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Note on the Codimension Two Splitting Problem

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Dedicated to Professor Akio Kawauchi with warm thanks for his long friendship

Abstract. Let W and V be manifolds of dimension $m+2$, M a locally flat submanifold of V whose dimension is m. Let $f : W \to V$ be a homotopy equivalence. The problem we study in this paper is the following: When is f homotopic to another homotopy equivalence $g : W \to V$ such that g is transverse regular along M and such that $g|g^{-1}(M) : g^{-1}(M) \to M$ is a simple homotopy equivalence? López de Medrano (1970) called this problem the *weak h-regularity problem.* We solve this problem applying the *codimension two surgery theory* developed by the author (1973). We will work in higher dimensions, assuming that $m \geqq 5$.

1. Introduction

In this paper, we study *the weak h-regularity problem* in the sense of López de Medrano [14], or *the codimension two splitting problem*, whose precise formulation will be given in §2. The main results will clarify the role of *relatively non-singular Hermitian K-groups* [3, 18] in the codimension two splitting problem. This type of K-groups (called *P-groups* in our theory) are defined over a surjective ring homomorphism between (not necessarily commutative) rings

$$A \to B$$

and are, roughly speaking, the Witt groups of *relatively non-singular Hermitian forms over $A \to B$*, meaning that they are defined over A and become non-singular over B. For example, P-groups over $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ are isomorphic[1] to Levine's knot

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[1] A purely algebraic proof of these isomorphisms is given in [21].

cobordism groups [11]. See [17]. Cappell and Shaneson [5, 6], defined similar but slightly different Hermitian K-groups (called Γ-*groups* by them), and using these groups they studied the codimension two placement problems extensively. However, in the author's opinion, our formulation is simpler than theirs. For example, we can give a unique element in our P-group as the obstruction to codimension two splitting (see Theorem 2.5 below), while in Cappell and Shaneson's theory, the obstruction to the same splitting problem is given in two steps involving two different (but mutually related) groups (see [4], [5, §8], [6, p.438]).

Relatively non-singular Hermitian K-theory is an interesting region of mathematics, but compared to the usual (non-singular) Hermitian K-theory[2] (cf. [1, 25, 27, 31, 32]), it is still under-developed. Actually, after Ranicki's remarkable work (see [28, §7.8 and §7.9], [29]), the relationship between our P-groups and Cappell-Shaneson's Γ-groups is still unclear. In Kato's problem list [9], C. T. C. Wall proposed the problem of computing these groups [9, Problem 6.2, p.426].

Another interesting problem is to find a Künneth type formula for relatively non-singular Hermitian K-groups. The problem is to find the formula describing the K-groups over $A \otimes \Lambda \to B \otimes \Lambda$ in terms of K-groups over $A \to B$, where $\Lambda = \mathbb{Z}[s, s^{-1}]$. It is known that Shaneson's type of Künneth formula [30] cannot be expected here. See [19].

There should be close relationships between relatively non-singular Hermitian K-theory and algebraic number theory. In fact, Milnor's and Levine's papers [12, 24] seem to suggest certain connections of it to the class field theory.

Axiomatic foundations of relatively non-singular Hermitian K-groups are found in [3, 18, 28, 29]. See also [21].

For geometric applications of our theory, see [17, 19, 20, 22].

Recently there was remarkable progress related to [19] concerning spineless 4-manifolds. Our example constructed in [19] was a compact PL-spineless 4-manifold homotopy equivalent to a 2-torus T^2. In 2018, using Heegaad Floer d invariants, A. S. Levine and T. Lidman [13] constructed compact PL-spineless 4-manifolds homotopy equivalent to a 2-sphere S^2, and in 2019, H. J. Kim and D. Ruberman [10] proved that some of the Levine-Lidman manifolds admit tame topological spines.

We remark here that M. H. Freedman [7] independently discovered the same Seifert forms as ours (see §5 and Appendix of the present paper). Applying his codimension two surgery theory, he found higher dimensional counterexamples to the generalized Thom conjecture concerning the Betti numbers of smooth hypersurfaces in the complex projective spaces [7].

The present paper is based on the author's old note [16], which has been unpublished for more than forty years. The author hopes that the note would be still worth publishing, but an apology for such a long delay would be necessary. An explanation is given in **Acknowledgments and Postscript** at the end of this paper.

[2] The usual Hermitian K-theory is a special case of relatively non-singular Hermitian K-theory, where the basic surjection $A \to B$ is the identity $id : A \to A$.

The (almost) verbatim reproduction of the old note [16] starts in the next paragraph after $\int\ \int\ \int$. In the reproduction, we have updated the references[3]. (In fact, in the old note, even the references [8] and [17] were cited as "to appear". The papers [5], [6] were not available even in the preprint form. At that time, the only papers of Cappell and Shaneson that were available to the author were [4] in preprint form.) Also we have added some footnotes.

Now the reproduction starts.

$$\int \quad \int \quad \int$$

In our previous paper [17], we introduced ambient surgery obstruction groups $P_m(\mathcal{E})$ in codimension two. There we introduced them in order to describe the obstruction to finding locally flat spines of $(m+2)$-manifolds which are simple homotopy equivalent to a Poincaré complex of formal dimension m.

The groups $P_m(\mathcal{E})$ work, however, as the obstruction groups for the weak h-regularity problem in the sense of López de Medrano [14] as well (i.e. the splitting problem in codimension two). The purpose of this note is to give a detailed proof of it.

Cappell and Shaneson [4] treated the same problem independently from homology surgery point of view. They state their obstruction in terms of Γ-groups introduced by them. Naturally their Γ-groups and our P-groups seem to be related to each other very closely. The relationship will be discussed elsewhere.

2. Definitions and Statement of Results

Throughout the paper, we will work in the PL-category[4]. All manifolds are compact connected and oriented. All submanifolds are locally flat unless the contrary is stated. If a submanifold M of W has a boundary, we always assume that it is properly embedded, i.e., $j^{-1}(\partial W) = \partial M$, where $j : M \to W$ is the inclusion.

The dimension of a manifold is indicated by a superscript.

Definition 2.1.([8, 17]) Let M^m be a submanifold of W^{m+2} with a regular neighborhood N. (If $\partial M \neq \emptyset$, N is assumed so that $N \cap \partial W$ is a regular neighborhood of ∂M in ∂W.) The closed complement $E = \overline{W - N}$ is called the *exterior*, $E \cap N$ is called the *frontier* of N and is denoted by $\mathcal{F}N$. If $\pi_i(E, \mathcal{F}N) = 0$ for $i \leqq k$, M^m is said to be *exterior k-connected*.

Suppose M^m is exterior 2-connected in W^{m+2}, then we have isomorphisms $\pi_1(\mathcal{F}N) \overset{\cong}{\to} \pi_1(E)$ and $\pi_1(M) \overset{\cong}{\to} \pi_1(N) \overset{\cong}{\to} \pi_1(W)$. The proof is not difficult. (See [17, Lemma 1.3].) The former group is denoted by π, and the latter by π'. Since

[3] The original references in [16] were [4, 8, 14, 15, 17, 23, 26, 32].

[4] All of our results, however, remain valid in the smooth or the topological categories. (cf. [32])

$\mathcal{F}N \to M$ is an S^1-bundle, there is an exact sequence:

$$\pi_2(M) \to \pi_1(S^1) \to \pi_1(\mathcal{F}N) \to \pi_1(M) \to 1.$$

From this, we have the exact sequence

$$1 \to C \to \pi \to \pi' \to 1,$$

where $C = \mathrm{Coker}(\pi_2(M) \to \pi_1(S^1))$, and C is a cyclic group in the center of π [17, Lemma 1.4]. We can specify a generator t of C. The generator t is canonically determined by the orientations of M^m and W^{m+2} [17, Lemma 1.4].

Definition 2.2.([17]) The above exact sequence is denoted by $\mathcal{E}$, and is said to *be associated with the exterior* 2*-connected manifold pair* (W^{m+2}, M^m).

Let P, Q be two manifolds, K a submanifold of Q, Let $h : P \to Q$ be a continuous map.

Definition 2.3.([14]) The map h is *weakly h-regular along* K if

(1) h is t-regular along K and

(2) $h|h^{-1}(K) : h^{-1}(K) \to K$ is a simple homotopy equivalence.

Now we can state our problem.

Suppose that we are given a diagram of weak h-regularity problem $(*)$ satisfying the following conditions:

$$(*) \qquad f : (W^{m+2}, \partial W) \to \begin{array}{c}(V^{m+2}, \partial V)\\ \cup \\ (M^m, \partial M).\end{array}$$

(C.1) $f|W : W \to V$ is a (not necessarily simple) homotopy equivalence.

(C.2) $f|\partial W : \partial W \to \partial V$ is weakly h-regular along ∂M.

(C.3) $f|\partial W - f^{-1}(\partial M) : \partial W - f^{-1}(\partial M) \to \partial V - \partial M$ is a Λ'-homology equivalence, where $\Lambda' = \mathbb{Z}[\pi']$ with $\pi' = \pi_1(W)$. The Λ'*-homology group* of $\partial W - f^{-1}(\partial M)$ means the integral homology of $\pi^{-1}(\partial W - f^{-1}(\partial M))$, where $\pi : \widetilde{W} \to W$ is the universal covering.

(C.4) M is exterior 2-connected in V.

Problem 2.4.(The weak h-regularity problem (W.H.-R.P.)) *When is f homotopic (rel. the boundary) to $g : (W, \partial W) \to (V, \partial V)$ which is weakly h-regular along M?*

Our purpose is to prove the following

Theorem 2.5. *There is a unique obstruction element $\gamma(f)$ in $P_m(\mathcal{E})$ which vanishes if and only if f is homotopic (rel. ∂W) to such a map g ($m \geqq 5$). Here $\mathcal{E}$ is the short exact sequence associated with (V^{m+2}, M^m).*

For the algebraic definition of the group $P_m(\mathcal{E})$, see §5.1. Since $P_{2n+1}(\mathcal{E}) \cong L_{2n+1}(\pi')$ ([17], the right-hand side being the Wall group [32]), we have

Corollary 2.6. *If $m = \text{odd} \geqq 5$, the obstruction is an element of the odd-dimensional Wall group $L_m(\pi')$.*

This is independently obtained by Cappell and Shaneson [4]. The obstruction in this note and the one in our previous paper [17] are related as follows.

Theorem 2.7.(Restatement of Theorem 5.1 and Lemmas 5.3 and 5.4 of the present paper)

(1) *Suppose we are given a diagram $(*)$ of the weak h-regularity problem. Let T denote a tubular neighborhood of M^{2n} in V^{2n+2}. Then f is homotopic (rel. ∂W) to a map g which is t-regular along $\mathcal{F}T$ with $g^{-1}(T) \to T$ a simple homotopy equivalence.*

(2) *Clearly $g^{-1}(T)$ is a Poincaré thickening in the sense of [17], and the obstruction $\eta(g^{-1}(T))$ to finding a locally flat spine is defined in $P_{2n}(\mathcal{E})$. The obstruction $\gamma(f)$ to weak h-regularity is defined to be $\eta(g^{-1}(T))$.*

In §6, we will prove the following "periodicity theorem" and the "invariance theorem under L-equivalence".

Theorem 2.8.(Restatement of Theorem 6.1) *Let $id_{\mathbb{C}P_2} \times f$ be the diagram*

$$\begin{array}{ccc} \mathbb{C}P_2 \times (W^{m+2}, \partial W) & \overset{\mathrm{id}\times f}{\to} & \mathbb{C}P_2 \times (V^{m+2}, \partial V) \\ & & \cup \\ & & \mathbb{C}P_2 \times (M^m, \partial M), \end{array}$$

then we have

$$\gamma(\mathrm{id}_{\mathbb{C}P_2} \times f) = \rho(\gamma(f)),$$

where $\rho : P_m(\mathcal{E}) \to P_{m+4}(\mathcal{E})$ is the algebraic periodicity isomorphism.

Theorem 2.9.(Restatement of Theorem 6.2) *If M_1^m and M_2^m are L-equivalent (rel. ∂), then $\gamma_{M_1}(f) = \gamma_{M_2}(f)$. (The notation $\gamma_M(f)$ is used to emphasize the submanifold M.)*

3. Surgery below the Middle Dimension

Suppose the diagram $(*)$ (on the previous page) is given. In this section we will perform surgery on $f^{-1}(M)$ below the middle dimension. The middle dimension will be studied in §4 and §5. In what follows, whenever we consider the preimage $f^{-1}(M)$, we are assuming that f is t-regular along M.

We will introduce the following notation:

$L^m = f^{-1}(M^m)$,
N; a regular neighborhood of L^m in W^{m+2},
E; the exterior of N,
$\mathcal{F}N$; the frontier of N,
T; a regular neighborhood of M^m in V^{m+2}, (We assume $N = f^{-1}(T)$.)
F; the exterior of T,
$\mathcal{F}T$; the frontier of T, and Φ denotes the quadruple $\begin{pmatrix} E & \longrightarrow & F \\ \Big\uparrow & & \Big\uparrow \\ \mathcal{F}N & \longrightarrow & \mathcal{F}T \end{pmatrix}$.

Let α be an element of $\pi_{i+2}(\Phi)$. Suppose $\partial\alpha \in \pi_{i+1}(E, \mathcal{F}N)$ is represented by a normally embedded $(i+1)$-handle H in W attached to L : (See [8])

$$\begin{aligned} &H = D^{i+1} \times D^{m-i} \subset W^{m+2}, \\ &H \cap L^m = \partial D^{i+1} \times D^{m-i}. \end{aligned}$$

($\partial\alpha$ denotes the image of α under the homomorphism $\partial : \pi_{i+2}(\Phi) \to \pi_{i+1}(E, \mathcal{F}N)$.) Then we have

Lemma 3.1. *There is a homotopy (rel. boundary) from f to f' which satisfies $f'^{-1}(M) = (L - Int(\partial D^{i+1} \times D^{m-i})) \cup D^{i+1} \times \partial D^{m-i}$. In other words, the surgery can be performed by a homotopy of f.*

Although the proof is not difficult, it is long and tedious, so we omit it. An analogous argument is done in [15].

3.1. To make $L^m = f^{-1}(M^m)$ Exterior 2-connected.

Suppose $m \geqq 4$. Since $\pi_i(F, \mathcal{F}T) = 0$ $(i \leqq 2)$ (this is our hypothesis (C.4)), we have

$$\begin{aligned} &\partial : \pi_2(\Phi) \xrightarrow{\cong} \pi_1(E, \mathcal{F}N), \quad \text{and} \\ &\partial : \pi_3(\Phi) \to \pi_2(E, \mathcal{F}N) \to 0. \end{aligned}$$

(These are obtained by the homotopy sequence of quadruple Φ.)

Thus any element in $\pi_i(E, \mathcal{F}N) (i \leqq 2)$ is of the form $\partial\alpha$, where $\alpha \in \pi_{i+1}(\Phi)$, and by dimension reasons, $\partial\alpha$ can be represented by a normally embedded i-handle. Then Lemma 3.1 applies.

The effect of this surgery is to kill $\partial\alpha$. Thus by successive surgeries, one can kill the whole sets $\pi_i(E, \mathcal{F}N)$ $(i \leqq 2)$. For a detailed proof, see [8].

Hereafter we will assume that $f^{-1}(M) = L$ is exterior 2-connected and that the following diagram holds

$$\begin{tikzcd}
 & \pi_1(E) \arrow[r] & \pi_1(F) & \\
 & & & \Big\}\ \pi \\
 & \pi_1(\mathcal{F}N) \arrow[r] \arrow[uu, "\cong"'] \arrow[dd] & \pi_1(\mathcal{F}T) \arrow[uu, "\cong"'] \arrow[dd] & \\
 & & & \\
(\pi_1(W) &)\pi_1(L) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \pi_1(M) & \}\ \pi' .
\end{tikzcd}$$

3.2. Surgery below the Middle Dimension.

Since $f : W^{m+2} \to V^{m+2}$ is a homotopy equivalence, the homomorphisms

$$\begin{aligned}
l(f|N)_* : H_*(N;\Lambda') &\to H_*(T;\Lambda'), \quad (\Lambda' = \mathbb{Z}[\pi']) \\
(f|E)_* : H_*(E;\Lambda) &\to H_*(F;\Lambda), \quad (\Lambda = \mathbb{Z}[\pi]) \\
(f|\mathcal{F}N)_* : H_*(\mathcal{F}N;\Lambda) &\to H_*(\mathcal{F}T;\Lambda), \qquad \text{etc.}
\end{aligned}$$

are surjective.

The corresponding kernels are denoted by $K_*(N;\Lambda')$, $K_*(E;\Lambda)$, $K_*(\mathcal{F}N;\Lambda)$, etc. (*The meaning of the* Λ-*homology*: For example, $H_*(E;\Lambda)$ denotes the integral homology of $\widetilde{E}$, the universal covering of E. Recall $\pi_1(E) \cong \pi$. Similarly $H_*(N;\Lambda')$, the Λ'-homology group, is defined by the integral homology of $\widetilde{N}$, the universal covering of N, where $\pi' = \pi_1(N)$.)

Assume $3 \leqq i+1 \leqq \left[\frac{m}{2}\right]$, and suppose that an element of $\pi_{i+1}(E,\mathcal{F}N)$ is of the form $\partial\alpha$, where $\alpha \in \pi_{i+2}(\Phi)$, and is represented by a normally embedded handle H. Then Lemma 3.1 applies. f is deformed by a homotopy (rel. ∂) to f' with $f'^{-1}(M)$ a new submaifold which is obtained by the surgery along H. The corresponding new exterior (or frontier) is denoted by E' (or $\mathcal{F}N'$). Then we have,

Lemma 3.2.

$$\begin{aligned}
\pi_j(E',\mathcal{F}N') &= 0 \qquad (j \leqq 2) \\
K_j(E',\mathcal{F}N';\Lambda) &\cong K_j(E,\mathcal{F}N;\Lambda) \quad (j < i+1) \\
K_{i+1}(E',\mathcal{F}N';\Lambda) &\cong K_{i+1}(E,\mathcal{F}N;\Lambda)/\Lambda(\partial\alpha).
\end{aligned}$$

If K_* is replaced by H_*, this is a standard result in codimension two surgery theory (cf. [8, 17]). The "standard proof" can apply to Lemma 3.2 under appropriate modifications.

Denote the restrictions of f, $f|E : E \to F$, $f|\mathcal{F}N : \mathcal{F}N \to \mathcal{F}T$ and $f|L : L \to M$ by f_E, $f_{\mathcal{F}}$, f_L respectively.

The purpose of this paragraph is to prove the following:

Theorem 3.3. *If $m \geqq 4$, we can perform ambient surgery on $L^m = f^{-1}(M^m)$ via homotopy* (*rel.* ∂) *of f to obtain a new map* (*again denoted by f*) *so that the new submanifold has the following properties*

(i) $L^m = f^{-1}(M^m)$ *remains exterior* 2*-connected,,*

(ii) $\pi_i(\Phi) = 0 \quad (i \leqq \left[\frac{m}{2}\right] + 1)$,

(iii) $\pi_i(f_E) = 0 \quad (i \leqq \left[\frac{m}{2}\right] + 1)$,

(iv) $\pi_i(f_{\mathcal{F}}) = 0 \quad (i \leqq \left[\frac{m}{2}\right])$ *and*

(v) $\pi_i(f_L) = 0 \quad (i \leqq \left[\frac{m}{2}\right])$.

The next lemma is useful in proving Theorem 3.3.

Lemma 3.4. *Let* $L^m = f^{-1}(M^m)$ *be exterior* 2*-connected. Let* ℓ *be an integer* $\geqq 1$*. Suppose*

(a) $\pi_i(\Phi) = 0$ *for* $i \leqq \ell + 2$,

(b) $\pi_i(f_E) = 0$ *for* $i \leqq \ell$ *and*

(c) $\pi_i(f_L) = 0$ *for* $i \leqq \ell$.

Then it follows that

(d) $\pi_i(f_E) = 0$ *for* $i \leqq \ell + 2$,

(e) $\pi_i(f_{\mathcal{F}}) = 0$ *for* $i \leqq \ell + 1$ *and*

(f) $\pi_i(f_L) = 0$ *for* $i \leqq \ell + 1$.

Proof. From the homotopy exact sequence of Φ, we have

$$\pi_i(f_{\mathcal{F}}) \overset{\cong}{\to} \pi_i(f_E) \quad \text{for} \quad i \leqq \ell + 1. \tag{3.1}$$

This implies

$$\pi_i(f_{\mathcal{F}}) = 0 \quad \text{for} \quad i \leqq \ell, \quad \text{by} \quad (b). \tag{3.2}$$

Let $\varpi_0 : \widehat{W} \to W$ (or $\varpi_1 : \widehat{V} \to V$) be the universal covering space of W (or V), and let $\hat{E} = {\varpi_0}^{-1}(E)$, $\widehat{\mathcal{F}N} = {\varpi_0}^{-1}(\mathcal{F}N)$, $\hat{F} = \varpi_1^{-1}(F)$ and $\widehat{\mathcal{F}T} = \varpi_1^{-1}(\mathcal{F}T)$. Their fundamental groups are isomorphic to the cyclic group C with generator t.

Let $\hat{N} = \varpi_0^{-1}(N)$, $\hat{L} = \varpi_0^{-1}(L)$, $\hat{T} = \varpi_1^{-1}(T)$ and $\hat{M} = \varpi_1^{-1}(M)$. These are connected and simply-connected.

Let $\hat{f}_{\mathcal{F}} : \widehat{\mathcal{F}N} \to \widehat{\mathcal{F}T}$, $\hat{f}_E : \hat{E} \to \hat{F}$, $\hat{f}_L : \hat{L} \to \hat{M}$ be the liftings of $f_{\mathcal{F}}$, f_E, f_L, respectively.

By (3.2) and hypothesis (b) together with the Hurewicz theorem, we have

$$\begin{cases} H_{\ell+1}(\hat{f}_E) \cong \pi_{\ell+1}(\hat{f}_E)/(1-t)\pi_{\ell+1}(\hat{f}_E), \quad \text{and} \\ H_{\ell+1}(\hat{f}_{\mathcal{F}}) \cong \pi_{\ell+1}(\hat{f}_{\mathcal{F}})/(1-t)\pi_{\ell+1}(\hat{f}_{\mathcal{F}}). \end{cases} \tag{3.3}$$

Here $(1-t)$ is an element of $\mathbb{Z}[C]$, the integral group-ring of C, and we consider $\pi_{\ell+1}(\hat{f}_E)$ and $\pi_{\ell+1}(\hat{f}_{\mathcal{F}})$ as $\mathbb{Z}[C]$-modules.

Now note that $\widehat{\mathcal{F}N}$ (resp. $\widehat{\mathcal{F}T}$) is the total space of an S^1-bundle over $\hat{L}$ (resp. $\hat{M}$), so the homomorphism $\pi_i(\widehat{\mathcal{F}N}) \to \pi_i(\hat{L})$ (resp. $\pi_i(\widehat{\mathcal{F}T}) \to \pi_i(\hat{M})$) induced by the projection is isomorphism for $i \geqq 3$, and injective for $i = 2$.

From this and the homotopy exact sequences of $\hat{f}_{\mathcal{F}} : \widehat{\mathcal{F}N} \to \widehat{\mathcal{F}T}$ and $\hat{f}_L : \hat{L} \to \hat{M}$, we have

$$(3.4) \qquad \begin{cases} \pi_i(\hat{f}_{\mathcal{F}}) \to \pi_i(\hat{f}_L) \quad \text{is injective for } \ i \geqq 2. \\ \text{If } \quad i \geqq 3, \text{ this is an isomorphism,} \\ \text{where the map is induced by the projection of } S^1\text{-bundles.} \end{cases}$$

The generator t of C is represented by a fiber of the S^1-bundle, thus t acts trivially on $\pi_i(\hat{f}_L)$. Therefore, by (3.4), t also acts trivially on $\pi_i(\hat{f}_{\mathcal{F}})$, in particular, on $\pi_{\ell+1}(\hat{f}_{\mathcal{F}})$. The isomorphism (3.1) implies that the action of t is also trivial on $\pi_{\ell+1}(\hat{f}_E)$. This implies $(1-t)\pi_{\ell+1}(\hat{f}_{\mathcal{F}}) = 0$ and $(1-t)\pi_{\ell+1}(\hat{f}_E) = 0$. Therefore we have from (3.3) and (3.1)

$$(3.5) \qquad \begin{cases} H_{\ell+1}(\hat{f}_E) & \cong & \pi_{\ell+1}(\hat{f}_E) & \cong & \pi_{\ell+1}(f_E) \\ & & & & \wr\!\! \| \ (3.1) \\ H_{\ell+1}(\hat{f}_{\mathcal{F}}) & \cong & \pi_{\ell+1}(\hat{f}_{\mathcal{F}}) & \cong & \pi_{\ell+1}(f_{\mathcal{F}}). \end{cases}$$

Since the restrictions of f, f_E, $f_{\mathcal{F}}$, f_N, f_L are of degree 1, the homomorphisms $(\hat{f}_E)_* : H_i(\hat{E};\mathbb{Z}) \to H_i(\hat{F};\mathbb{Z})$, $(\hat{f}_{\mathcal{F}})_* : H_i(\widehat{\mathcal{F}N};\mathbb{Z}) \to H_i(\widehat{\mathcal{F}T};\mathbb{Z})$, $(\hat{f}_N)_* : H_i(\hat{N};\mathbb{Z}) \to H_i(\hat{T};\mathbb{Z})$ and $(\hat{f}_L)_* : H_i(\hat{L};\mathbb{Z}) \to H_i(\hat{M};\mathbb{Z})$ are all surjective. Let $K_i(\hat{E})$, $K_i(\widehat{\mathcal{F}N})$. $K_i(\hat{N})$ and $K_i(\hat{L})$ be the corresponding kernels.

Consider the following diagram obtained by Mayer-Vietoris sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & K_i(\widehat{\mathcal{F}N}) \arrow[r] \arrow[d] & K_i(\hat{E}) \oplus K_i(\hat{N}) \arrow[d] & \\
H_{i+1}(\hat{W}) \arrow[r] \arrow[d, "f_*"', "\cong"] & H_i(\widehat{\mathcal{F}N}) \arrow[r] \arrow[d] & H_i(\hat{E}) \oplus H_i(\hat{N}) \arrow[r] \arrow[d] & H_i(\hat{W}) \arrow[d, "f_*"', "\cong"] \\
H_{i+1}(\hat{V}) \arrow[r] & H_i(\widehat{\mathcal{F}T}) \arrow[r] \arrow[d] & H_i(\hat{F}) \oplus H_i(\hat{T}) \arrow[r] \arrow[d] & H_i(\hat{V}) \\
 & 0 & 0 &
\end{tikzcd}$$

Here we have used the hypothesis that $f : W \to V$ is a homotopy equivalence

(C.1). By diagram chasing we see that

$$K_i(\widehat{\mathcal{F}N}) \to K_i(\hat{E}) \oplus K_i(\hat{N}) \quad \text{is surjective } (\forall i). \tag{3.6}$$

On the other hand, from the homology exact sequences it follows that $K_i(\widehat{\mathcal{F}N}) \cong H_{i+1}(\hat{f}_{\mathcal{F}})$, $K_i(\hat{E}) \cong H_{i+1}(\hat{f}_E)$ and $K_i(\hat{N}) \cong H_{i+1}(\hat{f}_N)$.

This and (3.5) imply

$$K_\ell(\widehat{\mathcal{F}N}) \cong K_\ell(\hat{E}) \tag{3.7}$$

We need an elementary algebraic lemma.

Lemma 3.5. *Let H, G_1, G_2 be abelian groups, $\varphi : H \to G_1$, $\psi : H \to G_2$ homomorphisms. Suppose that φ is injective and that $\varphi \oplus \psi : H \to G_1 \oplus G_2$ is surjective. Then $G_2 \cong \{0\}$.*

Proof. Suppose there were a non-zero element $x \in G_2$, and let $y \in H$ be mapped under $\varphi \oplus \psi$ to $0 \oplus x$. Then $\varphi(y) = 0$. Since φ is injective, $y = 0$ so $x = \psi(y) = 0$. This is a contradiction. □

By Lemma 3.5 together with (3.6) and (3.7), we have

$$K_\ell(\hat{N}) = 0. \tag{3.8}$$

Note that $0 \cong K_\ell(\hat{N}) \cong K_\ell(\hat{L}) \cong H_{\ell+1}(\hat{f}_L)$. Our hypothesis (c): $\pi_i(f_L) = 0$ for $i \leqq \ell$ and the Hurewicz theorem imply $\pi_{\ell+1}(\hat{f}_L) \cong H_{\ell+1}(\hat{f}_L) \cong 0$. Thus

$$\pi_i(f_L) \cong 0 \quad (i \leqq \ell + 1). \tag{3.9}$$

This is the conclusion (*f*). From (3.9) and (3.4) follows

$$\pi_i(f_{\mathcal{F}}) \cong \pi_i(\hat{f}_{\mathcal{F}}) \cong 0 \quad \text{for} \quad i \leqq \ell + 1. \tag{3.10}$$

This is the conclusion (e). From (3.10) and (3.5) follows

$$\pi_i(f_E) \cong 0 \quad \text{for} \quad i \leqq \ell + 1. \tag{3.11}$$

We see that

$$\begin{aligned} K_{\ell+1}(\widehat{\mathcal{F}N}) &\cong H_{\ell+2}(\hat{f}_{\mathcal{F}}) \\ &\cong \pi_{\ell+2}(\hat{f}_{\mathcal{F}}) \\ & \text{(Hurewicz. Recall that the } t\text{-action on } \pi_{\ell+2}(\hat{f}_{\mathcal{F}}) \text{ is trivial.)} \\ &\cong \pi_{\ell+2}(\hat{f}_L) \quad (\ell + 2 \geqq 3) \\ &\cong H_{\ell+2}(\hat{f}_L) \quad \text{(Hurewicz)} \\ &\cong K_{\ell+1}(\hat{L}) \\ &\cong K_{\ell+1}(\hat{N}). \end{aligned}$$

Apply this to (3.6) and use Lemma 3.5. Then we have

$$K_{\ell+1}(\hat{E}) = 0. \tag{3.12}$$

Consider the exact sequence of Φ:

$$\pi_{\ell+2}(\hat{f}_{\mathcal{F}}) \to \pi_{\ell+2}(\hat{f}_E) \to \pi_{\ell+2}(\hat{\Phi}) \cong 0 \quad \text{by} \quad (a).$$

From this we see that the t-action on $\pi_{\ell+2}(\hat{f}_E)$ is trivial, because its action on $\pi_{\ell+2}(\hat{f}_{\mathcal{F}})$ is trivial. Thus $0 = K_{\ell+1}(\hat{E}) \cong H_{\ell+2}(\hat{f}_E) \cong \pi_{\ell+2}(\hat{f}_E) \cong \pi_{\ell+2}(f_E)$. This together with (3.11) implies the conclusion (d). This completes the proof of Lemma 3.4. □

Proof of Theorem 3.3.

$\boxed{\text{Cases where } m = 4 \text{ or } 5.}$

First recall Namioka's theorem [26]. We state it in our situation.

Theorem 3.6.(Namioka's Theorem) *Let* $\Phi = \begin{pmatrix} E & \xrightarrow{f_E} & F \\ \uparrow & & \uparrow \\ \mathcal{F}N & \xrightarrow{f_{\mathcal{F}}} & \mathcal{F}T \end{pmatrix}$. *Suppose* $\pi_i(F, \mathcal{F}T) = 0$ *for* $i \leqq 2$ *and* $\pi_i(f_E) = 0$ *for* $i \leqq k$ $(k \geqq 1)$.

(I) $\pi_i(\Phi) = 0$ *for* $1 < i \leqq r$ *if and only if* $H_i(\Phi; \Lambda) = 0$ *for* $i \leqq r$. *Here* $1 < r \leqq k+2$.

(II) *If* $1 < r \leqq k+1$ *and* $\pi_i(\Phi) = 0$ *for* $i \leqq r$, *the Hurewicz map*

$$h : \pi_i(\Phi) \to H_i(\Phi; \Lambda)$$

is isomorphic for $i \leqq r+1$.

We now want to prove that our condition "L^m and M^m are exterior 2-connected"implies (ii)∼(v) of Theorem 3.3 for $m = 4, 5$ ($\left[\frac{m}{2}\right] = 2$).

From Λ-homology exact sequence of Φ, we have

$$H_{i+1}(\Phi; \Lambda) \cong K_i(E, \mathcal{F}N; \Lambda).$$

Since $\pi_i(E, \mathcal{F}N) \cong \pi_i(F, \mathcal{F}T) = 0$ $(i \leqq 2)$,

$$H_i(E, \mathcal{F}N; \Lambda) \cong H_i(F, \mathcal{F}T; \Lambda) \cong 0 \quad (i \leqq 2)$$

by the Hurewicz theorem.

Thus $H_i(\Phi) \cong K_{i-1}(E, \mathcal{F}N; \Lambda) \cong 0$ for $i \leqq 3$. Since $\pi_1(f_E) \cong 0$, we have $\pi_i(\Phi) \cong 0$ for $i \leqq 3$ by Namioka's theorem (I). ($\pi_1(f_E) \cong 0$ follows from the diagram at the end of §3.1.) Then apply Lemma 3.4 with $\ell = 1$ to obtain $\pi_i(f_E) = 0$ $(i \leqq 3)$, $\pi_i(f_{\mathcal{F}}) = 0$ $(i \leqq 2)$ and $\pi_i(f_L) = 0$ $(i \leqq 2)$. This proves Theorem 3.3 for $m = 4, 5$.

$\boxed{\text{Cases where } m \geqq 6.}$

Inductively we assume $\pi_i(\Phi) = 0$ $(i \leqq \ell+2)$, $\pi_i(f_E) = 0$ $(i \leqq \ell+2)$, $\pi_i(f_{\mathcal{F}}) = 0$ $(i \leqq \ell+1)$ and $\pi_i(f_L) = 0$ $(i \leqq \ell+1)$. These assumptions in fact hold when $\ell = 1$. Apply Namioka's theorem (II) with $k = \ell+1$, $r = \ell+2$. Then the Hurewicz map

$$h : \pi_{\ell+3}(\Phi) \to H_{\ell+3}(\Phi; \Lambda)$$

is an isomorphism. Therefore any element of $K_{\ell+2}(E, \mathcal{F}N; \Lambda) \cong H_{\ell+3}\ (\Phi; \Lambda)$ can be represented by an element α of $\pi_{\ell+3}(\Phi)$. The "boundary" $\partial\alpha$ in $\pi_{\ell+2}(E, \mathcal{F}N)$ is proved to be represented by a normally embedded $(\ell+2)$-handle H attached to $L^m = f^{-1}(M)$, provided that $\ell + 2 \leqq \left[\frac{m}{2}\right]$ (See [8, Lemma 3.3]).

Perform ambient codimension 2 handle exchange along H via homotopy of f (Lemma 3.1), then we have (by Lemma 3.2)

$$H_i(\Phi'; \Lambda) \cong K_{i-1}(E', \mathcal{F}N'; \Lambda) \cong 0 \quad (i \leqq \ell+2),$$

and

$$H_{\ell+3}(\Phi'; \Lambda) \cong K_{\ell+2}(E', \mathcal{F}N'; \Lambda) \cong K_{\ell+2}(E, \mathcal{F}N; \Lambda)/(\partial\alpha).$$

Since $H_{\ell+3}(\Phi; \Lambda)$ is finitely generated over Λ, we will have $H_{\ell+3}(\Phi'; \Lambda) \cong 0$ after a finite number of the procedures above.

Apply Namioka's theorem (I) with $k = \ell+2$, $r = \ell+3$. Then we have

$$\pi_i(\Phi') = 0 \quad \text{for} \quad i \leqq \ell+3.$$

Apply Lemma 3.4 with $\ell+1$ in place of ℓ, then we have $\pi_i(f'_E) = 0$ $(i \leqq \ell+3)$, $\pi_i(f'_{\mathcal{F}}) = 0$ $(i \leqq \ell+2)$ and $\pi_i(f'_L) = 0$ $(i \leqq \ell+2)$.

Proceeding inductively we will have

$$\begin{aligned} \pi_i(\Phi) &= 0 \ \ (i \leqq \left[\frac{m}{2}\right] + 1), \quad \pi_i(f_E) = 0 \ \ (i \leqq \left[\frac{m}{2}\right] + 1), \\ \pi_i(f_{\mathcal{F}}) &= 0 \ \ (i \leqq \left[\frac{m}{2}\right]), \quad \text{and} \quad \pi_i(f_L) = 0 \ \ (i \leqq \left[\frac{m}{2}\right]). \end{aligned}$$

This completes the proof of Theorem 3.3. □

4. Proof of Theorem 2.5 in the Odd Dimensional Case

Suppose $m = \dim L^m = 2n+1 \geqq 5$; $n = \left[\frac{m}{2}\right]$. Let $\Phi = \begin{pmatrix} E & \xrightarrow{f_E} & F \\ \uparrow & & \uparrow \\ \mathcal{F}N & \xrightarrow{f_{\mathcal{F}}} & \mathcal{F}T \end{pmatrix}$.

There are two exact sequences which contain $\pi_{n+2}(\Phi)$;

$\alpha)$ $\pi_{n+2}(\Phi) \xrightarrow{\partial} \pi_{n+1}(E, \mathcal{F}N) \to \pi_{n+1}(F, \mathcal{F}T) \to \cdots,$

β) $\pi_{n+2}(\Phi) \xrightarrow{\partial'} \pi_{n+1}(f_{\mathcal{F}}) \to \pi_{n+1}(f_E) \to \cdots$.

In §3 we performed surgery below the middle dimension, thus we may assume

$$\begin{aligned} \pi_i(f_L) &= 0 \quad \text{for} \quad i \leqq n, \\ \pi_i(f_E) &= 0 \quad \text{for} \quad i \leqq n+1, \\ \pi_i(f_{\mathcal{F}}) &= 0 \quad \text{for} \quad i \leqq n \quad \text{(Theorem 3.3).} \end{aligned}$$

Then from β), we have

β') $\pi_{n+2}(\Phi) \xrightarrow{\partial'} \pi_{n+1}(f_{\mathcal{F}}) \to 0$.

By (3.4) in the proof of Lemma 3.4, we have $\pi_{n+1}(f_{\mathcal{F}}) \cong \pi_{n+1}(f_L) \cong H_{n+1}(f_L;\Lambda') \cong K_n(L;\Lambda')$. Thus β') becomes

β'') $\pi_{n+2}(\Phi) \xrightarrow{\partial'} K_n(L;\Lambda') \to 0$. ($\partial'$ is used by abuse of the notation.)

Our task is to make f_L a simple homotopy equivalence, and f_L is already n-connected. Thus we have only to kill $K_n(L;\Lambda')$. Wall's surgery theory [32] tells us that there is a unique obstruction $\theta(f_L)$ in $L_m(\pi')$ to killing the group $K_n(L;\Lambda')$ by abstract Wall surgery. $\theta(f_L)$ is defined by the surgery obstruction of the following normal map diagram:

$$(\overset{*}{**}) \qquad \begin{array}{ccc} \nu_L & \longrightarrow & (\overline{f})^*\nu_W | M \oplus T \\ \big\downarrow & & \big\downarrow \\ L^m & \xrightarrow{f_L} & M^m \end{array}$$

where $\overline{f} : V \to W$ is a homotopy inverse of $f : W \to V$ and ν_L (resp. ν_W) the normal bundle of L (resp. W).

We will show that $\theta(f_L)$ is also the obstruction to make f_L a simple homotopy equivalence by codimension 2 ambient surgery through a homotopy of f (rel. the boundary).

First suppose $\theta(f_L) = 0$. Then following [32], we can find a finite number of disjoint embeddings $g_i : S^n \times D^{n+1} \to L^m$, each joined by a path to a base point, having the property that if we perform surgery on them, the resulting $f_L : L \to M$ will be a simple homotopy equivalence. All of them represent elements of $K_n(L;\Lambda')$. Denote the elements by $[g_i]$. By β''), there are elements $\tilde{g}_i$ in $\Phi_{n+2}(\Phi)$ such that $\partial'\tilde{g}_i = [g_i]$. Let ∂ be the boundary homomorphism in α). Then $\partial\tilde{g}_i$ are elements of $\pi_{n+1}(E, \mathcal{F}N)$ which are represented by a map $h_i : (D^{n+1}, S^n) \to (E, \mathcal{F}N)$. In [8, Lemma 3.3] we proved that in the odd dimensional case where $m = 2n+1$ h_i's can be represented by normally embedded $(n+1)$-handles $H_i = D_i^{n+1} \times D_i^{n+1}$.

Moreover, it is easily verified that the attaching framed n-spheres $H_i \cap L^m = \partial D_i^{n+1} \times D_i^{n+1}$ can be taken to coincide with any framed n-spheres in L^m if they represent the same homotopy classes as $H_i \cap L^m$'s. Thus we may assume $H_i \cap L^m = g_i(S^n \times D^{n+1})$. Then Lemma 3.1 can be applied, and there is a homotopy (rel. ∂W)

of f to a map f' which realizes the surgery on the embedded spheres $g_i(S^n \times D^{n+1})$ as the ambient codimension 2 surgery along the $(n+1)$-handles H_i;

$$f'^{-1}(M^m) = (L^m - \bigcup_i \operatorname{Int} g_i(S^n \times D^{n+1})) \cup \bigcup_i D_i^{n+1} \times \partial D_i^{n+1}.$$

The resulting $f_{L'} : L' \to M$ is clearly a simple homotopy equivalence.

Conversely, suppose that f is homotopic (rel. ∂W) to a map f' such that

$$f'|f'^{-1}(M) : f'^{-1}(M) \to M$$

is a simple homotopy equivalence. Then by the transverse regularity theorem, we can construct a normal cobordism between ($\overset{*}{**}$) and a normal map which is a simple homotopy equivalence. Thus $\theta'(f_L) = 0$. This completes the proof of Theorem 2.5 in the odd dimensional cases. □

5. The Even Dimensional Case

Suppose $m = \dim L^m = 2n \geqq 4$. The sequences α) β) β') β'') remain valid in this case.

In order to prove Theorem 2.5 in the even dimensional case, we first introduce the notion of *geometric free cores* (Cf. [17, Proof of Lemma 5.8]).

Let $\Lambda' = \mathbb{Z}[\pi']$ as before. Wall [32, Lemma 2.3] proved that $K_n(L;\Lambda')$ is a stably free Λ'-module with a preferred equivalence class of s-basis. We may assume that $K_n(L;\Lambda')$ is actually Λ'-free, for after performing codimension 2 surgery on some trivial n-handles, we can change $K_n(L;\Lambda')$ into $K_n(L;\Lambda') \oplus$(standard planes). Here "A trivial n-handle" means an n-handle representing the zero element of $\pi_n(E, \mathcal{F}N)$. See Wall [32, Lemma 5.5]. Let $e_1, e_2, \ldots, e_r$ be the preferred Λ'-basis of $K_n(L;\Lambda')$. Then they are lifted to elements $\widetilde{e_1}, \widetilde{e_2}, \ldots, \widetilde{e_r}$ of $\pi_{n+2}(\Phi)$. See sequence β''). By the standard technique, we can represent the elements $\partial\widetilde{e_1}, \partial\widetilde{e_2}, \ldots, \partial\widetilde{e_r}$ of $\pi_{n+1}(E, \mathcal{F}N)$ by "pathed" disjoint embeddings $g_i : (D^{n+1}, \partial D^{n+1}) \to (E, \mathcal{F}N)$, $i = 1, \ldots, r$. (This technique is explained in [32, pp. 39–40].) Take a regular neighborhood R_i of $g_i(D^{n+1})$ in E and construct a submanifold $N \cup (\cup_{i=1}^r R_i)$ in W^{m+2}. This submanifold is called a *geometric free core* of $f : W \to V$ and is denoted by W^*. A geometric free core has some useful properties.

In order to state the properties we introduce some notations:

$U^* = \overline{W - W^*}$, the complement of $\operatorname{Int} W^*$,
$\mathcal{F}W^* = U^* \cap W^*$, the frontier of W^*
$N^* = \frac{1}{2}N$, the tubular neighborhood of L^m in W^*
(we may assume that the radius of N^* is a half of that of N),
$E^* = \overline{W^* - N^*}$, the exterior of N^* in W^*,
$\mathcal{F}N^* = N^* \cap E^*$, the frontier of N^*.

Theorem 5.1.

(i) *$f: W \to V$ is homotopic (rel. ∂W) to a map (again denoted by f) which is t-regular along $\mathcal{F}T$ with $f^{-1}(T) = W^*$.*

(ii) *$f|W^* : W^* \to T$ is a simple homotopy equivalence.*

(iii) *$f|\mathcal{F}W^* : \mathcal{F}W^* \to \mathcal{F}T$ is a Λ'-homology equivalence.*

(iv) *$f|U^* : U^* \to F$ is a Λ'-homology equivalence.*

(v) *$\pi_i(E^*, \mathcal{F}N^*) = 0$ for $i \leqq n$, and $\pi_{n+1}(E^*, \mathcal{F}N^*)$ is a free Λ-module with the basis $\partial\widetilde{e_1}, \ldots, \partial\widetilde{e_r}$.*

Remark 5.2. Λ'-homology of F is defined to be $H_*(\hat{F}, \mathbb{Z})$ in the notation of the proof of Lemma , similarly for Λ'-homologies of U^*, $\mathcal{F}W^*$ or $\mathcal{F}T$.

Proof of Theorem 5.1
(i) This was done in the proof of "Fundamental Lemma" of [15, § 2.1]. We will state the result in our present situation; we proved there that if the core disks of R_i's (representing elements of $\pi_{n+1}(E, \mathcal{F}N)$) are mapped to zero in $\pi_{n+1}(F, \mathcal{F}T)$ by f, then we can construct the desired homotopy which "splits" along T.

The condition is satisfied in our situation; the core disks of R_i's are of the form $\partial\widetilde{e_i}$, $\widetilde{e_i} \in \pi_{n+2}(\Phi)$, and so they are mapped to zero in $\pi_{n+1}(F, \mathcal{F}T)$. This follows by the exactness of (α).
(ii) A proof was given in [17, Lemma 5.2]. We repeat it here for completeness.

Let $\psi : L \to W^*$ be the inclusion, $\varphi : L \to T$ the composition $L \xrightarrow{f_L} M \xrightarrow{\sim} T$. Then we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(L) & \xrightarrow{\psi} & C_*(W^*) & \longrightarrow & C_*(\psi) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f|W^*} & & \downarrow{\scriptstyle \theta} & & ; \\
0 & \longrightarrow & C_*(L) & \xrightarrow{\varphi} & C_*(T) & \longrightarrow & C_*(\varphi) & \longrightarrow & 0
\end{array}$$

we identify the mapping cylinder of φ with T.

According to [32, Lemma 2.5], in order to prove that $f|W^*$ is a simple homotopy equivalence, we have only to show that θ is a simple homotopy equivalence. Let $\mathcal{H}$ be the Λ'-homology sequence of $C_*(\psi) \xrightarrow{\theta} C_*(\varphi)$, then we have

$$\tau(C_*(\varphi)) = \tau(C_*(\psi)) + \tau(\theta) + \tau(\mathcal{H})$$

([23, Theorem 3.2]), where τ denotes the Whitehead torsion. The only non-zero Λ'-homology of $C_*(\psi)$ is $H_{n+1}(\psi)$ which is based isomorphic to $K_n(L; \Lambda')$. This follows from the construction of W^*. On the other hand the only non-zero Λ'-homology of $C_*(\varphi)$ is $H_{n+1}(\varphi)$ which is only non vanishing kernel $K_n(L; \Lambda')$ of $f_L : L \to M$. θ_* induces the identity of these two $K_n(L; \Lambda')$'s by the construction of W^*. Thus we have $\tau(\mathcal{H}) = 0$.

The bases of $H_n(\varphi)$ and of $H_{n+1}(\psi)$ are chosen so that $\tau(C_*(\varphi)) = \tau(C_*(\psi)) = 0$ ([32, p. 27]). Hence $\tau(\theta) = 0$ follows as desired.

(iii) and (iv) follow easily from (ii) and the hypothesis that $f : W \to V$ is a homotopy equivalence.

(v) is obvious by the construction of W^*.

The proof of Theorem 5.1 is completed. □

Because of property (ii) of W^* (Theorem 5.1), W^* is an m-Poincaré thickening in the sense of [17, Definition 1.1]. Thus there is a unique obstruction $\eta(W^*)$ in $P_m(\mathcal{E})$ to finding a locally flat spine of W^* [17]. We assert that $\eta(W^*)$ serves as the obstruction of the weak h-regularity problem as well.

Lemma 5.3. *If $\eta(W^*) = 0$, f is homotopic (rel. ∂W) to a map denoted by the same letter f with $f|f^{-1}(M) : f^{-1}(M) \to M$ a simple homotopy equivalence.* ($m \geqq 6$).

Lemma 5.4. *The element $\eta(W^*) \in P_m(\mathcal{E})$ does not depend on a particular choice of a geometric free core W^* and depends only on the diagram*

$$\begin{array}{ccc} f : (W, \partial W) & \longrightarrow & (V, \partial V) \\ & & \cup \\ & & (M, \partial M). \end{array}$$

These lemmas will be proved in §5.2.

5.1. Seifert Forms

Let us recall the definition of Seifert $(-1)^n$-forms introduced in [17]. Let $\mathcal{E} = \{1 \to C \to \pi \overset{\varpi_*}{\to} \pi' \to 1\}$ be associated with the pair (W^*, L); $\pi = \pi_1(\mathcal{F}N^*) \cong \pi_1(E^*)$, and $\pi' = \pi_1(L)$. The homomorphism $\varpi_* : \pi \to \pi'$ is induced by the projection ϖ of the S^1-bundle $\mathcal{F}N^* \to L$. C is a cyclic group with a specified generator t. Let Λ be $\mathbb{Z}[\pi]$, and let Λ' be $\mathbb{Z}[\pi']$, as usual. Abelian groups $Q_n^t(\pi)$ and $Q_n(\pi')$ are defined by

$$\begin{array}{rcl} Q_n^t(\pi) & = & \Lambda/\{a - (-1)^n \overline{a} \cdot t \mid a \in \Lambda\}, \\ Q_n(\pi') & = & \Lambda'/\{b - (-1)^n \overline{b} \mid b \in \Lambda'\}, \end{array}$$

where $^-: \Lambda \to \Lambda$ (or $\Lambda' \to \Lambda'$) is induced by $g \mapsto g^{-1}$ for $g \in \pi$ or (by $g \mapsto g^{-1}$ for $g \in \pi'$). Here $Q_n(\pi')$ is Wall's notation [32]. The homomorphism $\varpi_* : \pi \to \pi'$ induces homomorphisms $\varpi_* : \Lambda \to \Lambda'$ and $\overline{\varpi}_* : Q_n^t(\pi) \to Q_n(\pi')$.

A triple (G, λ, μ) consisting of a finitely generated free Λ-module G and maps $\lambda : G \times G \to \Lambda$, $\mu : G \to Q_n^t(\pi)$ is called a *free Seifert $(-1)^n$-form over* $\mathcal{E}$ if it satisfies the following ([17]):

(a) $\lambda(x, y) = (-1)^n \overline{\lambda(y, x)} \cdot t$

(b) For any fixed y, $\lambda(*, y) : G \to \Lambda$ is a Λ-homomorphism.

(c) $\mu(x + y) = \mu(x) + \mu(y) + \lambda(x, y)$.

(d) $\lambda(x,x) = \mu(x) + (-1)^n\overline{\mu(x)} \cdot t$.

(e) $\mu(ax) = a\mu(x)\overline{a}$ for $\forall a \in \Lambda$.

(f) $\Lambda' \otimes_\Lambda G$ (denoted by G') is a free Λ'-module with a preferred basis $\{e_i\}$, and G' has a structure of Wall's special Hermitian $(-1)^n$-form (λ_0, μ_0).

(g) The following diagrams are commutative:

$$\begin{array}{ccc} G \times G & \xrightarrow{\lambda} & \Lambda \\ {\scriptstyle \partial\times\partial}\downarrow & & \downarrow{\scriptstyle \varpi_*} \\ G' \times G' & \xrightarrow{\lambda_0} & \Lambda' \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{\mu} & Q_n^t(\pi) \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \varpi_*}, \\ G' & \xrightarrow{\mu_0} & Q_n(\pi') \end{array}$$

where $\partial : G \to G'$ is defined by $\partial(x) = 1 \otimes x$. Sometimes (G', λ_0, μ_0) is denoted by $\Lambda \otimes_\Lambda (G, \lambda, \mu)$.

A free Seifert $(-1)^n$-form (G, λ, μ) is *null-cobordant* if there is a sub Λ-module H of G such that $\lambda(H \times H) = 0$, $\mu(H) = 0$ and H is mapped onto a subkernel of G' under $\partial : G \to G'$.

(G, λ, μ) is *stably null-cobordant* if a direct sum of (G, λ, μ) and a finite number of "standard planes" defined by $(\Lambda x \oplus \Lambda y, \lambda, \mu)$ with $\lambda(x,y) = 1$, $\lambda(y,x) = (-1)^n t$, $\mu(x) = \mu(y) = 0$, is null-cobordant. (Note that our "standard plane" is not the same as in the Wall's book [32].)

Definition 5.5.([17]) The group $P_{2n}(\mathcal{E})$ is defined to be the Grothendieck group of all free Seifert $(-1)^n$-forms over $\mathcal{E}$ modulo the subgroup generated by all stably null-cobordant forms. The group structure is given by the direct sum $\oplus$.

Remark 5.6. $P_{2n+1}(\mathcal{E})$ is defined to be $L_{2n+1}(\pi')$.

Remark 5.7. Note that (G, λ, μ) represents the zero of $P_{2n}(\mathcal{E})$ if and only if there is a stably null-cobordant form Y such that $(G, \lambda, \mu) \oplus Y$ is stably null-cobordant. However we have proved, in [17, Lemma 5.3], that it is equivalent to saying that (G, λ, μ) itself is stably null-cobordant.

5.2. Geometric Meaning of Seifert Forms

Let W^* be a free core of $f : W \to V$. We proved in [17] that the Λ-module $\pi_{n+1}(E^*, \mathcal{F}N^*)$ carries a structure of a Seifert $(-1)^n$-form (λ, μ). (See Appendix of the present paper.)

The following properties are important to our purpose:

(i) $\Lambda' \otimes_\Lambda \pi_{n+1}(E^*, \mathcal{F}N^*)$ is identified with $K_n(L;\Lambda')$, and $\Lambda' \otimes_\Lambda (\lambda,\mu)$ with (λ_0,μ_0), Wall's Hermitian form on $K_n(L;\Lambda')$.

(ii) Elements $x_1, x_2, \dots, x_s$ of $\pi_{n+1}(E^*, \mathcal{F}N^*)$ can be represented by mutually disjoint normally embedded $(n+1)$-handles if and only if $\mu(x_i)=0$ $(\forall i)$, and $\lambda(x_i,x_j)=0$ $(\forall i,\, j)$, $(2n \geqq 6)$.

We can now prove Lemma 5.3.

Proof of Lemma 5.3. The obstruction $\eta(W^*) \in P_{2n}(\mathcal{E})$ is represented by the free Seifert $(-1)^n$-form $(\pi_{n+1}(E^*, \mathcal{F}N^*), \lambda, \mu)$. Suppose $\eta(W^*)=0$, then by Remark 5.7 above, the Seifert form $(\pi_{n+1}(E^*, \mathcal{F}N^*), \lambda, \mu)$ is stably null-cobordant. After performing codimension 2 surgery along some trivial n-handles, we may suppose it is actually null-cobordant (See [17, Lemma 4.6].). Then there is a sub Λ-module H which is mapped onto a subkernel of $K_n(L;\Lambda')$, satisfying $\lambda(H\times H)=0$, $\mu(H)=0$. Let $e_1,\dots,e_r$ be the basis of the subkernel.

These $e_1,\dots,e_r$ are lifted to some $\widetilde{e_1},\dots,\widetilde{e_r}$ in $H \subset \pi_{n+1}(E^*, \mathcal{F}N^*)$. By the geometric property (ii) of Seifert forms, $\widetilde{e_1},\dots,\widetilde{e_r}$ can be represented by mutually disjoint normally embedded $(n+1)$-handles $H_1,\dots,H_r$. If F^*, T^* are defined by $T^* = \dfrac{1}{2}T$, $F^* = \overline{T-T^*}$, it is clear that $\pi_i(F^*, \mathcal{F}T^*)=0$ for all i. Thus the quadruple $\begin{pmatrix} E^* & \longrightarrow & F^* \\ \Big\uparrow & & \Big\uparrow \\ \mathcal{F}N^* & \longrightarrow & \mathcal{F}T^* \end{pmatrix}$ denoted by Φ^*, satisfies $\pi_{i+2}(\Phi^*) \overset{\cong}{\to} \pi_{i+1}(E^*, \mathcal{F}N^*)$ for all i. Lemma 3.1 can now be applied to $H_1,\dots,H_r$, and $f|W^* : W^* \to T$; we can find a homotopy (rel. ∂W^*) of $f|W^*$ to a map f' with $f'^{-1}(M)$ the resulting submanifold obtained by performing surgery on the framed n-spheres $L^m \cap H_i \cong S^n \times D^{n+1}$ representing $e_1,\dots,e_r$. Since $\{e_1,\dots,e_r\}$ is the preferred basis of a subkernel of $K_n(L;\Lambda')$, the map $f'^{-1}(M) \to M$ must be a simple homotopy equivalence [32]. Extending the homotopy of $f|W^*$to the whole of W by the identity, we will obtain the desired homotopy. □

Proof of Lemma 5.4. Let W_1^* be another free core of $f: W \to V$. We have to prove that $\eta(W^*) = \eta(W_1^*) \in P_{2n}(\mathcal{E})$. There is a homotopy of f (rel. ∂W) "between W^* and W_1^*"; this means that there is a map $H: W\times I \to V \times I$ with $f = H|W\times\{0\} : W\times\{0\} \to V\times\{0\}$, $f' = H|W\times\{1\} : W\times\{1\} \to V\times\{1\}$, and $f^{-1}(T\times\{0\}) = W^*$ and $f'^{-1}(T\times\{1\}) = W_1^*$ hold.

As we proved in [8], the submanifold $M\times I$ in $V\times I$ can be transformed into an exterior n-connected submanifold Y^{2n+1} by exchanging handles in codimension 2 (rel. $\partial(M\times I)$). Let U be a tubular neighborhood of Y in $V\times I$, B the exterior of U; $B = \overline{V\times I - U}$. U can be taken so that $U\cap(T\times\{i\}) = T^*\times\{i\}$ $(i=0,1)$. Let $\mathcal{F}U$ be the frontier of U; $\mathcal{F}U = U\cap B$.

Now consider the preimage $Z^{2n+1} = H^{-1}(Y) \subset W\times I$. By exchanging handles in codimension 2, Z^{2n+1} can be made exterior n-connected, and this surgery can be carried out as surgery through a homotopy of H (rel. $\partial(W\times I)$). This is because Y^{2n+1} is already exterior n-connected, so any normally embedded i-handle in $W \times I$ attached to Z^{2n+1} shrinks if it is mapped into $(B, \mathcal{F}U)$, provided that $i \leqq n$. Therefore Lemma 3.1 is applied.

Let Q denote a tubular neighborhood of Z in $W \times I$, $\mathcal{F}Q$ the frontier of Q, and let P be the exterior of Q.

Let $\Psi = \begin{pmatrix} E^* \cup E_1^* & \longrightarrow & P \\ \uparrow & & \uparrow \\ \mathcal{F}N^* \cup \mathcal{F}N_1^* & \longrightarrow & \mathcal{F}Q \end{pmatrix}$, $\Psi' = \begin{pmatrix} F^* \cup F_1^* & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathcal{F}T^* \cup \mathcal{F}T_1^* & \longrightarrow & \mathcal{F}U \end{pmatrix}$, where E_1^*, $\mathcal{F}N_1^*$ (or F_1^*, $\mathcal{F}T_1^*$) denote the exterior and the frontier in W_1^* (or T_1). By the exterior n-connectivity of Y and Z, we have

$$(*)\qquad \begin{array}{ccccccc} \begin{matrix} H_{n+1}(E^*, \mathcal{F}N^*; \Lambda) \\ \oplus \\ H_{n+1}(E_1^*, \mathcal{F}N_1^*; \Lambda) \end{matrix} & \longrightarrow & H_{n+1}(P, \mathcal{F}Q; \Lambda) & \longrightarrow & H_{n+1}(\Psi; \Lambda) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{n+1}(B, \mathcal{F}U; \Lambda) & \longrightarrow & H_{n+1}(\Psi'; \Lambda) & \longrightarrow & 0 \end{array}$$

Here we used the fact that $\pi_i(F^*, \mathcal{F}T^*) \cong \pi_i(F_1^*, \mathcal{F}T_1^*) = 0$ $(\forall i)$. Note that the map H is of degree one, hence the vertical maps are onto. We can kill the kernel of $H_{n+1}(\Psi; \Lambda) \to H_{n+1}(\Psi'; \Lambda)$ and make it an isomorphism. Here we will give an indication of it. By Namioka's theorem, any element of $H_{n+1}(\Psi; \Lambda)$ is represented by a suitable embedded $n+1$-disk $g : (D^{n+1}; D_+^n, D_-^n) \to (P; E^*, \mathcal{F}Q)$, where D_+^n (or D_-^n) is the upper (or lower) hemisphere of ∂D^{n+1}, with $f|D_+^n : (D_+^n, \partial D_+^n) \to (E^*, \mathcal{F}N^*)$ being null-homotopic. By "attaching a collar" to $g(D_-^n)$ we extend the embedding g to an embedding $\overline{g} : (D^{n+1}; D_+^n, D_-^n) \to (W \times I; W^*, Z)$ which satisfies $\overline{g}(D^{n+1}) \cap Z = \overline{g}(D_-^n)$. Then this extends to a"normally embedded knob" $\tilde{g} : (D^{n+1}; D_+^n, D_-^n) \times D^{n+1} \to (W \times I; W^*, Z)$ satisfying $\tilde{g}(D^{n+1} \times D^{n+1}) \cap Z = \tilde{g}(D_-^n \times D^{n+1})$. (See the proof of [17, Proposition 4.11].) If the first embedding $g : (D^{n+1}; D_+^n, D_-^n) \to (P; E^*, \mathcal{F}Q)$ represents an element in the kernel of $H_{n+1}(\Psi; \Lambda) \to H_{n+1}(\Psi'; \Lambda)$, it is proved that we can construct a homotopy (rel. $\overline{\partial(W \times I) - W^* \times \{0\}}$) from H to a map H' which satisfies

$$H'^{-1}(Y) = \overline{Z - \tilde{g}(D_-^n \times D^{n+1})} \cup \tilde{g}(D^{n+1} \times \partial D^{n+1}).$$

The construction of the homotopy is essentially the same as in the proof of Lemma which was omitted. Repeating the above "knob exchanging" process, we can kill the kernel of $H_{n+1}(\Psi; \Lambda) \to H_{n+1}(\Psi'; \Lambda)$.

The effect of the surgery on $H_{n+1}(E^*, \mathcal{F}N^*)$ is to add a direct sum of standard planes, and this does not affect the class in $P_{2n}(\mathcal{E})$ which it represents. (Cf. the proof of [17, Proposition 4.11].) And we remain using the same notations E^*, $\mathcal{F}^*N$.

Now we consider the diagram $(*)$ above again. Since we have made the vertical map on the right an isomorphism, we get the exact sequence (with Λ-coefficients):

$$H_{n+1}(E^*;\mathcal{F}N^*) \oplus H_{n+1}(E_1^*;\mathcal{F}N_1^*) \to H_{n+1}(P,\mathcal{F}Q) \to H_{n+1}(B,\mathcal{F}U) \to 0.$$

Note that the kernel of $H_{n+1}(P,\mathcal{F}Q) \to H_{n+1}(B,\mathcal{F}U)$ is $K_{n+1}(P,\mathcal{F}Q)$ by the definition, so we have

$$(**) \qquad H_{n+1}(E^*,\mathcal{F}N^*) \oplus H_{n+1}(E_1^*,\mathcal{F}N_1^*) \to K_{n+1}(P,\mathcal{F}Q) \to 0.$$

Let K denote the kernel of this surjection. We have

$$H_{n+1}(E^*,\mathcal{F}N^*) \oplus H_{n+1}(E_1^*,\mathcal{F}N_1^*) \cong \pi_{n+1}(E^*,\mathcal{F}N^*) \oplus \pi_{n+1}(E_1^*,\mathcal{F}N_1^*)$$

by the Hurewicz theorem, and it has the Seifert $(-1)^n$-form representing $\eta(W)^* - \eta(W_1)^*$. It is shown in [17, Theorem 3.5, Proposition 4.11] that the Seifert form vanishes on the sub Λ-module K.

Notice that $\Lambda' \otimes_\Lambda H_{n+1}(E^*,\mathcal{F}N^*) \cong K_n(L,\Lambda')$, $\Lambda' \otimes_\Lambda H_{n+1}(E_1^*,\mathcal{F}N_1^*) \cong K_n(L_1,\Lambda')$. Tensoring Λ' with $(**)$, we have

$$\Lambda' \otimes_\Lambda K \to K_n(L;\Lambda') \oplus K_n(L_1;\Lambda') \to K_n(Z,\Lambda') \to 0.$$

This shows (by Wall [32, Proof of 5.7]) that $\Lambda' \otimes_\Lambda K$ is mapped onto the subkernel of $K_n(L;\Lambda') \oplus K_n(L_1;\Lambda')$.

Thus the Seifert form on $\pi_{n+1}(E^*,\mathcal{F}N^*) \oplus \pi_{n+1}(E_1^*,\mathcal{F}N_1^*)$ is null-cobordant by definition, and the element $\eta(W^*) - \eta(W_1^*)$ which it represents is zero in $P_{2n}(\mathcal{E})$, i.e., $\eta(W^*) = \eta(W_1^*)$. This completes the proof of Lemma 5.4. $\square$

Now Theorem 2.5 follows immediately from Lemmas 5.3 and 5.4; we define $\gamma(f)$ in the theorem to be $\eta(W^*)$ in the even dimensional case. The proof of Theorem 2.5 is completed. $\square$

6. Some Properties of the Obstruction

6.1. Geometric Periodicity

Theorem 6.1. *Let* $\begin{pmatrix} f:(W^{m+2},\partial W) & \to & (V^{m+2},\partial V) \\ & & \cup \\ & & (M^m,\partial M) \end{pmatrix}$ *be a diagram of the weak h-regularity problem with* $m \geqq 4$ *satisfying* $(C.1) \sim (C.4)$ *in §2. Let* $\mathcal{E}$ *be the associated extension. Then we have*

$$\gamma(\mathrm{id}_{\mathbb{C}P_2} \times f) = \rho(\gamma(f)) \in P_{m+4}(\mathcal{E}),$$

where $\begin{pmatrix} \mathrm{id}_{\mathbb{C}P_2} \times f : \mathbb{C}P_2 \times (W^{m+2},\partial W) & \to & \mathbb{C}P_2 \times (V^{m+2},\partial V) \\ & & \cup \\ & & \mathbb{C}P_2 \times (M^m,\partial M) \end{pmatrix}$ *is the product with the complex projective plane* $\mathbb{C}P_2$*, and* $\rho : P_m(\mathcal{E}) \to P_{m+4}(\mathcal{E})$ *is the algebraic periodicity isomorphism.*

Proof. In the odd dimensional case, this is obvious by the definition of $\gamma(f) = \theta(f_L)$. In the even dimensional case, we have defined $\gamma(f)$ by $\eta(W^*)$ with W^* a geometric free core of f. The product $\mathbb{C}P_2 \times W^*$ is clearly a geometric free core of $\mathrm{id}_{\mathbb{C}P_2} \times f$. Therefore, we have

$$\begin{aligned} \gamma(\mathrm{id}_{\mathbb{C}P_2} \times f) &= \eta(\mathrm{id}_{\mathbb{C}P_2} \times W^*) \\ &= \rho(\eta(W^*)) \quad ([17, \text{Theorem } 5.12]) \\ &= \rho(f). \end{aligned}$$

□

6.2. The Invariance of $\gamma(f)$ under L-equivalence of M

Let f be as in Theorem 6.1. We will say that f *splits along* M if f is homotopic (rel. ∂W) to a map f' which is weakly h-regular along M. The obstruction $\gamma(f)$ in Theorem 2.5 is written here as $\gamma_M(f)$ to emphasize the submanifold M.

Theorem 6.2. *Let $(M_i^m, \partial M_i) \subset (V^{m+2}, \partial V)$ $(i = 1, 2)$ be exterior 2-connected submanifolds of $(V, \partial V)$ satisfying conditions $(C.1) \sim (C.4)$ (of §1). Suppose that $\partial M_1 = \partial M_2$ and that M_1 and M_2 are L-equivalent (rel. the boundary) to each other in the sense of Thom. Then we have*

$$\gamma_{M_1}(f) = \gamma_{M_2}(f) \in P_m(\mathcal{E}),$$

$(m \geqq 4)$.

Corollary 6.3. *If $m \geqq 5$, f splits along M_1 if and only if it splits along M_2.*

Outline of the proof of Theorem 6.2. Since M_1 and M_2 are L-equivalent, there exists a submanifold Y^{2n+1} in $V \times I$ such that

$$\begin{aligned} Y \cap V \times \{0\} &= M_1 \times \{0\}, \\ Y \cap V \times \{1\} &= M_2 \times \{1\}, \quad \text{and} \\ Y \cap \partial V \times I &= \partial M_1 \times I. \end{aligned}$$

Then the rest of the proof is completely the same as Lemma 5.4 or, in the odd dimensional case, as in §4. □

We will collect some known results on the structure of the group $P_m(\mathcal{E})$.

Theorem 6.4.

(i) $P_{2k+1}(1 \to C \to \pi \to \pi' \to 1) = L_{2k+1}(\pi')$

(ii) $P_m(1 \to 1 \to \pi \xrightarrow{\mathrm{id}} \pi \to 1) = L_m(\pi)$, *in particular*
$P_m(1 \to 1 \to 1 \to 1 \to 1) = L_m(1)$,

(iii) $P_m(1 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 1 \to 1) = C_{m-1}$ $(m \geqq 5)$
(C_{m-1} *is the knot cobordism group of* $(m-1, m+1)$*-knots.*)

The isomorphism (i) is the definition of $P_{2k+1}(\mathcal{E})$. The isomorphisms (ii) and (iii) are proved in [17].

Let $P_m(C)$ denote the group $P_m(1 \to C \to C \to 1 \to 1)$ for simplicity.

Theorem 6.5. *$P_{4k+2}(C)$ is infinitely generated if C is a cyclic group of even* (*or infinite*) *order.*

For the proof of Theorem 6.5, see [17].

Let $\mathcal{E}$ denote an extension $\{1 \to C \xrightarrow{j} \pi \to \pi' \to 1\}$ with j the inclusion. Let ρ be the morphism defined by the following diagram:

$$\rho : \begin{array}{ccccccccc} 1 & \longrightarrow & C & \xrightarrow{\text{id}} & C & \longrightarrow & 1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 1 & \longrightarrow & C & \xrightarrow{j} & \pi & \longrightarrow & \pi' & \longrightarrow & 1 \end{array} .$$

Theorem 6.6. *If π' is a finite group, then all elements in the kernel of $\rho_* : P_m(C) \to P_m(\mathcal{E})$ are of finite order whose orders divide the order of π'. Here ρ_* is the homomorphism induced by ρ.*

Corollary 6.7. *If $C = \mathbb{Z}$ and the order of π' is odd, then the homomorphism $\rho_* : P_m(\mathbb{Z}) \to P_m(\mathcal{E})$ is injective.*

Proof of Corollary 6.7. According to Theorem 6.4 (iii), $P_m(\mathbb{Z}) \cong C_{m-1}$. Levine [11, 12] has proved that C_{m-1} contains no odd torsions. Thus Corollary 6.7 follows from Theorem 6.6. □

Proof of Theorem 6.6. By the periodicity of $P_m(\mathcal{E})$, we may assume that m is sufficiently large. Let $\xi : E \to M^{m-1}$ be a closed 2-disk bundle over an $(m-1)$-manifold which "represents" $\mathcal{E}$, i.e., the short exact sequence derived from the homotopy exact sequence of the associated circle bundle $\partial E \to M$ is isomorphic to $\mathcal{E}$. In the following, M is identified with the zero section of $E \to M$.

It is proved in [17] that $P_m(\mathbb{Z}) \to P_m(C)$ is onto and any element σ of $P_m(C)$ is represented by a PL $(m-1, m+1)$-knot $\kappa = (\Sigma^{m-1}, S^{m+1})$. Taking a cone over $\kappa = (\Sigma^{m-1}, S^{m-1})$, we have a (not necessarily locally flat) disk pair (Δ^m, D^{m+2}). Consider a pairwise boundary connected sum of $(E \times I, M^{m-1} \times I)$ and (D^{m+2}, Δ^m). Then it is clear that $(W, \partial((M \times I)\natural\Delta^m)) = ((E \times I)\natural D^{m+2}, \partial((M^{m-1} \times I)\natural\Delta^m))$ is an m-Poincaré pair.

Here we recall the main result of [17]: An oriented manifold pair (W^{m+2}, K^{m-1}) with $K^{m-1} \subset \partial W^{m+2}$ is called an *m-Poincaré thickening pair* if K is locally flat in ∂W and the pair is simple homotopy equivalent to an m-Poincaré pair. A locally flat submanifold L^m of W^{m+2} is said to be a *(locally flat) spine* if $\partial L^m = K^{m-1}$ and the inclusion $L^m \to W^{m+2}$ is a simple homotopy equivalence. The main result of [17] states the following: *Given an m-Poincaré thickening pair (W^{m+2}, K^{m-1}) with $m \geqq 5$, there exists a well-defined obstruction element η in $P_m(\mathcal{E})$ which vanishes if and only if the pair (W^{m+2}, K^{m-1}) admits a spine (cf. Appendix below).*

Now returning to our present situation, we have $\eta = \rho_*(\sigma)$ with $\rho_* : P_m(C) \to P_m(\mathcal{E})$, where the element $\sigma \in P_m(C)$ is represented by the PL $(m-1, m+1)$-knot $\kappa = (\Sigma^{m-1}, S^{m+1})$ taken above. (This follows from the naturality of our obstruction [17, § 5, Complement 1] and the construction of $W = (E \times I)\natural D^m$.)

Suppose that the element $\sigma \in P_m(C)$ is in the kernel of $\rho_* : P_m(C) \to P_m(\mathcal{E})$. Then $\eta = \rho_*(\sigma) = 0$, thus by the main result of [17], a spine L^m of $(W, \partial((M \times I)\natural\Delta^m))$ can be found.

Let $\pi : \tilde{W} \to W$ be the universal covering of W. It is easy to see that the associated extension $\tilde{\mathcal{E}}$ of the m-Poincaré thickening $(\tilde{W}, \pi^{-1}(\partial L^m))$ is $1 \to C \to C \to 1 \to 1$. Clearly $\tilde{W} = \widetilde{E \times I}\natural\ell(D^{m+2}, \Delta^m)$ where $\ell = |\pi'|$ is the order of $\pi' = \pi_1(W) \cong \pi_1(M)$. Thus the obstruction to finding a spine of $(\tilde{W}, \pi^{-1}(\partial L^m))$ is $\ell \cdot \sigma \in P_m(C)$. However, we have already such a spine $\tilde{L} = \pi^{-1}(L^m)$. Therefore, $\ell \cdot \sigma = 0$. This completes the proof of Theorem 6.6. $\square$

A. Appendix

In [17] we show that a kind of intersection form can be defined on $\pi_{n+1}(E, \mathcal{F}N)$, which is associated with an even-dimensional submanifold of codimension two. The form is called the *Seifert form*, and it plays an essential role in the present paper. In this appendix, we will recall the geometric definition of it.

Let L^{2n} be a $2n$-dimensional (locally flat) submanifold of a $(2n+2)$-dimensional manifold W^{2n+2}. We suppose that it is exterior 2-connected. Thus a short exact sequence $\mathcal{E}$ is associated with the manifold pair:

$$\mathcal{E} = \{1 \to C \to \pi \to \pi' \to 1\},$$

where $\pi = \pi_1(W - L)$, $\pi' = \pi_1(W)$, $C = \mathrm{Coker}(\pi_2(W) \to \pi_1(S^1))$. (Cf. §2.) Let E be the exterior of a regular neighborhood N of L, $\mathcal{F}N$ the frontier of N : $\mathcal{F}N = N \cap E$. Then we have $\pi = \pi_1(E) \cong \pi_1(\mathcal{F}N)$, $\pi' = \pi_1(W) \cong \pi_1(L)$. A map $f : (D^{n+1}, S^n) \to (E^{2n+2}, \mathcal{F}N)$ is said to be a *nice immersion* if

(i) f is a generic immersion in the sense of Haefliger. Thus $f|\mathrm{Int}D^{n+1}$ has only a finite number of isolated double points at which $f(D^{n+1})$ intersects with itself transversely,

(ii) $f|S^n : S^n \to \mathcal{F}N$ is an embedding, and

(iii) the composition $\varpi \circ (f|S^n) : S^n \to L^{2n}$ is a generic immersion, where ϖ is the projection map of the S^1-bundle $\mathcal{F}N \to L^{2n}$.

A nice immersion f is *pathed* if a path $\gamma(f)$ in $\mathcal{F}N$ from a base point $* \in \mathcal{F}N$ to a point in the image $f(S^n)$ is specified.

Two nice immersions f, $g : (D^{n+1}, S^n) \to (E^{2n+2}, \mathcal{F}N)$ *intersect nicely* if

(i) $f(D^{n+1})$ and $g(D^{n+1})$ intersect in general position,

(ii) $f(S^n) \cap g(S^n) = \emptyset$ and

(iii) $\varpi \circ f(S^n)$ and $\varpi \circ g(S^n)$ intersect in general position.

Assume that two pathed nice immersions f and g intersect nicely. Then we will define a pairing $\lambda(f,g)$ as an element of $\Lambda = \mathbb{Z}[\pi]$ as follows:

Let $\{p_1, \cdots, p_k\}$ be the set of intersection points of $\varpi \circ f(S^n)$ and $\varpi \circ g(S^n)$ in L^{2n}. Let $\varepsilon_i(f,g)$ be the sign ± 1 of the intersection at p_i. We are assuming that the S^1-fiber of ϖ is oriented. The orientation convention will be described later. We take a following loop $\ell_i(f,g)$ in $\mathcal{F}N$:

$$\ell_i(f,g) = \{* \overset{\gamma(f)}{\to} p_i^f \to (\text{along the } S^1\text{-fiber } \varpi^{-1}(p_i) \text{ in the positive direction}) \to p_i^g \overset{\gamma(g)^{-1}}{\to} *\},$$

where p_i^f (or p_i^g) is the point of $f(S^n)$ (or $g(S^n)$) over p_i, i.e., $\{p_i^f\} = f(S^n) \cap \varpi^{-1}(p_i) \subset \mathcal{F}N$ (or $\{p_i^g\} = g(S^n) \cap \varpi^{-1}(p_i) \subset \mathcal{F}N$). Let $g_i(f,g) \in \pi_1(\mathcal{F}N)$ be represented by the loop $\ell_i(f,g)$.

An auxiliary pairing $\alpha(f,g) \in \mathbb{Z}[\pi]$ is defined by

$$\alpha(f,g) = \sum_{i=1}^{k} \varepsilon_i(f,g) g_i(f,g).$$

In order to define the pairing $\lambda(f,g)$ we need another auxiliary pairing $\beta(f,g)$, which is defined as follows: Let $\{q_1, \ldots, q_\ell\}$ be the set of intersection points of $f(D^{n+1})$ and $g(D^{n+1})$ in E^{2n+2}, ε'_i the sign ± 1 of the intersection at q_i.

Let $g'_i(f,g) \in \pi_1(E)$ ($\cong \pi_1(\mathcal{F}N)$) be defined by the following loop in E:

$$g'_i(f,g) = \{* \xrightarrow{\gamma(f)} q_i \xrightarrow{\gamma(g)^{-1}} *\}.$$

Then the pairing $\beta(f,g)$ is defined by

$$\beta(f,g) = \sum_{i=1}^{\ell} \varepsilon'_i(f,g) g'_i(f,g).$$

Now the pairing $\lambda(f,g)$ is defined by the following formula:

$$\text{(A.1)} \qquad \lambda(f,g) = \alpha(f,g) + (-1)^{n+1}(1-t)\beta(f,g).$$

Here t denotes the positive generator of the cyclic group C,

Next we will define an element $\mu(f) \in Q_n^t(\pi)$ which corresponds to the self-intersection of f.

Let $\alpha(f)$, $\beta(f) \in \mathbb{Z}[\pi]$ be defined analogously to the definition of $\alpha(f,g)$, $\beta(f,g)$. To define $\alpha(f)$ and $\beta(f)$ we have only to replace the "intersection points" in the definition of $\alpha(f,g)$ and $\beta(f,g)$ by the "self-intersection points" with an order of the two branches of $\varpi \circ f(S^n)$ (or $f(D^{n+1})$) arbitrarily fixed at each self-intersection point. If the order is reversed, $\alpha(f)$ and $\beta(f)$ change. However it is proved in [17] that the ambiguity of $\alpha(f)$ and $(1-t)\beta(f)$ is contained in the subgroup $I = \{a - (-1)^n \overline{a} \cdot t | a \in \Lambda\}$ of $\Lambda (= \mathbb{Z}[\pi])$, so $\alpha(f)$ and $(1-t)\beta(f)$ are well-defined as elements of $Q_n^t(\pi) = \Lambda / I$. An integer $\mathcal{O}(f) \in \mathbb{Z}$ is defined as follows: Let v be a non-singular vector field over $\mathcal{F}N$ which is along S^1-fibers in their positive directions.

By the condition (iii) of nice immersions, v is transverse to $f(S^n)$. Let $\mathcal{O}(f)$ be the obstruction to extending this non-zero cross-section of the normal bundle of $f(S^n)$ in $\mathcal{F}N$ to a non-zero cross-section of the normal bundle of $f(D^{n+1})$ in E. At this point we use the orientation conventions which will be stated in § A.1 below. Finally, $\mu(f) \in Q_n^t(\pi)$ is defined by the following:

$$\mu(f) = \alpha(f) + (-1)^{n+1}(1-t)\beta(f) + (-1)^{n+1}\mathcal{O}(f), \tag{A.2}$$

where $\mathcal{O}(f)$ is considered to be an element of $Q_n^t(\pi)$ via $\mathbb{Z} \to \mathbb{Z}e \subset \Lambda \to Q_n^t(\pi)$.

A.1. Orientation Conventions.

For an oriented manifold X, we will denote by $[X]$ its orientation, and by $[X]_p$ the local orientation at p. We are given $[L^{2n}]$ and $[W^{2n+2}]$. Then $[E]$ is defined by $[E]_p = [W]_p$ $(\forall p \in E)$. $[\mathcal{F}N]$ is defined by $[E]_p = [\mathcal{F}N]_p \times \mathcal{U}_p$, where $\mathcal{U}_p$ is the inward normal direction of E at $p \in \mathcal{F}N$. The *positive direction* $[S^1]$ of an S^1-fiber is given by $[\mathcal{F}N] = [L^{2n}] \times [S^1]$. The normal fiber $\mathbb{R}^{n+1}$ of $f(S^n)$ in $\mathcal{F}N$ is oriented by $[\mathcal{F}N] = [f(S^n)] \times [\mathbb{R}^{n+1}]$.

Proposition A.1.([17, Theorems 2.5, 2.9]) *Under the above orientation conventions, $\lambda(f, g)$ and $\mu(f)$ depend only on the homotopy classes of f and g.*

Thus we can define the following maps.

$$\begin{array}{rcl} \lambda & : & \pi_{n+1}(E, \mathcal{F}N) \times \pi_{n+1}(E, \mathcal{F}N) \to \Lambda, \\ \mu & : & \pi_{n+1}(E, \mathcal{F}N) \longrightarrow Q_n^t(\pi). \end{array}$$

This completes the geometric definition of Seifert forms.

Remark A.2. $\pi_{n+1}(E, \mathcal{F}N)$ is not necessarily a free Λ-module. Moreover, even if it is Λ-free, the bilinear form λ is not in general unimodular. The deviation from unimodularity "measures" the extent to which the submanifold L^{2n} is knotted. The "determinant" $\det \lambda$ is somewhat like the Alexander polynomial. The property (a) of Seifert forms (§5.1) corresponds to the reciprocity of Alexander polynomials, and property (f) corresponds to the fact $\Delta(1) = \pm 1$ in the classical theory.

Acknowledgments and Postscript. I forget exactly when I finished my old note [16], but seeing the bibliography of the note, I infer that it was not later than 1973. I submitted the note to *Topology.* But it was rejected. I was very much disappointed, and lost the will to publish this note. Thirty years later, the note gave a motivation to define the "pull back relation"for non-spherical knots, which was investigated in our joint paper [2]. There we cited the note [16] under the title *Note on the splitting problem in codimension two*, because the word *the weak h-regularity problem* contained in the original title had already become difficult to understand its meaning.

Since then I began considering the possibility of publishing the note somewhere. Akio Kawauchi encouraged me, and suggested Kyungpook Mathematical Journal having a long tradition in Korea as a possible journal to which my note should be submitted. The final impulse was given by Kokoro Tanaka and Reiko Shinjo, who kindly offered to convert my note (written on an old-fashioned typewriter) into a TeX file. I am very grateful to these friends of mine for their benevolent encouragement and cooperation, without which this paper would have never been published.

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