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Strong Roman Domination in Grid Graphs

Xue-Gang Chen
Department of Mathematics, North China Electric Power University, Beijing 102206, China
e-mail : gxcxdm@163.com

Moo Young Sohn*
Department of Mathematics, Changwon National University, Changwon 51140, Korea
e-mail : mysohn@changwon.ac.kr

Abstract. Consider a graph G of order n and maximum degree Δ. Let $f : V(G) \to \{0, 1, \cdots, \lceil\frac{\Delta}{2}\rceil + 1\}$ be a function that labels the vertices of G. Let $B_0 = \{v \in V(G) : f(v) = 0\}$. The function f is a strong Roman dominating function for G if every $v \in B_0$ has a neighbor w such that $f(w) \geq 1 + \lceil\frac{1}{2}|N(w) \cap B_0|\rceil$. In this paper, we study the bounds on strong Roman domination numbers of the Cartesian product $P_m \Box P_k$ of paths P_m and paths P_k. We compute the exact values for the strong Roman domination number of the Cartesian product $P_2 \Box P_k$ and $P_3 \Box P_k$. We also show that the strong Roman domination number of the Cartesian product $P_4 \Box P_k$ is between $\lceil\frac{1}{3}(8k - \lfloor\frac{k}{8}\rfloor + 1)\rceil$ and $\lceil\frac{8k}{3}\rceil$ for $k \geq 8$, and that both bounds are sharp bounds.

1. Introduction

Graph theory terminology not presented here can be found in [1]. Let $G = (V, E)$ be a graph with $|V| = n$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph G is clear from context, we simply write $d(v)$, $N(v)$ and $N[v]$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *diameter* $diam(G)$ of a connected graph G is the maximum distance between two vertices of G. The graph induced by $S \subseteq V$ is denoted by $G[S]$. A *path* on n vertices is denoted by P_n.

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For two graphs G_1 and G_2, the *Cartesian product* $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, where vertex (u_1, v_1) is adjacent to vertex (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$. $G = P_m \square P_k$ is called a grid graph.

Let $\{v_{ij} | 1 \leq i \leq m, 1 \leq j \leq k\}$ be the vertex set of $G = P_m \square P_k$ so that the subgraph induced by $\mathcal{R}_i = \{v_{i1}, v_{i2}, \cdots, v_{ik}\}$ is isomorphic to the path P_k for each $1 \leq i \leq m$ and the subgraph induced by $\mathcal{C}_j = \{v_{1j}, v_{2j}, \cdots, v_{mj}\}$ is isomorphic to the path P_m for each $1 \leq j \leq k$.

A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A dominating set of G with cardinality $\gamma(G)$ is called a γ-set of G.

Let $f : V \to \{0, 1, 2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. Such a function is called a *Roman dominating function*. The weight of a Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and is denoted $\gamma_R(G)$. Roman domination was defined and discussed by Stewart [4] in 1999. It was developed by ReVelle and Rosing [3] in 2000 and Cockayne et al. [2] in 2004. In order to deal with multiple simultaneous attacks, Álvarez-Ruiz et al. [1] in 2017 initiated the study of a new parameter related to Roman dominating function, which is called strong Roman domination.

Consider a graph G of order n and maximum degree Δ. Let $f : V(G) \to \{0, 1, \cdots, \lceil \frac{\Delta}{2} \rceil + 1\}$ be a function that labels the vertices of G. Let $B_0 = \{v \in V : f(v) = 0\}$. Then f is a *strong Roman dominating function* for G, if every $v \in B_0$ has a neighbor w, such that $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap B_0| \rceil$. The weight of a strong Roman dominating function is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a strong Roman dominating function on G is called the *strong Roman domination number* of G and is denoted $\gamma_{StR}(G)$. A strong Roman dominating function of G with weight $\gamma_{StR}(G)$ is called a γ_{StR}-function of G. For any $S \subseteq V$, define $f(S) = \sum_{v \in S} f(v)$. Let f be a γ_{StR}-function of G. Let $B_2 = \{v \in V : f(v) \geq 2\}$. For any $v \in B_0$, there exists a vertex $u \in B_2$ such that $vu \in E(G)$. We say that v is dominated by u or by B_2. If f is a strong Roman dominating function of G, then every vertex in B_0 is dominated by some vertex in B_2.

In this paper, we study the bounds on strong Roman domination numbers of the Cartesian product $P_m \square P_k$ of paths P_m and paths P_k. Exact values for the strong Roman domination number of the Cartesian product $P_2 \square P_k$ and $P_3 \square P_k$ are found, and it is shown that for the strong Roman domination number of the Cartesian product $P_4 \square P_k$ this number is between $\lceil \frac{1}{3}(8k - \lfloor \frac{k}{8} \rfloor + 1) \rceil$ and $\lceil \frac{8k}{3} \rceil$ for $k \geq 8$, and both bounds are sharp bounds.

2. Bounds of Strong Roman Domination Number of $P_m \square P_k$

In this section, we present upper and lower bounds on the strong Roman domination number of the Cartesian product of paths P_m and paths P_k.

Observation 2.1. *For any positive integers m, k such that $k \equiv 0(mod\ 3)$, $\gamma_{StR}(P_m \Box P_k) \leq \frac{2mk}{3}$.*

Proof. Let $G = P_m \Box P_k$, where $k = 3t$ for a positive integer t. Let $V_2 = \{v_{ij} | 1 \leq i \leq m, j = 3l-1, 1 \leq l \leq t\}$, $V_1 = \varnothing$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Clearly, (V_0, V_1, V_2) is a partition of $V(G)$. Define f on $V(G)$ by $f(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 2$. It is obvious that f is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq |f(V(G))| = \frac{2mk}{3}$. □

Observation 2.2. *For any positive integers m, k such that $k \equiv 1(mod\ 3)$, $\gamma_{StR}(P_m \Box P_k) \leq \frac{m(2k+1)}{3}$.*

Proof. Let $G = P_m \Box P_k$, where $k = 3t+1$ for a positive integer t. Let $V_2 = \{v_{ij} | 1 \leq i \leq m, j = 3l-1, 1 \leq l \leq t\}$, $V_1 = \{v_{ik} | 1 \leq i \leq m\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Clearly, (V_0, V_1, V_2) is a partition of $V(G)$. Define f on $V(G)$ by $f(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 2$. It is obvious that f is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq |f(V(G))| = \frac{m(2k+1)}{3}$. □

Observation 2.3. *For any positive integers m, k such that $k \equiv 2(mod\ 3)$, then*

$$\gamma_{StR}(P_m \Box P_k) \leq \begin{cases} \frac{m(4k+1)}{6} & \text{if } m \equiv 0\ (mod\ 2) \\ \frac{m(4k+1)+3}{6} & \text{if } m \equiv 1\ (mod\ 2). \end{cases}$$

Proof. Let $G = P_m \Box P_k$, where $k = 3t+2$ for a positive integer t. Suppose $m \equiv 0\ (mod\ 2)$. Let $V_2 = \{v_{ij} | 1 \leq i \leq m, j = 3l-1, 1 \leq l \leq t\} \cup \{v_{jk} | j = 4l+1, 0 \leq l \leq \lfloor \frac{m-1}{4} \rfloor\} \cup \{v_{jk} | j = 4l, 1 \leq l \leq \lfloor \frac{m}{4} \rfloor\}$, $V_1 = \{v_{j(k-1)} | j = 4l+2, 0 \leq l \leq \lfloor \frac{m-1}{4} \rfloor\} \cup \{v_{j(k-1)} | j = 4l-1, 1 \leq l \leq \lfloor \frac{m}{4} \rfloor\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Clearly, (V_0, V_1, V_2) is a partition of $V(G)$. Define f on $V(G)$ by $f(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 2$. It is obvious that f is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq f(V(G)) = \frac{m(4k+1)}{6}$.

Suppose that $m \equiv 1\ (mod\ 2)$. Let $V_2 = \{v_{ij} | 1 \leq i \leq m, j = 3l-1, 1 \leq l \leq t\} \cup \{v_{jk} | j = 4l+1, 0 \leq l \leq \lfloor \frac{m-2}{4} \rfloor\} \cup \{v_{jk} | j = 4l, 1 \leq l \leq \lfloor \frac{m-1}{4} \rfloor\}$, $V_1 = \{v_{j(k-1)} | j = 4l+2, 0 \leq l \leq \lfloor \frac{m-2}{4} \rfloor\} \cup \{v_{j(k-1)} | j = 4l-1, 1 \leq l \leq \lfloor \frac{m-1}{4} \rfloor\} \cup \{v_{m(k-1)}, v_{mk}\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Clearly, (V_0, V_1, V_2) is a partition of $V(G)$. Define f on $V(G)$ by $f(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 2$. It is obvious that f is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq f(V(G)) = \frac{m(4k+1)+3}{6}$. □

Lemma 2.4.([1]) *Let G be a connected graph of order n. Then $\gamma_{StR}(G) \geq \lceil \frac{n+1}{2} \rceil$.*

By the following result, we improve the above result for a connected graph G with $\Delta(G) \leq 4$.

Theorem 2.5. *Let G be a connected graph of order n with $\Delta(G) \leq 4$. Then $\gamma_{StR}(G) \geq \lceil \frac{3n}{5} \rceil$.*

Proof. Let f be a γ_{StR}-function of G, and let $B_0 = \{w \in V(G) |\ f(w) = 0\}$, $B_1 = \{w \in V(G) |\ f(w) = 1\}$ and $B_2 = \{w \in V(G) |\ f(w) \geq 2\}$. Let $B_2^i = \{w \in$

$B_2| \; |N(w) \cap B_0| = i\}$ for $i = 1, 2, 3, 4$. Let $B_0^1 = \{w \in B_0| \; |N(w) \cap B_2| = 1\}$ and $B_0^2 = \{w \in B_0| \; |N(w) \cap B_2| \geq 2\}$. Clearly (B_0, B_1, B_2) is a partition of $V(G)$, $(B_2^1, B_2^2, B_2^3, B_2^4)$ is a partition of B_2 and (B_0^1, B_0^2) is a partition of B_0. Hence, $n = |B_0| + |B_1| + |B_2|$, $|B_2| = |B_2^1| + |B_2^2| + |B_2^3| + |B_2^4|$ and $|B_0| = |B_0^1| + |B_0^2|$. Among all γ_{StR}-function of G, let f be chosen so that $|B_1|$ is maximized.

Claim 1. $B_2^1 = \emptyset$.

Proof. Suppose that $B_2^1 \neq \emptyset$. Say $v \in B_2^1$ and $N(v) \cap B_0 = \{u\}$. Define f' on $V(G)$ by $f'(x) = f(x)$ for $x \in V(G) - \{u, v\}$, $f'(u) = 1$ and $f'(v) = 1$. Obviously f' is a γ_{StR}-function of G with $|B_1|$ more than f, which is a contradiction. □

Claim 2. $B_2^3 = \emptyset$.

Proof. Suppose that $B_2^3 \neq \emptyset$. Say $v \in B_2^3$ and $N(v) \cap B_0 = \{u_1, u_2, u_3\}$. Define f' on $V(G)$ by $f'(x) = f(x)$ for $x \in V(G) - \{u_1, v\}$, $f'(u_1) = 1$ and $f'(v) = 2$. Obviously f' is a γ_{StR}-function of G with $|B_1|$ more than f, which is a contradiction. □

Claim 3. Let $u \in B_2^2$. Then $N(u) \cap B_0 \subseteq B_0^1$.

Proof. Say $N(u) \cap B_0 = \{w_1, w_2\}$. Suppose that $w_1 \notin B_0^1$. So $w_1 \in B_0^2$. Define f' on $V(G)$ by $f'(x) = f(x)$ for $x \in V(G) - \{u, w_2\}$, $f'(u) = 1$ and $f'(w_2) = 1$. Obviously f' is a γ_{StR}-function of G with $|B_1|$ more than f, which is a contradiction. □

By Claim 1, $|B_2| = |B_2^2| + |B_2^4|$. Let $E(B_0, B_2)$ denote the edge set between B_0 and B_2. It is obvious that $|B_0^1| + 2|B_0^2| \leq |E(B_0, B_2)| \leq 2|B_2^2| + 4|B_2^4|$. So, $|B_0| + |B_0^2| \leq 2|B_2^2| + 4|B_2^4|$. Hence, $n + |B_0^2| \leq |B_1| + 3|B_2^2| + 5|B_2^4|$. Hence

$$\begin{aligned} \gamma_{StR}(G) &= |B_1| + 2|B_2^2| + 3|B_2^4| \\ &= \frac{2}{3}(n + \frac{1}{2}|B_1| + 2|B_2^2| + \frac{7}{2}|B_2^4| - |B_0|) \\ &\geq \frac{2}{3}(n + |B_0^2| + \frac{1}{2}(|B_1| - |B_2^4|)). \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \gamma_{StR}(G) &= |B_1| + 2|B_2^2| + 3|B_2^4| \\ &= |B_1| + 3(\frac{2|B_2^2|}{3}) + 5(\frac{3|B_2^4|}{5}) \\ &\geq \frac{3}{5}(|B_1| + 3|B_2^2| + 5|B_2^4|) \\ &\geq \frac{3}{5}(n + |B_0^2|) \\ &\geq \frac{3}{5}n. \end{aligned}$$

Therefore, the result follows, since $\gamma_{StR}(G)$ is an integer number. □

3. Strong Roman Domination Number of $P_m \Box P_k$

In this section, we investigate the strong Roman domination number of $P_m \Box P_k$.

Theorem 3.1. *For any positive integer k, $\gamma_{StR}(P_2 \Box P_k) = \lceil \frac{4k}{3} \rceil$.*

Proof. By Observation 2.1, if $k \equiv 0(mod\ 3)$, $\gamma_{StR}(P_2 \Box P_k) \leq \frac{4k}{3} = \lceil \frac{4k}{3} \rceil$. By Observation 2.2, if $k \equiv 1(mod\ 3)$, $\gamma_{StR}(P_2 \Box P_k) \leq \frac{2(2k+1)}{3} = \lceil \frac{4k}{3} \rceil$. By Observation 2.3, if $k \equiv 2(mod\ 3)$, $\gamma_{StR}(P_2 \Box P_k) \leq \frac{4k+1}{3} = \lceil \frac{4k}{3} \rceil$. Hence, in any case, $\gamma_{StR}(P_2 \Box P_k) \leq \lceil \frac{4k}{3} \rceil$. Among all γ_{StR}-function of $P_2 \Box P_k$, let f be chosen so that $|B_1|$ is maximized.

It is obvious that $B_2^4 = \emptyset$. By inequality (2.1) in Theorem 2.5, it follows that

$$\begin{aligned} \gamma_{StR}(P_2 \Box P_k) &\geq \tfrac{2}{3}(2k + |B_0^2| + \tfrac{1}{2}|B_1|) \\ &\geq \lceil \tfrac{4k}{3} \rceil. \end{aligned}$$

Therefore, $\gamma_{StR}(P_2 \Box P_k) = \lceil \frac{4k}{3} \rceil$. □

Theorem 3.2. *For any positive integer k, $\gamma_{StR}(P_3 \Box P_k) = 2k$.*

Proof. By Observation 2.1, $\gamma_{StR}(P_3 \Box P_k) = \gamma_{StR}(P_k \Box P_3) \leq 2k$.

Among all γ_{StR}-function of $P_3 \Box P_k$, let f be chosen so that $|B_1|$ is maximized. By Claim 2 in Theorem 2.5, $B_2^3 = \emptyset$.

By inequality (2.1) in Theorem 2.5, it follows that

$$\gamma_{StR}(P_3 \Box P_k) \geq \frac{2}{3}(3k + |B_0^2| + \frac{1}{2}(|B_1| - |B_2^4|)).$$

In order to prove $\gamma_{StR}(P_3 \Box P_k) \geq 2k$, it is sufficient to prove that $2|B_0^2| + |B_1| \geq |B_2^4|$. If $B_2^4 = \emptyset$, then it holds obviously. Hence, we may assume that $B_2^4 \neq \emptyset$. It is obvious that $B_2^4 \subseteq \mathcal{R}_2$. Now we define a function $g : B_2^4 \to B_0^2 \cup B_1$ as follows:

For any $u, v \in B_2^4$, $d(u, v) \geq 2$. Suppose that $u = v_{2i}$, $v = v_{2j}$ and $(\bigcup_{i+1 \leq l \leq j-1} \mathcal{C}_l) \cap B_2^4 = \emptyset$, where $j - i \geq 2$. We discuss it from the following cases.

Case 1: $j = i + 2$. That is $u = v_{2i}$ and $v = v_{2(i+2)}$. Then $v_{2(i+1)} \in B_0^2$. Define $g(u) = v_{2(i+1)}$.

Case 2: $j = i + 3$. That is $u = v_{2i}$ and $v = v_{2(i+3)}$. If $v_{1(i+1)} \in B_1$, then define $g(u) = v_{1(i+1)}$. If $f(v_{1(i+1)}) \geq 2$, then $v_{2(i+1)} \in B_0^2$ and define $g(u) = v_{2(i+1)}$. If $v_{1(i+1)} \in B_0$, then $f(v_{1(i+2)}) = 3$ and $v_{1(i+2)} \in B_2^3$, which is a contradiction.

Case 3: $j \geq i+4$. If $v_{1(i+1)} \in B_1$, then define $g(u) = v_{1(i+1)}$. If $f(v_{1(i+1)}) \geq 2$, then $v_{2(i+1)} \in B_0^2$ and define $g(u) = v_{2(i+1)}$. We may assume that $f(v_{1(i+1)}) = 0$. Then $2 \leq f(v_{1(i+2)}) \leq 3$. Since $B_2^3 = \emptyset$, $f(v_{1(i+2)}) = 2$. Without loss of generality, we may assume that $f(v_{3(i+1)}) = 0$ and $f(v_{3(i+2)}) = 2$. If $f(v_{2(i+2)}) = 1$, then define $g(u) = v_{2(i+2)}$. If $f(v_{2(i+2)}) = 0$, then $v_{2(i+2)} \in B_0^2$ and define $g(u) = v_{2(i+2)}$. If $f(v_{2(i+2)}) \geq 2$, then $v_{2(i+1)} \in B_0^2$ and define $g(u) = v_{2(i+1)}$.

Let $h = \max\{j | v_{2j} \in B_2^4\}$. If $h \leq k-2$, then by a similar way as Case 3, there exists a vertex v_{ij} such that $v_{ij} \in B_0^2 \cup B_1$, where $i \in \{1, 2, 3\}$ and $j \in \{h+1, h+2\}$. So we define $g(v_{2h}) = v_{ij}$. If $h = k-1$, then $f(v_{1k}) = 1$ and $v_{1k} \in B_1$. So we define $g(v_{2h}) = v_{1k}$.

Hence, for any $u \in B_2^4$, there exists $w \in B_0^2 \cup B_1$ such that $g(u) = w$. Furthermore, for any $u, v \in B_2^4$, if $u \neq v$, then $g(u) \neq g(v)$. Hence, $|B_0^2| + |B_1| \geq |B_2^4|$. So

$$\gamma_{StR}(P_3 \Box P_k) \geq \frac{2}{3}(3k + |B_0^2| + \frac{1}{2}(|B_1| - |B_2^4|)) \geq 2k.$$

Therefore, $\gamma_{StR}(P_3 \Box P_k) = 2k$. □

Lemma 3.3. *For any positive integer k, $\gamma_{StR}(P_4 \Box P_k) \leq \lceil \frac{8k}{3} \rceil$.*

Proof. Let $G = P_4 \Box P_k$. By Observation 2.1, if $k \equiv 0 (mod\, 3)$, $\gamma_{StR}(P_4 \Box P_k) \leq \frac{8k}{3} = \lceil \frac{8k}{3} \rceil$. By Observation 2.2, if $k \equiv 1 (mod\, 3)$, $\gamma_{StR}(P_4 \Box P_n) \leq \frac{4(2k+1)}{3} = \lceil \frac{8k}{3} \rceil + 1$. By Observation 2.3, if $k \equiv 2 (mod\, 3)$, $\gamma_{StR}(P_4 \Box P_n) \leq \frac{4(4k+1)}{6} = \lceil \frac{8k}{3} \rceil$.

Let $G = P_4 \Box P_4$. Let $V_3 = \{v_{32}\}$, $V_2 = \{v_{11}, v_{14}, v_{44}\}$, $V_1 = \{v_{23}, v_{41}\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Clearly, (V_0, V_1, V_2, V_3) is a partition of $V(G)$. Define f_4 on $V(G)$ by $f_4(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 3$. It is obvious that f_4 is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq |f_4(V(G))| = 11 = \lceil \frac{8k}{3} \rceil$, where $k = 4$.

Let $G = P_4 \Box P_7$. Let $V_3 = \{v_{21}, v_{35}\}$, $V_2 = \{v_{13}, v_{14}, v_{17}, v_{42}, v_{43}, v_{47}\}$, $V_1 = \{v_{26}\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Clearly, (V_0, V_1, V_2, V_3) is a partition of $V(G)$. Define f_7 on $V(G)$ by $f_7(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 3$. It is obvious that f_7 is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq |f_7(V(G))| = 19 = \lceil \frac{8k}{3} \rceil$, where $k = 7$.

For $k \geq 10$, let $k = 7+3t$. Let $V_3 = \{v_{21}, v_{35}\}$, $V_2 = \{v_{13}, v_{14}, v_{17}, v_{42}, v_{43}, v_{47}\} \cup \{v_{1(2+6j)}, v_{2(4+6j)}, v_{3(4+6j)}, v_{4(2+6j)} | j = 1, 2, \cdots, \lceil \frac{t}{2} \rceil\} \cup \{v_{1(7+6j)}, v_{2(5+6j)}, v_{3(5+6j)}, v_{4(7+6j)} | j = 1, 2, \cdots, \lceil \frac{t+1}{2} \rceil - 1\}$, $V_1 = \{v_{26}\}$ and $V_0 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Clearly, (V_0, V_1, V_2, V_3) is a partition of $V(G)$. Define f_k on $V(G)$ by $f_k(v) = i$ for any $v \in V_i$, where $0 \leq i \leq 3$. It is obvious that f_k is a strong Roman dominating function of G. Therefore, $\gamma_{StR}(G) \leq |f_k(V(G))| = \lceil \frac{8k}{3} \rceil$. □

Lemma 3.4. *For any positive integer $k \geq 4$,*

$$\gamma_{StR}(P_4 \Box P_k) \geq \begin{cases} \lceil \frac{8k}{3} \rceil & \text{if } k = 4, 5, 6, 7 \\ \lceil \frac{1}{3}(8k - \lfloor \frac{k}{8} \rfloor + 1) \rceil & \text{if } k \geq 8. \end{cases}$$

Proof. Among all γ_{StR}-function of $P_4 \Box P_k$, let f be chosen so that $|B_1|$ is maximized. Then $B_2^1 = B_2^3 = \emptyset$. By inequality (2.1) in Theorem 2.5, it follows that

$$\gamma_{StR}(P_4 \Box P_k) \geq \frac{2}{3}(4k + |B_0^2| + \frac{1}{2}(|B_1| - |B_2^4|)).$$

If $B_2^4 = \emptyset$, then $\gamma_{StR}(P_4 \Box P_k) \geq \lceil \frac{8k}{3} \rceil$. Hence, we may assume that $B_2^4 \neq \emptyset$. It is obvious that $B_2^4 \subseteq \mathcal{R}_2 \cup \mathcal{R}_3$.

Claim 1. Suppose $\mathcal{C}_j \cap B_2^4 \neq \emptyset$ for $j \in \{2, 3, \cdots, k-1\}$. Then $|(\mathcal{C}_{j-1} \cup \mathcal{C}_j \cup \mathcal{C}_{j+1}) \cap B_2^4| = 1$.

Proof. Without loss of generality, we can assume that $v_{2j} \in B_2^4$. Then $v_{2(j-1)}, v_{2(j+1)}, v_{3j} \in B_0$. If $v_{3(j+1)} \in B_2^4$, then $d(v_{3(j+1)}) = 4$ and $j \leq k-2$. Define a function f' on $V(G)$ by $f'(x) = f(x)$ for $x \in V(G) - \{v_{3(j+1)}, v_{3(j+2)}, v_{4(j+1)}\}$, $f'(v_{3(j+1)}) = 1$, $f'(v_{3(j+2)}) = 1$ and $f'(v_{4(j+1)}) = 1$. Then f' is γ_{StR}-function of $P_4 \Box P_k$ with $|B_1|$ more than $|B_1|$ in f, which is a contradiction. Hence $v_{3(j+1)} \notin B_2^4$. Similarly, $v_{3(j-1)} \notin B_2^4$. So, $|(\mathcal{C}_{j-1} \cup \mathcal{C}_j \cup \mathcal{C}_{j+1}) \cap B_2^4| = 1$. $\Box$

Claim 2. Let $h = min\{j | v_{ij} \in B_2^4\}$. Then $(\bigcup_{1 \leq i \leq h} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Proof. Without loss of generality, we can assume that $v_{2h} \in B_2^4$. Then $2 \leq h \leq k-1$. If $h = 2$, then $v_{11} \in B_1$. So $(\bigcup_{1 \leq i \leq h} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Suppose that $h \geq 3$ and $(\bigcup_{1 \leq i \leq h} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$. Hence, for any vertex $v_{ij} \in (\bigcup_{1 \leq i \leq h} \mathcal{C}_i) \setminus \{v_{2h}\}$, $v_{ij} \in B_0^1$ or $v_{ij} \in B_2^2$. By Claim 3 in Theorem 2.5, $v_{1(h-1)}, v_{2(h-2)}, v_{3(h-1)}, v_{4h} \in B_0^1$. In order to dominate $v_{1(h-1)}$, $f(v_{1(h-2)}) = 2$. Hence, $v_{3(h-2)} \in B_0^1$. In order to dominate $v_{3(h-1)}$, $f(v_{4(h-1)}) = 2$. Hence, $f(v_{4(h-2)}) = 2$. If $h = 3$, then $v_{4(h-2)} \in B_2^1$, which is a contradiction. If $h = 4$, then $f(v_{1(h-3)}) = 2$ and $v_{2(h-3)}, v_{3(h-3)}, v_{4(h-3)} \in B_0^1$. Then $v_{1(h-3)} \in B_2^1$, which is a contradiction. If $h \geq 5$, then $v_{1(h-4)}, v_{2(h-4)}, , v_{4(h-4)} \in B_0^1$. Hence, $f(v_{3(h-4)}) = 3$ and $v_{3(h-4)} \in B_2^4$, which is a contradiction. Hence, $\bigcup_{1 \leq i \leq h} \mathcal{C}_i \cap (B_0^2 \cup B_1) \neq \emptyset$. $\Box$

Claim 3. Let $l = max\{j | v_{ij} \in B_2^4\}$. Then $(\bigcup_{l+1 \leq i \leq k} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Proof. Without loss of generality, we can assume that $v_{2l} \in B_2^4$. Then $l \leq k-1$. If $l = k-1$, then $v_{1k} \in B_1$. So $(\bigcup_{l+1 \leq i \leq k} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Suppose that $l \leq k-2$ and $(\bigcup_{l+1 \leq i \leq k} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$. Hence, for any vertex $v_{ij} \in (\bigcup_{l+1 \leq i \leq k} \mathcal{C}_i)$, $v_{ij} \in B_0^1$ or $v_{ij} \in B_2^2$. Then $v_{1(l+1)}, v_{2(l+2)}, v_{3(l+1)} \in B_0^1$. In order to dominate $v_{1(l+1)}$, $f(v_{1(l+2)}) = 2$. Hence, $v_{3(l+2)} \in B_0^1$. In order to dominate $v_{3(l+1)}$, $f(v_{4(l+1)}) = 2$. If $l = k-2$, then $f(v_{4(l+2)}) = 2$ and $v_{4(l+2)} \in B_2^1$, which is a contradiction. If $l = k-3$, then $f(v_{1(l+3)}) = 2$ and $v_{2(l+3)}, v_{3(l+3)} \in B_0^1$. So, $v_{1(l+3)} \in B_2^1$, which is a contradiction. If $l \leq k-4$, then $f(v_{1(l+3)}) = 2$ and $v_{2(l+3)}, v_{3(l+3)}, v_{1(l+4)}, v_{2(l+4)} \in B_0^1$. In order to dominate $v_{3(l+2)}$, $f(v_{4(l+2)}) = 2$. Hence, $v_{4(l+3)}, v_{4(l+4)} \in B_0^1$. Hence, $f(v_{3(l+4)}) = 3$ and $v_{3(l+4)} \in B_2^4$, which is a contradiction. $\Box$

Claim 4. Suppose that $\mathcal{C}_j \cap B_2^4 \neq \emptyset$, $\mathcal{C}_r \cap B_2^4 \neq \emptyset$ and $(\bigcup_{j+1 \leq i \leq r-1} \mathcal{C}_i) \cap B_2^4 = \emptyset$. If $r - j \geq 5$ or $2 \leq r - j \leq 3$, then $(\bigcup_{j+1 \leq i \leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Proof. If $r - j \geq 5$, then $(\bigcup_{r-4 \leq i \leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$ by a similar proof as Claim 2. Since $r - j \geq 5$, $r - 4 \geq j + 1$. Hence, $(\bigcup_{j+1 \leq i \leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Suppose that $r - j = 2$. Without loss of generality, we can assume that $v_{2j} \in B_2^4$. If $v_{2(j+2)} \in B_2^4$, then $v_{2(j+1)} \in B_0^2$. So $(\bigcup_{j+1 \leq i \leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. Suppose that $v_{3(j+2)} \in B_2^4$. If $f(v_{1(j+2)}) \geq 1$, then $(\bigcup_{j+1 \leq i \leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. If $f(v_{1(j+2)}) = 0$, then $v_{1(j+1)} \in B_1$. So, $(\bigcup_{j+1 \leq i \leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Suppose that $r-j=3$. Without loss of generality, we can assume that $v_{2j} \in B_2^4$. Assume that $v_{2(j+3)} \in B_2^4$. If $f(v_{1(j+1)}) \geq 1$ or $f(v_{1(j+2)}) \geq 1$, then $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. If $f(v_{1(j+1)}) = 0$ and $f(v_{1(j+2)}) = 0$, then $v_{1(j+2)}$ is not dominated by B_2, which is a contradiction. Hence, $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Without loss of generality, we can assume that $v_{3(j+3)} \in B_2^4$. If $f(v_{1(j+1)}) \geq 1$, $f(v_{2(j+2)}) \geq 1$ or $f(v_{1(j+3)}) \geq 1$, then $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. If $f(v_{1(j+1)}) = 0$, $f(v_{2(j+2)}) = 0$ and $f(v_{1(j+3)}) = 0$, then $f(v_{1(j+2)}) = 3$ and $v_{1(j+2)} \in B_2^3$, which is a contradiction. Hence, $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. □

Remark. if $r = j+4$, then $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$ may be hold. Assume that $v_{2j} \in B_2^4$. If $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$, then $f(v)$ is fixed for any $v \in (\bigcup_{j\leq i\leq r} \mathcal{C}_i)$. That is, $v_{3(j+4)} \in B_2^4$, $\{v_{1(j+2)}, v_{1(j+3)}, v_{4(j+1)}, v_{4(j+2)}\} \subseteq B_2^2$ and $(\bigcup_{j\leq i\leq r} \mathcal{C}_i) \setminus \{v_{3(j+4)}, v_{2j}, v_{1(j+2)}, v_{1(j+3)}, v_{4(j+1)}, v_{4(j+2)}\} \subseteq B_0^1$.

Claim 5. Suppose that $\mathcal{C}_{j-4} \cap B_2^4 \neq \emptyset$, $\mathcal{C}_j \cap B_2^4 \neq \emptyset$, $(\bigcup_{j-3\leq i\leq j-1} \mathcal{C}_i) \cap B_2^4 = \emptyset$ and $(\bigcup_{j-3\leq i\leq j} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$. If $\mathcal{C}_l \cap B_2^4 \neq \emptyset$ for $l \geq j+2$ and $(\bigcup_{j+1\leq i\leq l-1} \mathcal{C}_i) \cap B_2^4 = \emptyset$, then $(\bigcup_{j+1\leq i\leq l} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$.

Proof. Assume that $v_{2(j-4)} \in B_2^4$. By remark, $v_{3j} \in B_2^4$, $v_{1(j-1)} \in B_2^2$ and $v_{1j} \in B_0^1$. If $l-j \geq 5$ or $2 \leq l-j \leq 3$, then $(\bigcup_{j+1\leq i\leq l} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$ by Claim 4. Without loss of generality, we can assume that $l = j+4$. Suppose that $v_{2(j+4)} \in B_2^4$. If $v_{1(j+1)} \in B_1$, then $(\bigcup_{j+1\leq i\leq l} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. Hence, $v_{1(j+1)} \in B_0^1$.

If $f(v_{2(j+2)}) \geq 1$ or $f(v_{1(j+3)}) \geq 1$, then $(\bigcup_{j+1\leq i\leq l} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. If $f(v_{2(j+2)}) = 0$ and $f(v_{1(j+3)}) = 0$, then $v_{1(j+2)} \in B_2^3$, which is a contradiction.

Without loss of generality, we can assume that $v_{3(j+4)} \in B_2^4$. If $f(v_{4(j+1)}) \geq 1$, $f(v_{3(j+2)}) \geq 1$ or $f(v_{4(j+3)}) \geq 1$, then $(\bigcup_{j+1\leq i\leq l} \mathcal{C}_i) \cap (B_0^2 \cup B_1) \neq \emptyset$. If $f(v_{4(j+1)}) = 0$, $f(v_{3(j+2)}) = 0$ and $f(v_{4(j+3)}) = 0$, then $f(v_{4(j+2)}) = 3$ and $v_{4(j+2)} \in B_2^3$, which is a contradiction. □

Suppose that there exist two positive integer j and r such that $\mathcal{C}_{j-4} \cap B_2^4 \neq \emptyset$, $\mathcal{C}_j \cap B_2^4 \neq \emptyset$, $\mathcal{C}_r \cap B_2^4 \neq \emptyset$, $\mathcal{C}_{r+4} \cap B_2^4 \neq \emptyset$, $(\bigcup_{j-3\leq i\leq j-1} \mathcal{C}_i) \cap B_2^4 = \emptyset$, $(\bigcup_{r+1\leq i\leq r+3} \mathcal{C}_i) \cap B_2^4 = \emptyset$, $(\bigcup_{j+1\leq i\leq r-1} \mathcal{C}_i) \cap B_2^4 = \emptyset$, $(\bigcup_{j-3\leq i\leq j} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$ and $(\bigcup_{r+1\leq i\leq r+4} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$, where $r \geq j+2$. If $(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap B_0^2 = \emptyset$ and $|(\bigcup_{j+1\leq i\leq r} \mathcal{C}_i) \cap B_1| = 1$, then $r - j \geq 4$.

Suppose that there exist a positive integer j such that $\mathcal{C}_j \cap B_2^4 \neq \emptyset$, $\mathcal{C}_{j+4} \cap B_2^4 \neq \emptyset$, $(\bigcup_{j+1\leq i\leq j+4} \mathcal{C}_i) \cap (B_0^2 \cup B_1) = \emptyset$ and $(\bigcup_{1\leq i\leq j-1} \mathcal{C}_i) \cap B_2^4 = \emptyset$. If $(\bigcup_{1\leq i\leq j} \mathcal{C}_i) \cap B_0^2 = \emptyset$ and $|(\bigcup_{1\leq i\leq j} \mathcal{C}_i) \cap B_1| = 1$, then $j \geq 3$.

By Claims 2-5, it follows that if $k = 4, 5, 6$ or 7, then $|B_0^2| + |B_1| - |B_2^4| \geq 0$. So, $\gamma_{StR}(P_4 \Box P_k) \geq \frac{8k}{3}$. If $k \geq 8$, then $|B_0^2| + |B_1| - |B_2^4| \geq -\lfloor \frac{k}{8} \rfloor + 1$. Hence,

$$\begin{aligned}\gamma_{StR}(P_4 \Box P_k) &\geq \tfrac{2}{3}(4k + |B_0^2| + \tfrac{1}{2}(|B_1| - |B_2^4|)) \\ &= \tfrac{8k}{3} + \tfrac{1}{3}(2|B_0^2| + |B_1| - |B_2^4|) \\ &\geq \tfrac{1}{3}(8k - \lfloor \tfrac{k}{8} \rfloor + 1).\end{aligned}$$

Therefore, the result follows, since $\gamma_{StR}(G)$ is an integer number. □

By Lemma 3.3 and Lemma 3.4, we give the following.

Corollary 3.5. *For positive integer $k \in \{4, 5, 6, 7\}$, $\gamma_{StR}(P_4 \Box P_k) = \lceil \frac{8k}{3} \rceil$.*

Theorem 3.6. *For any positive integer $k \geq 8$, $\lceil \frac{1}{3}(8k - \lfloor \frac{k}{8} \rfloor + 1) \rceil \leq \gamma_{StR}(P_4 \Box P_k) \leq \lceil \frac{8k}{3} \rceil$, and both bounds are sharp.*

Remark 3.7. In order to show the lower bound is sharp, define a function f on $P_4 \Box P_{17}$ by

$$\begin{aligned}
f(\mathcal{R}_1) &= \{f(v_{11}), f(v_{12}), \cdots, f(v_{1(17)})\} = \{2,0,0,0,2,2,0,0,2,0,0,2,2,0,0,0,2\},\\
f(\mathcal{R}_2) &= \{f(v_{21}), f(v_{22}), \cdots, f(v_{2(17)})\} = \{0,0,3,0,0,0,0,0,2,0,0,0,0,0,3,0,0\},\\
f(\mathcal{R}_3) &= \{f(v_{31}), f(v_{32}), \cdots, f(v_{3(17)})\} = \{0,1,0,0,0,0,3,0,1,0,3,0,0,0,0,1,0\},\\
f(\mathcal{R}_4) &= \{f(v_{41}), f(v_{42}), \cdots, f(v_{4(17)})\} = \{2,0,0,2,2,0,0,0,2,0,0,0,2,2,0,0,2\}.
\end{aligned}$$

It is obvious that f is a strong Roman dominating function of $P_4 \Box P_{17}$ and $\gamma_{StR}(P_4 \Box P_{17}) \leq 45$. Since $\lceil \frac{1}{3}(8k - \lfloor \frac{k}{8} \rfloor + 1) \rceil = 45$ for $k = 17$, it follows that $\gamma_{StR}(P_4 \Box P_{17}) = \lceil \frac{1}{3}(8k - \lfloor \frac{k}{8} \rfloor + 1) \rceil$.

In order to show the upper bound is sharp, define a function f on $P_4 \Box P_9$ by

$$\begin{aligned}
f(\mathcal{R}_1) &= \{f(v_{11}), f(v_{12}), \cdots, f(v_{19})\} = \{2,0,0,0,2,2,0,0,2\},\\
f(\mathcal{R}_2) &= \{f(v_{21}), f(v_{22}), \cdots, f(v_{29})\} = \{0,0,3,0,0,0,0,1,0\},\\
f(\mathcal{R}_3) &= \{f(v_{31}), f(v_{32}), \cdots, f(v_{39})\} = \{0,1,0,0,0,0,3,0,0\},\\
f(\mathcal{R}_4) &= \{f(v_{41}), f(v_{42}), \cdots, f(v_{49})\} = \{2,0,0,2,2,0,0,0,2\}.
\end{aligned}$$

It is obvious that f is a strong Roman dominating function of $P_4 \Box P_9$ and $\gamma_{StR}(P_4 \Box P_9) \leq 24 = \lceil \frac{8k}{3} \rceil$. Since $\lceil \frac{1}{3}(8k - \lfloor \frac{k}{8} \rfloor + 1) \rceil = 24$ for $k = 9$, it follows that $\gamma_{StR}(P_4 \Box P_9) = \lceil \frac{8k}{3} \rceil$.

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