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Hopf Hypersurfaces in Complex Two–plane Grassmannians with Generalized Tanaka–Webster Reeb–parallel Structure Jacobi Operator

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Abstract. In relation to the generalized Tanaka-Webster connection, we consider a new notion of parallel structure Jacobi operator for real hypersurfaces in complex two-plane Grassmannians and prove the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka-Webster parallel structure Jacobi operator.

1. Introduction

In complex projective spaces or in quaternionic projective spaces, many differential geometers studied real hypersurfaces with parallel curvature tensor [8, 9, 10, 14, 15, 16]. Taking a new perspective, we look to classify real hypersurfaces in complex two-plane Grassmannians with parallel structure Jacobi operator; that

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is, having $\nabla R_\xi = 0$ [6, 7, 12, 14].

As an ambient space, a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. There are two natural geometric conditions to consider for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. The first is that a 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and a 3-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M [2], where the *Reeb* vector field ξ is defined by $\xi = -JN$, and N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The second is that the *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$.

Using a result from Alekseevskii [1], Berndt and Suh [2] proved the following:

Theorem A. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 2 [11] it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Now, instead of the Levi-Civita connection, we consider the *generalized Tanaka-Webster* connection $\hat{\nabla}$ for contact Riemannian manifolds introduced by Tanno [18]. The original *Tanaka-Webster connection* [17, 19] is given as a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifolds which associated with the almost contact structure. Cho [4, 5] defined the generalized Tanaka-Webster connection for a real hypersurface of a Kähler manifold as

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y,$$

where $k \in \mathbb{R} \setminus \{0\}$.

We put the Reeb vector field ξ into the curvature tensor R of a real hypersurface M in $G_2(C^{m+2})$. Then for any tangent vector field X on M, the structure Jacobi operator R_ξ is defined by

$$R_\xi(X) = R(X, \xi)\xi.$$

Using this structure Jacobi operator R_ξ, in [6] and [7] the authors proved non-existence theorems. On the other hand, using the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$, we considered the notion of $\mathfrak{D}^{\perp}$*-parallel structure Jacobi operator in the generalized Tanaka-Webster connection*, that is, $(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$ for any $X \in \mathfrak{D}^{\perp}$ and any tangent vector field Y in M. We gave a classification theorem as follows (see [13]):

Theorem B. *Let M be a connected orientable Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the structure Jacobi operator R_ξ is $\mathfrak{D}^{\perp}$-parallel in the generalized Tanaka-Webster connection, M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

In the present paper, motivated by Theorem B, we consider another new notion for generalized Tanaka-Webster parallelism of the structure Jacobi operator on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, when the structure Jacobi operator R_ξ of M satisfies $(\hat{\nabla}^{(k)}_\xi R_\xi)Y = 0$ for any tangent vector field Y in M. In this case, the stucture Jacobi operator is said to be a *Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.* We can give a non-existence theorem as follows:

Main Theorem. *There does not exist any Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

On the other hand, we consider another new notion for generalized Tanaka-Webster parallelism of the structure Jacobi operator on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. If the structure Jacobi operator R_ξ of M satisfies $(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$ for any tangent vector fields X and Y in M, then the the structure Jacobi operator is said to be *parallel structure Jacobi operator in the generalized Tanaka-Webster connection.* Naturally, we see that this notion of parallel structure Jacobi operator in the generalized Tanaka-Webster connection is stronger than Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Related to this notion, we have the following corollary.

Corollary. *There does not exist any Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

We refer to [1, 2, 3] and [11, section 1] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ and [11, section 2] for basic formulas of tangent space at $p \in M$ of real hypersufaces M in $G_2(\mathbb{C}^{m+2})$.

2. Key Lemma

Let us denote by $R(X,Y)Z$ the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$. Then the structure Jacobi operator R_ξ of M in $G_2(\mathbb{C}^{m+2})$ can be defined by $R_\xi X = R(X,\xi)\xi$ for any vector field $X \in T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$, $x \in M$. In [6] and [7], by using the structure Jacobi operator R_ξ, the authors obtained

$$
\begin{aligned}
(2.1)\qquad &(\nabla_X R_\xi)Y\\
&= -g(\phi AX,Y)\xi - \eta(Y)\phi AX\\
&\quad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX,Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\\
&\qquad + 3\Big\{g(\phi_\nu AX,\phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi\\
&\qquad\quad + \eta_\nu(\phi Y)\big(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu\big)\Big\}\\
&\qquad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX,Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big]\\
&\quad + \eta\big((\nabla_X A)\xi\big)AY + \alpha(\nabla_X A)Y - \eta\big((\nabla_X A)Y\big)A\xi\\
&\quad - g(AY,\phi AX)A\xi - \eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX.
\end{aligned}
$$

On the other hand, by using the generalized Tanaka-Webster connection, we have

$$
\begin{aligned}
(2.2)\qquad (\hat\nabla^{(k)}_X R_\xi)Y &= \hat\nabla^{(k)}_X(R_\xi Y) - R_\xi(\hat\nabla^{(k)}_X Y)\\
&= \nabla_X(R_\xi Y) + g(\phi AX, R_\xi Y)\xi - \eta(R_\xi Y)\phi AX - k\eta(X)\phi R_\xi Y\\
&\quad - R_\xi(\nabla_X Y + g(\phi AX,Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y).
\end{aligned}
$$

From this, together with the fact that M is Hopf, it becomes

$$
\begin{aligned}
(2.3)\qquad &(\hat\nabla^{(k)}_X R_\xi)Y\\
&= -\sum_{\nu=1}^{3}\Big[g(\phi_\nu AX,Y)\xi_\nu - \eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\\
&\qquad + 3\Big\{g(\phi_\nu AX,\phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi\\
&\qquad + \eta_\nu(\phi Y)\Big(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu\Big)\Big\}\\
&\qquad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX,Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\\
&\qquad + \eta_\nu(Y)\eta_\nu(\phi AX)\xi - \eta_\nu(\xi)\eta(Y)\eta_\nu(\phi AX)\xi\\
&\qquad + 3\eta(\phi_\nu Y)g(\phi AX,\phi_\nu\xi)\xi + \eta_\nu(\xi)g(\phi AX,\phi_\nu\phi Y)\xi\\
&\qquad - \eta_\nu(Y)\eta_\nu(\xi)\phi AX + \eta_\nu^2(\xi)\eta(Y)\phi AX - \eta_\nu(\xi)\eta(\phi_\nu\phi Y)\phi AX\\
&\qquad - k\eta(X)\eta_\nu(Y)\phi\xi_\nu - 4k\eta(X)\eta(\phi_\nu Y)\eta_\nu(\xi)\xi - 4k\eta(X)\eta(\phi_\nu Y)\xi_\nu\\
&\qquad + 3\eta(Y)\eta(\phi_\nu\phi AX)\phi_\nu\xi - \eta(Y)\eta_\nu(\xi)\phi_\nu AX + \alpha\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi\\
&\qquad + 3k\eta(X)\eta(\phi_\nu\phi Y)\phi_\nu\xi + k\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi\Big]\\
&\quad + \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi\\
&\quad - \alpha\eta(Y)(\nabla_X A)\xi - \alpha k\eta(X)\phi AY + \alpha k\eta(X)A\phi Y
\end{aligned}
$$

for any tangent vector fields X and Y on M. Let us assume that the structure Jacobi operator R_ξ on a Hopf hypersurface M in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ is *Reeb-parallel* in the generalized Tanaka-Webster connection, that is,

$$(\hat{\nabla}^{(k)}_{\xi} R_\xi)Y = 0 \tag{*}$$

for any tangent vector field Y on M.

Here, it is a main goal to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}^\perp$ or orthogonal complement of $\mathfrak{D}^\perp$ (i.e., $\mathfrak{D}$) such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the structure Jacobi operator is Reeb-parallel in the generalized Tanaka-Webster connection.

From now on, unless otherwise stated in the present section, we may put the Reeb vector field ξ as follows :

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{**}$$

for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

Putting $X = \xi$ in (2.3) and using the condition (*), we have

$$\begin{aligned}
0 =&(\hat{\nabla}^{(k)}_{\xi} R_\xi)Y \\
=&-\sum_{\nu=1}^{3}\Big[\alpha g(\phi_\nu\xi, Y)\xi_\nu + \alpha\eta_\nu(Y)\phi_\nu\xi \\
&\qquad + 3\Big\{\alpha g(\phi_\nu\xi, \phi Y)\phi_\nu\xi + \alpha\eta(Y)\eta_\nu(\xi)\phi_\nu\xi - \alpha\eta_\nu(\phi Y)\xi_\nu\Big\} \\
&\qquad + 4\eta_\nu(\xi)\Big\{\alpha\eta_\nu(\phi Y)\xi - \alpha g(\xi, Y)\phi_\nu\xi\Big\} \\
&\qquad - k\eta_\nu(Y)\phi\xi_\nu - 4k\eta(\phi_\nu Y)\eta_\nu(\xi)\xi - 4k\eta(\phi_\nu Y)\xi_\nu \\
&\quad + 3k\eta(\phi_\nu\phi Y)\phi_\nu\xi + k\eta(Y)\eta_\nu(\xi)\phi_\nu\xi\Big] \\
&+ \eta((\nabla_\xi A)\xi)AY + \alpha(\nabla_\xi A)Y - \alpha\eta((\nabla_\xi A)Y)\xi \\
&- \alpha\eta(Y)(\nabla_\xi A)\xi - \alpha k\phi AY + \alpha kA\phi Y
\end{aligned} \tag{2.4}$$

for any tangent vector field Y on M.

Now, using these facts, we prove the following Lemma.

Lemma 2.1. *Let M be a Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. By taking the inner product with ξ in (2.4), it becomes

$$\begin{aligned} 0 = &- \sum_{\nu=1}^{3} \Big\{ \alpha g(\phi_\nu \xi, Y)\eta_\nu(\xi) - 3\alpha\eta_\nu(\phi Y)\eta_\nu(\xi) + 4\alpha\eta_\nu(\xi)\eta_\nu(\phi Y) \\ &\qquad 4k\eta_\nu(\phi Y)\eta_\nu(\xi) - 4k\eta(\phi_\nu Y)\eta_\nu(\xi) \Big\} \\ &+ \alpha\eta((\nabla_\xi A)\xi)\eta(Y) + \alpha\eta((\nabla_\xi A)Y) - \alpha\eta((\nabla_\xi A)Y) - \alpha\eta(Y)\eta((\nabla_\xi A)\xi) \\ = &\ \ 8k\eta(\phi_1 Y)\eta_1(\xi) \\ = &- 8kg(Y, \phi_1\xi)\eta_1(\xi) \\ = &- 8k\eta(X_0)\eta(\xi_1)g(Y, \phi_1 X_0) \end{aligned}$$

for any tangent vector field Y on M, since $\phi\xi_1 = \eta(X_0)\phi_1 X_0$.
Thus substituting Y with $\phi_1 X_0$, it follows

$$k\eta(X_0)\eta(\xi_1) = 0.$$

Since k is a nonzero real number, we get $\eta(X_0)\eta_1(\xi) = 0$, that is, $\eta(X_0) = 0$ or $\eta_1(\xi) = 0$. It means that ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$. Accordingly, it completes the proof of our Lemma. □

3. Proof of The Main Theorem

Let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Reeb-parallel structure Jacobi operator R_ξ in the generalized Tanaka-Webster connection, that is, $(\hat{\nabla}^{(k)}_\xi R_\xi)Y = 0$ for any vector field Y on M. Then by Lemma 2.1 we shall divide our consideration in two cases of which the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}^\perp$ or the distribution $\mathfrak{D}$.

First of all, we consider the case $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 3.1. *If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then there does not exist any Hopf hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

Proof. Since our assumption ξ belongs to the distribution $\mathfrak{D}^\perp$, using (2.4), we have

$$\begin{aligned} 0 = &- \Big\{ \alpha g(\phi_2\xi, Y)\xi_2 + \alpha g(\phi_3\xi, Y)\xi_3 + \alpha\eta_2(Y)\phi_2\xi + \alpha\eta_3(Y)\phi_3\xi \\ &\quad + 3\alpha g(\phi_2\xi, \phi Y)\phi_2\xi + 3\alpha g(\phi_3\xi, \phi Y)\phi_3\xi - 3\alpha\eta_2(\phi Y)\xi_2 \\ &\quad - 3\alpha\eta_3(\phi Y)\xi_3 - k\eta_2(Y)\phi\xi_2 - k\eta_3(Y)\phi\xi_3 - 4k\eta(\phi_2 Y)\xi_2 \\ &\quad - 4k\eta(\phi_3 Y)\xi_3 + 3k\eta(\phi_2\phi Y)\phi_2\xi + 3k\eta(\phi_3\phi Y)\phi_3\xi \Big\} \\ &+ \eta((\nabla_\xi A)\xi)AY + \alpha(\nabla_\xi A)Y - \alpha\eta((\nabla_\xi A)Y)\xi \\ &- \alpha\eta(Y)(\nabla_\xi A)\xi - \alpha k\phi AY + \alpha kA\phi Y \end{aligned}$$

$$\begin{aligned}&= -8k\eta_2(Y)\xi_3 + 8k\eta_3(Y)\xi_2 + \eta((\nabla_\xi A)\xi)AY + \alpha(\nabla_\xi A)Y\\&\quad -\alpha\eta((\nabla_\xi A)Y)\xi - \alpha\eta(Y)(\nabla_\xi A)\xi - \alpha k\phi AY + \alpha kA\phi Y\end{aligned}$$

for any tangent vector field Y on M. Taking the inner product with X, we have

$$\begin{aligned}0 &= g((\hat{\nabla}^{(k)}_\xi R_\xi)Y, X)\\&= -8k\eta_2(Y)\eta_3(X) + 8k\eta_3(Y)\eta_2(X) + \eta((\nabla_\xi A)\xi)g(AY, X)\\&\quad + \alpha g((\nabla_\xi A)Y, X) - \alpha\eta(X)\eta((\nabla_\xi A)Y) - \alpha\eta(Y)g((\nabla_\xi A)\xi, X)\\&\quad - \alpha kg(\phi AY, X) + \alpha kg(A\phi Y, X)\end{aligned} \tag{3.5}$$

for any tangent vector fields X and Y on M. Interchanging X with Y in above equation, we get

$$\begin{aligned}0 &= g((\hat{\nabla}^{(k)}_\xi R_\xi)X, Y)\\&= -8k\eta_2(X)\eta_3(Y) + 8k\eta_3(X)\eta_2(Y) + \eta((\nabla_\xi A)\xi)g(AX, Y)\\&\quad + \alpha g((\nabla_\xi A)X, Y) - \alpha\eta(Y)\eta((\nabla_\xi A)X) - \alpha\eta(X)g((\nabla_\xi A)\xi, Y)\\&\quad - \alpha kg(\phi AX, Y) + \alpha kg(A\phi X, Y)\end{aligned} \tag{3.6}$$

for any tangent vector fields X and Y on M. Thus subtracting (3.6) from (3.5), we obtain

$$\begin{aligned}0 &= g((\hat{\nabla}^{(k)}_\xi R_\xi)Y, X) - g((\hat{\nabla}^{(k)}_\xi R_\xi)X, Y)\\&= 16k\eta_3(Y)\eta_2(X) - 16k\eta_2(Y)\eta_3(X)\end{aligned} \tag{3.7}$$

for any tangent vector fields X and Y on M. Since k is a nonzero real number, the equation (3.7) reduces to

$$\eta_3(Y)\eta_2(X) - \eta_2(Y)\eta_3(X) = 0 \tag{3.8}$$

for any tangent vector fields X and Y on M. Replacing X with ξ_2 and Y with ξ_3, we have

$$\eta_3(\xi_3) = 0. \tag{3.9}$$

Let $\{e_1,\ e_2, \cdots,\ e_{4m-4},\ e_{4m-3},\ e_{4m-2},\ e_{4m-1}\}$ be an orthonormal basis for a tangent vector space T_xM at any point $x \in M$. Without loss of generality, we may put $e_{4m-3} = \xi_1,\ e_{4m-2} = \xi_2$ and $e_{4m-1} = \xi_3$. Since the dimension of M is equal to $4m-1$, above equation (3.9) gives a contradiction. So, we can assert our Lemma 3.1. □

Next we consider the case $\xi \in \mathfrak{D}$. Using Theorem A, Lee and Suh [11] gave a characterization of real hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Lemma 3.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$.*

From the above two Lemmas 3.1 and 3.2 and the classification theorem given by Theorem A in this paper, we see that M is locally congruent to a model space of type (B) in Theorem A under the assumption of our Main Theorem given in the introduction.

Hence it remains to check that whether the stucture Jacobi operator R_ξ of real hypersurfaces of type (B) satisfies the condition (*) for any tangent vector field Y on M or not. In order to do this, we introduce a proposition related to eigenspaces of the model space of type (B) with respect to the shape operator. As the following proposition [2] is well known, a real hypersurface M of type (B) has five distinct constant principal curvatures as follows:

Proposition 3.3. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \operatorname{Span}\{\xi\},\\
&T_\beta = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_\nu \mid \nu = 1,2,3\},\\
&T_\gamma = \mathfrak{J}\xi = \operatorname{Span}\{\phi_\nu\xi \mid \nu = 1,2,3\},\\
&T_\lambda,\\
&T_\mu,
\end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$ where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

To check this problem, we suppose that M has a Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Putting $X = \xi \in \mathfrak{D}$ in (2.4), it becomes

$$
\begin{aligned}
(3.10)\qquad & -\sum_{\nu=1}^{3}\Big[\alpha g(\phi_\nu\xi,Y)\xi_\nu+\alpha\eta_\nu(Y)\phi_\nu\xi+3\Big\{\alpha g(\phi_\nu\xi,\phi Y)\phi_\nu\xi-\alpha\eta_\nu(\phi Y)\xi_\nu\Big\}\\
&\qquad -k\eta_\nu(Y)\phi\xi_\nu-4k\eta(\phi_\nu Y)\xi_\nu+3k\eta(\phi_\nu\phi Y)\phi_\nu\xi\Big]\\
&+\eta((\nabla_\xi A)\xi)AY+\alpha(\nabla_\xi A)Y-\alpha\eta((\nabla_\xi A)Y)\xi\\
&-\alpha\eta(Y)(\nabla_\xi A)\xi-\alpha k\phi AY+\alpha kA\phi Y=0
\end{aligned}
$$

for any tangent vector field Y on M. Replacing Y into $\xi_2\in T_\beta$, we get

$$
\begin{aligned}
0=&-\sum_{\nu=1}^{3}\Big[\alpha\eta_\nu(\xi_2)\phi_\nu\xi+3\alpha g(\phi_\nu\xi,\phi\xi_2)\phi_\nu\xi-k\eta_\nu(\xi_2)\phi\xi_\nu-3k\eta_\nu(\xi_2)\phi_\nu\xi\Big]\\
&+\alpha(\nabla_\xi A)\xi_2-\alpha\eta((\nabla_\xi A)\xi_2)\xi-\alpha k\phi A\xi_2\\
=&-4\alpha\phi\xi_2+4k\phi\xi_2+\alpha^2\beta\phi\xi_2-\alpha\beta k\phi\xi_2
\end{aligned}
$$

because of $(\nabla_\xi A)\xi=0$, $(\nabla_\xi A)\xi_2=\alpha\beta\phi\xi_2$, $\gamma=0$ and equations [13, (1.4) and (1.6)]. Taking the inner product with $\phi_2\xi$, we have

$$(\alpha-k)(-4+\alpha\beta)=0.$$

Since $\alpha\beta=-4$ by virtue of Proposition, it follows that

$$\alpha=k. \tag{3.11}$$

On the other hand, putting $Y\in T_\lambda$ in (3.10), we get

$$\alpha(\nabla_\xi A)Y-\alpha\eta((\nabla_\xi A)Y)\xi-\alpha k\phi AY+\alpha kA\phi Y=0 \tag{3.12}$$

Using the equation of Codazzi [13, (1.10)], we know

$$
\begin{aligned}
(\nabla_\xi A)Y&=(\nabla_Y A)\xi+\phi Y\\
&=\alpha\phi AY-A\phi AY+\phi Y.
\end{aligned}
$$

Thus the equation (3.12) can be written as

$$\alpha^2\lambda\phi Y-\alpha\lambda\mu\phi Y+\alpha\phi Y-\alpha\lambda k\phi Y+\alpha\mu k\phi Y=0, \tag{3.13}$$

because of $\phi Y\in T_\mu$. Therefore, inserting (3.11) in (3.13) we have

$$-\alpha\lambda\mu\phi Y+\alpha\phi Y+\alpha^2\mu\phi Y=0.$$

Taking the inner product with ϕY, we obtain

$$-\alpha\lambda\mu+\alpha+\alpha^2\mu=0.$$

Since $\alpha = -2\tan(2r)$, $\lambda = \cot(r)$, $\mu = -\tan(r)$ with some $r \in (0, \pi/4)$, from Proposition, we get $\tan^2(r) = -1$. This gives a contradiction. So this case can not occur.

Hence summing up these assertions, we give a complete proof of our main theorem in the introduction.

On the other hand, we consider a new notion which is different from Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. The *parallel stucture Jacobi operator in the generalized Tanaka-Webster connection* can be defined in such a way that

$$(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$$

for any tangent vector fields X and Y on M. From this notion, together with Lemmas 2.1, 3.1, 3.2 and the classification theorem given by Theorem A in the introduction, we see that M is locally congruent to a model space of type (B) in Theorem A. Hence we can check that whether the stucture Jacobi operator R_ξ of real hypersurfaces of type (B) satisfies the condition (*) for any tangent vector fields X and Y in M or not.

To check this problem, we suppose that M has a parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Putting $X = \xi_2 \in T_\beta$ and $Y = \xi \in \mathfrak{D}$ in (2.3) , it becomes

$$\begin{aligned} 0 &= (\hat{\nabla}^{(k)}_{\xi_2} R_\xi)\xi \\ &= -\sum_{\nu=1}^{3} \Big[\beta g(\phi_\nu\xi_2, \xi)\xi_\nu - \beta\eta_\nu(\phi\xi_2)\xi_\nu \\ &\qquad + 3\beta\eta_\nu(\xi_2)\phi_\nu\xi + 3\beta\eta(\phi_\nu\phi\xi_2)\phi_\nu\xi\Big] \\ &= -6\beta\phi_2\xi. \end{aligned}$$

By taking the inner product with $\phi_2\xi$, we have $\beta = 0$. It gives a contradiction. Accordingly, we give a complete proof of our Corollary in the introduction.

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