq \cdots \leq r_n$.

Claim 3.5. $r_1 \geq b$, $r_2 \geq b$ and $C/G \cong \mathbb{P}^1$.

Both P_∞ and Q_∞ are ramification points of $\varphi_\infty : C \to \mathbb{P}^1 = C/\langle\sigma_\infty\rangle$ with ramification number b. Since $\langle\sigma_\infty\rangle$ is a subgroup of G, the map π can be factored as $\pi : C \to C/\langle\sigma_\infty\rangle \to C/G$. Hence, $r_1 \geq b$ and $r_2 \geq b$. Since $C/\langle\sigma_\infty\rangle \cong \mathbb{P}^1$, we have $C/G \cong \mathbb{P}^1$.

Claim 3.6. $b = 2a - 1$.

By the Riemann-Hurwitz formula, $2g-2 = \#G(-2) + \sum_{i=1}^n \#G/r_i \cdot (r_i - 1)$. Let $T := -2 + \sum_{i=1}^n (1 - 1/r_i)$. Then, $\#G = (2g-2)/T$, and T is a positive rational number. When $n=3$ and $(r_1,r_2,r_3)=(b,b,2)$, we have the minimum of T and therefore the maximum of $\#G$. In this case, $T=-2+2(b-1)/b+1/2=(b-4)/(2b)$. Hence, $\#G\le 2b/(b-4)\cdot(2g-2)=2b(ab-a-b-1)/(b-4)$. By $b^2\le\#G$, we have $b^2(b-4)\le 2b(ab-a-b-1)$, so $b^2-2(a+1)b+2(a+1)\le 0$. Hence, $b\le a+1+\sqrt{(a+1)^2-2(a+1)}<2a+1$. By Claim 3.1, we have $b=2a-1$.

Since P_∞ is a weak GW point with $H(P_\infty)=\langle a,b\rangle$, $b=2a-1$ and $P_1+\cdots+P_b$ is a fiber of φ_∞, by Lemma 2.2, there exist a nonsingular plane curve $D_\infty\subset\mathbb{P}^2$ and a double covering $f_\infty: D_\infty\to C$ such that $\mathrm{Branch}(f_\infty)=\{P_\infty,P_1,\ldots,P_b\}$ and $\bar{P}_\infty:=f_\infty^{-1}(P_\infty)$ is an inner Galois point for D_∞. We also have that P_1 is a weak GW point with $H(P_1)=\langle a,b\rangle$, $b=2a-1$ and $P_2+\cdots+P_b+P_\infty$ is a fiber of φ_1. Hence, there also exist a nonsingular plane curve $D_1\subset\mathbb{P}^2$ and a double covering $f_1: D_1\to C$ such that $\mathrm{Branch}(f_1)=\{P_\infty,P_1,\ldots,P_b\}$ and $\bar{P}_1:=f_1^{-1}(P_1)$ is an inner Galois point for D_1.

Claim 3.7. The two double coverings f_∞ and f_1 are isomorphic. Namely, there exists an isomorphism $h: D_\infty\to D_1$ such that $f_\infty=f_1\circ h$.

The ramification locus of f_1 coincides with the ramification locus of f_∞, and f_1 is obtained by $\mathcal{O}(aP_1)$. Since $aP_\infty\sim aP_1$, we have an isomorphism $\mathcal{O}(aP_\infty)\cong\mathcal{O}(aP_1)$. From the isomorphism between the line bundles, we have h, and we conclude Claim 3.7.

Since D_∞ and D_1 are nonsingular plane curves of degree $2a>3$, by [3], the isomorphism $h: D_\infty\to D_1$ can be extended to a projective transformation of $\mathbb{P}^2$. Namely, there exists $H\in\mathrm{Aut}(\mathbb{P}^2)$ such that $H(D_\infty)=D_1$ and $H|_{D_\infty}=h$. Then, $H(\bar{P}_\infty)$ must be an inner Galois point for D_1. By Theorem 1.1, the number of inner Galois points for each D_∞ and D_1 is at most one. Hence, $h(\bar{P}_\infty)=\bar{P}_1$. Since $f_\infty=f_1\circ h$, we have $P_\infty=P_1$, and this is a contradiction.

Now we conclude Theorem 1.3(2).

3.3. Proof of Theorem 1.3(3)

Let P be a weak GW point with $H(P)=\langle a,b\rangle$ and $a,b\in\mathrm{degGW}(P)$. Especially, $b\in\mathrm{degGW}(P)$, thus by Proposition 2.2, there exist rational functions x and $y\in k(C)$ with $(x)_\infty=aP$ and $(y)_\infty=bP$ such that $k(C)=k(x,y)$ and $x^b=\prod_{i=1}^{a}(y-c_i)$, where each $c_1,\ldots,c_a$ are mutually distinct. By Theorem 2.1, a base point free a-gonal pencil is unique, it must be given by x. By Remark 2.1, there exist $\alpha,\beta\in k$ such that $\prod_{i=1}^{a}(y-c_i)-\alpha^b=(y-\beta)^a$. Note that $\alpha\ne 0$. Let $\bar{x}:=x/\alpha$ and $\bar{y}:=(y-\beta)\cdot\alpha^{-b/a}$. Now we obtain the relation $\bar{x}^b=\bar{y}^a+1$. Hence, C is birationally equivalent to the singular plane curve defined by $X^b=Y^aZ^{b-a}+Z^b$.

We prove the converse. Let C' be a singular plane curve given by $X^b=Y^aZ^{b-a}+Z^b$ and $P':=(0:1:0)$. We have $\mathrm{Sing}(C')=\{P'\}$, where P' is a cusp. Let $\rho: C\to C'$ be the resolution of singularity and $P:=\rho^{-1}(P')\in C$. We have that $g:=g(C)=(a-1)(b-1)/2$. Let $x:=X/Z|_{C'}, y:=Y/Z|_{C'}\in k(C')=k(C)$. Then, $(x)_\infty=aP$ and $(y)_\infty=bP$. Namely, $a,b\in H(P)$.

Since $g = (a-1)(b-1)/2 = \#(\mathbb{Z}_{\geq 0} \setminus \langle a, b\rangle) = \#(\mathbb{Z}_{\geq 0} \setminus H(P))$, we have that $H(P) = \langle a, b\rangle$. The morphism $\Phi_{|aP|}$ is given by x. Since $y^a = x^b - 1$, where $\Phi_{|aP|}$ is Galois and P is a total ramification point. Let $f : C \to \mathbb{P}^1$ be the morphism given by y. Since $x^b = y^a + 1$, we have that $\deg f = b$, f is Galois, and P is a total ramification point of f. Therefore, P is a weak GW point with $H(P) = \langle a, b\rangle$ and $a, b \in \mathrm{degGW}(P)$.

4. Examples

The following is an example in which a curve has one GW point with semigroup $\langle a, b\rangle$.

Example 4.1. Let C' be the singular plane curve $Y^3Z^4 = X^7 - XZ^6$ and $P'_\infty := (0 : 1 : 0)$. Then, $\mathrm{Sing}(C') = \{P'_\infty\}$ and P'_∞ is a cusp. Let $\rho : C \to C'$ be the resolution of singularity, and $P_\infty := \rho^{-1}(P'_\infty)$. We see the genus $g(C) = 6$. Let $x := (X/Z)|_{C'}, y := (Y/Z)|_{C'} \in k(C') = k(C)$. Then, $k(C) = k(x, y)$, $(x)_\infty = 3P_\infty$, and $(y)_\infty = 7P_\infty$. Hence, $H(P_\infty) = \langle 3, 7\rangle$.

The morphism $\Phi_{|3P_\infty|} : C \to \mathbb{P}^1$ corresponds to x. Since $y^3 = x^7 - x$, we have that $\Phi_{|3P_\infty|}$ is Galois, i.e., P_∞ is a GW point with $H(P_\infty) = \langle 3, 7\rangle$. Let $P'_1 := (0 : 0 : 1)$, $P'_i := (\zeta_6^{i-2} : 0 : 1)$ $(i = 2, \dots, 7)$, where ζ_6 is a primitive 6th root of unity. Let $P_i := \rho^{-1}(P'_i)$ $(i = 1, \dots, 7)$. Then, the ramification locus of $\Phi_{|3P_\infty|}$ is $\{P_\infty, P_1, \dots, P_7\}$. Hence, P_i $(i = 1, \dots, 7)$ is also a GW point. However, $H(P_i) \neq \langle 3, 7\rangle$. In fact, $(1/x)_\infty = 3P_1$, $(y/x^3)_\infty = 8P_1$, $(y^2/x^5)_\infty = 13P_1$, $(1/(x - \zeta_6^{i-2}))_\infty = 3P_i$, $(y/(x - \zeta_6^{i-2})^3)_\infty = 8P_i$ and $(y^2/(x - \zeta_6^{i-2})^5)_\infty = 13P_i$, where $i = 2, \dots, 7$. Hence, $H(P_i) = \langle 3, 8, 13\rangle$ since $g(C) = 6 = \#(\mathbb{Z}_{\geq 0} \setminus H(P_i))$.

We see $7 \notin \mathrm{degGW}(P_\infty)$. To prove this by contradiction, let us assume that there exists a Galois covering $f : C \to \mathbb{P}^1$ of degree 7 such that P_∞ is a total ramification point of f. Then, f is given by a linear subsystem of $|7P_\infty|$, hence f is the composition of $\Phi_{|7P_\infty|} : C \to \mathbb{P}^3$ and the projection $\pi_L : \mathbb{P}^3 \cdots \to \mathbb{P}^1$ from some line L. Let $(X_0 : X_1 : X_2 : X_3)$ be a system of homogeneous coordinates of $\mathbb{P}^3$. Let $\Phi_{|7P_\infty|}$ be given by $P \mapsto (1 : x(P) : x^2(P) : y(P))$. We see that P_i $(i = 1, \dots, 7)$ is not a ramification point of f. Indeed, if P_i is a ramification point of f, then $7P_\infty \sim 7P_i$. Since $3P_\infty \sim 3P_i$, we have $P_\infty \sim P_i$, and this contradicts $C \not\cong \mathbb{P}^1$. Since a point in $f^{-1}(f(P_i))$ is a point with the Weierstrass semigroup $\langle 3, 8, 13\rangle$. By Theorem 2.1, all points with the Weierstrass semigroup $\langle 3, 8, 13\rangle$ are $P_1, \dots, P_7$. We have that the line L and 7 points $\Phi_{|7P_\infty|}(P_1), \dots, \Phi_{|7P_\infty|}(P_7)$ lie on a single hyperplane in $\mathbb{P}^3$, i.e., L lies on the hyperplane $X_3 = 0$. Since P_∞ is a total ramification point of f, we have that L lies on the projective tangent plane at $\Phi_{|7P_\infty|}(P_\infty)$ to $\Phi_{|7P_\infty|}(C)$, that is L lies on the hyperplane $X_0 = 0$. Therefore, the line L is given by $X_0 = X_3 = 0$, and $\pi_L : (X_0 : X_1 : X_2 : X_3) \mapsto (X_0 : X_3)$, therefore f necessarily corresponds to y. According to the Riemann-Hurwitz formula, since f is Galois, the number of branch points of f must be equal to 4. However, from $x^7 - x - y^3 = 0$, we can find 19 branch points of f, i.e., $y = \infty$ and 18 roots of equations $x^7 - x - y^3 = 0$ and $7x^6 - 1 = 0$. This is a contradiction.

The following is an example in which a curve has one pseudo-GW point with semigroup $\langle a, b\rangle$.

Example 4.2. Let C' be the singular plane curve $X^5 = Y^3Z^2 - YZ^4$ and $P'_\infty := (0:1:0)$. Then, $\mathrm{Sing}(C') = \{P'_\infty\}$ and P'_∞ is a cusp. Let $\rho : C \to C'$ be the resolution of singularity, and $P_\infty := \rho^{-1}(P'_\infty)$. We see the genus $g(C) = 4$. Let $x := (X/Z)|_{C'}, y := (Y/Z)|_{C'} \in k(C') = k(C)$. Then, $k(C) = k(x, y)$, $(x)_\infty = 3P_\infty$ and $(y)_\infty = 5P_\infty$. Hence, $H(P_\infty) = \langle 3, 5\rangle$.

The morphism $\Phi_{|3P_\infty|} : C \to \mathbb{P}^1$ corresponds to x. We see that $\Phi_{|3P_\infty|}$ is not Galois, i.e., P_∞ is not a GW point. Indeed, if $\Phi_{|3P_\infty|}$ is Galois, then the ramification index at every ramification point of $\Phi_{|3P_\infty|}$ equals three. By the Riemann-Hurwitz formula, the number of branch points of $\Phi_{|3P_\infty|}$ must be equal to six. However, we can find 11 branch points of $\Phi_{|3P_\infty|}$, i.e., $x = \infty$ and 10 roots of equations $y^3 - y - x^5 = 0$ and $3y^2 - 1 = 0$. This is a contradiction.

Let $f : C \to \mathbb{P}^1$ be the morphism corresponding to y. Then, f is Galois since $x^5 - y^3 + y = 0$, and P_∞ is a total ramification point. Hence, P_∞ is a pseudo-GW point with $H(P_\infty) = \langle 3, 5\rangle$.

The following is an example in which a curve has $b + 1$ GW points with semigroup $\langle a, b\rangle$.

Example 4.3. Let C' be the singular plane curve $X^5 - 5X^3Z^2 + 4XZ^4 = Y^3Z^2$ and $P'_\infty := (0:1:0)$. Then, $\mathrm{Sing}(C') = \{P'_\infty\}$ and P'_∞ is a cusp. Let $\rho : C \to C'$ be the resolution of singularity, and $P_\infty := \rho^{-1}(P'_\infty)$. Let $P'_1 := (0:0:1)$, $P'_2 = (1:0:1)$, $P'_3 = (-1:0:1)$, $P'_4 = (2:0:1)$, $P'_5 = (-2:0:1)$, and $P_i := \rho^{-1}(P'_i)$ $(i = \infty, 1, \dots, 5)$. Then, $H(P_i) = \langle 3, 5\rangle$ $(i = \infty, 1, \dots, 5)$. Let $x := (X/Z)|_{C'}, y := (Y/Z)|_{C'} \in k(C') = k(C)$. Then, $\Phi_{|3P_\infty|} : C \to \mathbb{P}^1$ corresponds to x. The point P_i $(i = \infty, 1, \dots, 5)$ is a total ramification point of $\Phi_{|3P_\infty|}$ and $3P_\infty \sim 3P_i$. Since $y^3 = x(x-1)(x+1)(x-2)(x+2)$, $\Phi_{|3P_\infty|}$ is Galois. Hence, the 6 points in P_i $(i = \infty, 1, \dots, 5)$ are GW points with $H(P_i) = \langle 3, 5\rangle$.

Furthermore, we see that $5 \notin \mathrm{degGW}(P_i)$ for $i = \infty, 1, \dots, 5$. Here, we prove this for $i = \infty$ and 1 as below. We can prove this for the other cases by using a similar argument.

Assume that $5 \in \mathrm{degGW}(P_\infty)$. Let $f : C \to \mathbb{P}^1$ be a Galois covering of degree 5 such that P_∞ is a total ramification point of f. Then, f is given by a linear subsystem of $|5P_\infty|$, that is, f can be expressed as the composition of $\Phi_{|5P_\infty|} : C \to \mathbb{P}^2$ and the projection $\pi_Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from some point Q. Remark that $\Phi_{|5P_\infty|}(C) = C'$ and $\Phi_{|5P_\infty|} = \rho$. We see that P_i $(i = 1, \dots, 5)$ is not a ramification point of f. Indeed, if P_i is a ramification point of f, then $5P_\infty \sim 5P_i$. Since $3P_\infty \sim 3P_i$, we have $P_\infty \sim P_i$, and this contradicts $C \not\cong \mathbb{P}^1$. By Theorem 1.3, all GW points with $H(P) = \langle 3, 5\rangle$ are $P_1, \dots, P_5$, and P_∞. Because a point in $f^{-1}(f(P_i))$ is also a GW point with $H(P) = \langle 3, 5\rangle$, we have that $\Phi_{|5P_\infty|}(P_1), \dots, \Phi_{|5P_\infty|}(P_5)$ and Q are collinear. Since P_∞ is a total ramification point of f, we have that Q is on the projective tangent line at P'_∞ to C'. Therefore, Q must be $(1:0:0)$ and f must be given by y. By the Riemann-Hurwitz formula, the number of branch points of f must be equal to four. However, from $x^5-5x^3+4x-y^3=0$, we can find 13 branch points of f, i.e., $y=\infty$ and 12 roots of equations $x^5-5x^3+4x-y^3=0$ and $5x^4-15x^2+4=0$. This is a contradiction.

Assume that $5 \in \mathrm{degGW}(P_1)$. Let $f : C \to \mathbb{P}^1$ be a Galois covering of degree 5 such that P_1 is a total ramification point of f. Then, f is given by a linear subsystem of $|5P_1|$, that is, f can be expressed as the composition of $\Phi_{|5P_1|} : C \to \mathbb{P}^2$ and the projection $\pi_Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from some point Q. Let $\bar{C} := \Phi_{|5P_1|}(C)$. Since $(1/x)_\infty = 3P_1$ and $(y/x^2)_\infty = 5P_1$, we have that $\Phi_{|5P_1|}$ is given by $(1 : 1/x : y/x^2)$. Then, $\bar{C}$ is defined by $X_1(X_0-2X_1)(X_0+2X_1)(X_0-X_1)(X_0+X_1) = X_0^2X_2^3$, where $(X_0 : X_1 : X_2)$ is a system of homogeneous coordinates of $\mathbb{P}^2$. We also have that $\Phi_{|5P_1|}(P_1) = (0:0:1)$, $\Phi_{|5P_1|}(P_2) = (1:1:0)$, $\Phi_{|5P_1|}(P_3) = (1:-1:0)$, $\Phi_{|5P_1|}(P_4) = (2:1:0)$, $\Phi_{|5P_1|}(P_5) = (2:-1:0)$ and $\Phi_{|5P_1|}(P_\infty) = (1:0:0)$. We see that P_i ($i = 2,\dots,5,\infty$) is not a ramification point of f. Indeed, if P_i is a ramification point of f, then $5P_1 \sim 5P_i$. Since $3P_1 \sim 3P_i$, we have $P_1 \sim P_i$, and this contradicts $C \not\cong \mathbb{P}^1$. By Theorem 1.3, all GW points with $H(P) = \langle 3,5\rangle$ are $P_1,\dots,P_5$, and P_∞. Because a point in $f^{-1}(f(P_i))$ is also a GW point with $H(P) = \langle 3,5\rangle$, we have that $\Phi_{|5P_1|}(P_2),\dots,\Phi_{|5P_1|}(P_5), \Phi_{|5P_1|}(P_\infty)$ and Q are collinear. Since P_1 is a total ramification point of f, we have that Q is on the projective tangent line at $\Phi_{|5P_1|}(P_1)$ to $\bar{C}$. Therefore, Q must be $(0:1:0)$ and f must be given by y/x^2. By the Riemann-Hurwitz formula, the number of branch points of f must be equal to four. However, from $4(1/x)^5-(1/x)^3+(1/x)-(y/x^2)^3=0$, we can find 13 branch points of f, i.e., $y/x^2=\infty$ and 12 roots of equations $4(1/x)^5-(1/x)^3+(1/x)-(y/x^2)^3=0$ and $20(1/x)^4-3(1/x)^2+1=0$. This is a contradiction.

In the following example a curve has a weak GW point P with semigroup $\langle a,b\rangle$ and $a,b \in \mathrm{degGW}(P)$.

Example 4.4. Let C' be the singular plane curve $X^5 = Y^3Z^2 + Z^5$ and $P'_\infty := (0:1:0)$. Then, $\mathrm{Sing}(C') = \{P'_\infty\}$ and P'_∞ is a cusp. Let $\rho : C \to C'$ be the resolution of singularity. We see the genus $g(C) = 4$. Let $P'_i := (\zeta_5^{i-1} : 0 : 1)$ ($i = 1,\dots,5$), where ζ_5 is the primitive 5th root of unity, and $P_i := \rho^{-1}(P'_i)$ ($i = \infty,1,\dots,5$). Then, $H(P_i) = \langle 3,5\rangle$ ($i = \infty,1,\dots,5$). Let $x := (X/Z)|_{C'}, y := (Y/Z)|_{C'} \in k(C') = k(C)$. Then, $\Phi_{|3P_\infty|} : C \to \mathbb{P}^1$ corresponds to x. The point P_i ($i = \infty,1,\dots,5$) is a total ramification point of $\Phi_{|3P_\infty|}$. Since $y^3 = x^5-1$, $\Phi_{|3P_\infty|}$ is Galois. Thus, P_i ($i = \infty,1,\dots,5$) is a GW point with $H(P_i) = \langle 3,5\rangle$. Let $f : C \to \mathbb{P}^1$ be the morphism corresponding to y. Then, f is Galois and P_∞ is a total ramification point. Therefore, P_∞ is a weak GW point with $3,5 \in \mathrm{degGW}(P_\infty)$ and $H(P_\infty) = \langle 3,5\rangle$. By Theorem 1.3, we have that $5 \notin \mathrm{degGW}(P_i)$ ($i = 1,\dots,5$).

Acknowledgements. The authors would like to express their sincere gratitude toward Professor Marc Coppens. Coppens brought Theorems 1.4 and 2.1 to the attention of the authors after they uploaded their preprint [8] to the e-print server arXiv.org. We would like to thank Editage (www.editage.jp) for English language editing.

References

[1] W. Barth, C. Peters, and A. van de Ven, *Compact Complex Surfaces*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **4**, Springer-Verlag, Berlin, 2004. https://doi.org/10.1007/978-3-642-57739-0

[2] W. Castryck and F. Cools, *Linear pencils encoded in the Newton polygon*, Int. Math. Res. Not. IMRN **2017** (2017), no. 10, 2998–3049. https://doi.org/10.1093/imrn/rnw082

[3] H. C. Chang, *On plane algebraic curves*, Chinese J. Math. **6** (1978), no. 2, 185–189.

[4] M. Coppens, *The uniqueness of Weierstrass points with semigroup* $\langle a; b\rangle$ *and related semigroups*, Abh. Math. Semin. Univ. Hambg., published online: 15 February 2019. https://doi.org/10.1007/s12188-019-00201-y

[5] S. Fukasawa, *On the number of Galois points for a plane curve in positive characteristic. III*, Geom. Dedicata **146** (2010), 9–20. https://doi.org/10.1007/s10711-009-9422-x

[6] S. Fukasawa and H. Yoshihara, *List of problems*, available at http://hyoshihara.web.fc2.com/openquestion.html.

[7] J. Komeda and T. Takahashi, *Relating Galois points to weak Galois Weierstrass points through double coverings of curves*, J. Korean Math. Soc. **54** (2017), no. 1, 69–86. https://doi.org/10.4134/JKMS.j150593

[8] ———, *Galois Weierstrass points whose Weierstrass semigroups are generated by two elements*, arXiv:1703.09416 (2017).

[9] K. Miura and H. Yoshihara, *Field theory for function fields of plane quartic curves*, J. Algebra **226** (2000), no. 1, 283–294. https://doi.org/10.1006/jabr.1999.8173

[10] I. Morrison and H. Pinkham, *Galois Weierstrass points and Hurwitz characters*, Ann. of Math. (2) **124** (1986), no. 3, 591–625. https://doi.org/10.2307/2007094

[11] H. Yoshihara, *Function field theory of plane curves by dual curves*, J. Algebra **239** (2001), no. 1, 340–355. https://doi.org/10.1006/jabr.2000.8675

Jiryo Komeda
Department of Mathematics
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology
Atsugi, Kanagawa, 243-0292, Japan
Email address: komeda@gen.kanagawa-it.ac.jp

Takeshi Takahashi
Education Center for Engineering and Technology
Faculty of Engineering
Niigata University
Niigata 950-2181, Japan
Email address: takeshi@eng.niigata-u.ac.jp