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Initial Maclaurin Coefficient Bounds for New Subclasses of Analytic and m-Fold Symmetric Bi-Univalent Functions Defined by a Linear Combination

Hari M. Srivastava
Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada
Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China
e-mail : harimsri@math.uvic.ca

Abbas Kareem Wanas*
Department of Mathematics, College of Science, University of Al-Qadisiyah, Iraq
e-mail : abbas.kareem.w@qu.edu.iq

Abstract. In the present investigation, we define two new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination in the open unit disk U. Furthermore, for functions in each of the subclasses introduced here, we establish upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Also, we indicate certain special cases for our results.

1. Introduction

Let $\mathcal{A}$ stands the class of functions f that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, are normalized by the conditions $f(0) = f'(0) - 1 = 0$, and have the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let S be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in U. The Koebe one-quarter theorem (see [4]) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every

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function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$(1.2)\quad g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of bi-univalent functions in U satisfying (1.1). In fact, Srivastava et al. [15] has apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [6], Goyal and Goswami [7], Srivastava and Bansal [9] and others (see, for example [3, 10, 11, 12, 14]).

For each function $f \in S$, the function $h(z) = (f(z^m))^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [8]) if it has the following normalized form:

$$(1.3)\qquad f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1},\ (z \in U, m \in \mathbb{N}).$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the class S are one-fold symmetric.

In [16] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1} \\ (1.4)\quad &- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots , \end{aligned}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}},\ \left[\frac{1}{2}\log\left(\frac{1 + z^m}{1 - z^m}\right)\right]^{\frac{1}{m}} \ and\ \left[-\log\left(1 - z^m\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1 + w^m}\right)^{\frac{1}{m}},\ \left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1}\right)^{\frac{1}{m}} \ and\ \left(\frac{e^{w^m} - 1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [1, 2, 5, 13, 16, 17, 18]).

The purpose of the present paper is to introduce the new subclasses $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ and $\mathcal{WS}^*_{\Sigma_m}(\lambda,\gamma,\delta;\beta)$ of Σ_m, which involve a linear combination of the following three expressions

$$\frac{f(z)}{z}, \quad f'(z) \quad and \quad zf''(z)$$

and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to prove our main results, we require the following lemma.

Lemma 1.1.([4]) *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in U for which*

$$Re\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

2. Coefficient Estimates for the Function Class $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.3) is said to be *in the class* $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ if it satisfies the following conditions:

$$\left|arg\left(1+\frac{1}{\delta}\left[\lambda\gamma\left(zf''(z)-2\right)+\left(\gamma(\lambda+1)+\lambda\right)f'(z)+(1-\lambda)(1-\gamma)\frac{f(z)}{z}-1\right]\right)\right| < \frac{\alpha\pi}{2}, \tag{2.1}$$

and

$$\left|arg\left(1+\frac{1}{\delta}\left[\lambda\gamma\left(wg''(w)-2\right)+\left(\gamma(\lambda+1)+\lambda\right)g'(w)+(1-\lambda)(1-\gamma)\frac{g(w)}{w}-1\right]\right)\right| < \frac{\alpha\pi}{2}, \tag{2.2}$$

$$\left(z,w \in U, 0<\alpha\leq 1,\ \lambda\geq 0,\ 0\leq\gamma\leq 1,\ \delta\in\mathbb{C}\backslash\{0\},\ m\in\mathbb{N}\right),$$

where the function $g = f^{-1}$ is given by (1.4).

Remark 2.1. It should be remarked that the class $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ is a generalization of well-known classes consider earlier. These classes are:

(1) For $\gamma = 0$, the class $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ reduce to the class $\mathcal{B}_{\Sigma_m}(\tau,\lambda;\alpha)$ which was introduced recently by Srivastava et al. [13];

(2) For $\gamma = 0$ and $\delta = 1$, the class $\mathcal{WS}_{\Sigma_m}(\lambda,\gamma,\delta;\alpha)$ reduce to the class $\mathcal{A}^{\alpha,\lambda}_{\Sigma,m}$ which was investigated recently by Eker [5];

(3) For $\gamma = 0$ and $\lambda = \delta = 1$, the class $\mathcal{WS}_{\Sigma_m}(\lambda, \gamma, \delta; \alpha)$ reduce to the class $\mathcal{H}^{\alpha}_{\Sigma,m}$ which was given by Srivastava et al. [16].

Remark 2.2. For one-fold symmetric bi-univalent functions, we denote the class $\mathcal{WS}_{\Sigma_1}(\lambda, \gamma, \delta; \alpha) = \mathcal{WS}_{\Sigma}(\lambda, \gamma, \delta; \alpha)$. Special cases of this class illustrated below:

(1) For $\gamma = 0$ and $\delta = 1$, the class $\mathcal{WS}_{\Sigma}(\lambda, \gamma, \delta; \alpha)$ reduce to the class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$ which was investigated recently by Frasin and Aouf [6];

(2) For $\gamma = 0$ and $\lambda = \delta = 1$, the class $\mathcal{WS}_{\Sigma}(\lambda, \gamma, \delta; \alpha)$ reduce to the class $\mathcal{H}^{\alpha}_{\Sigma}$ which was given by Srivastava et al. [15].

Theorem 2.1. *Let* $f \in \mathcal{WS}_{\Sigma_m}(\lambda, \gamma, \delta; \alpha)$ $(0 < \alpha \leq 1,\ \lambda \geq 0,\ 0 \leq \gamma \leq 1,\ \delta \in \mathbb{C}\setminus\{0\},\ m \in \mathbb{N})$ *be given by (1.3). Then*

(2.3)
$$|a_{m+1}| \leq \frac{2\alpha\,|\delta|}{\sqrt{|\alpha\delta(m+1)\left[\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1\right]+(1-\alpha)\Omega(\lambda,\gamma,m)|}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2\,|\delta|^2\,(m+1)}{\Omega(\lambda,\gamma,m)} + \frac{2\alpha\,|\delta|}{\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1}, \tag{2.4}$$

where

$$\Omega(\lambda,\gamma,m) = \left[\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1\right]^2.$$

Proof. It follows from conditions (2.1) and (2.2) that

$$1+\frac{1}{\delta}\left[\lambda\gamma\left(zf''(z)-2\right)+\left(\gamma(\lambda+1)+\lambda\right)f'(z)+(1-\lambda)(1-\gamma)\frac{f(z)}{z}-1\right] = [p(z)]^{\alpha} \tag{2.5}$$

and

$$1+\frac{1}{\delta}\left[\lambda\gamma\left(wg''(w)-2\right)+\left(\gamma(\lambda+1)+\lambda\right)g'(w)+(1-\lambda)(1-\gamma)\frac{g(w)}{w}-1\right] = [q(w)]^{\alpha}, \tag{2.6}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$\frac{\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1}{\delta}a_{m+1}=\alpha p_m, \tag{2.9}$$

$$\frac{\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1}{\delta}a_{2m+1}=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-\frac{\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1}{\delta}a_{m+1}=\alpha q_m \tag{2.11}$$

and

$$\begin{aligned}&\frac{\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1}{\delta}\left((m+1)a_{m+1}^2-a_{2m+1}\right)\\&=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2.\end{aligned} \tag{2.12}$$

In view of (2.9) and (2.11), we find that

$$p_m=-q_m \tag{2.13}$$

and

$$\frac{2\left[\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1\right]^2}{\delta^2}a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we obtain

$$\begin{aligned}&(m+1)\,\frac{\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1}{\delta}a_{m+1}^2\\&=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right)\\&=\alpha(p_{2m}+q_{2m})+\frac{(\alpha-1)\left[\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1\right]^2}{\alpha\delta^2}a_{m+1}^2.\end{aligned}$$

Therefore, we have

$$a_{m+1}^2=\frac{\alpha^2\delta^2(p_{2m}+q_{2m})}{\alpha\delta(m+1)\left[\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1\right]+(1-\alpha)\Omega(\lambda,\gamma,m)}. \tag{2.15}$$

Now, taking the absolute value of (2.15) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we deduce that

$$|a_{m+1}|\leq\frac{2\alpha\,|\delta|}{\sqrt{|\alpha\delta(m+1)\left[\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1\right]+(1-\alpha)\Omega(\lambda,\gamma,m)|}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.3).
In order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$\begin{aligned}&\frac{2\left[\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1\right]}{\delta}a_{2m+1}\\&-(m+1)\frac{\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1}{\delta}a_{m+1}^2\\&=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right).\end{aligned}\tag{2.16}$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1}=\frac{\alpha^2\delta^2(m+1)\left(p_m^2+q_m^2\right)}{4\Omega(\lambda,\gamma,m)}+\frac{\alpha\delta\left(p_{2m}-q_{2m}\right)}{2\left[\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1\right]}.\tag{2.17}$$

Taking the absolute value of (2.17) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{2\alpha^2\left|\delta\right|^2(m+1)}{\Omega(\lambda,\gamma,m)}+\frac{2\alpha\left|\delta\right|}{\lambda\gamma\left(4m(m+1)+2\right)+2m(\lambda+\gamma)+1},$$

which completes the proof of Theorem 2.1. □

Remark 2.3. In Theorem 2.1, if we choose

(1) $\gamma=0$, then we obtain the results which was proven by Srivastava et al. [13, Theorem 2.1];

(2) $\gamma=0$ and $\delta=1$, then we obtain the results which was obtained by Eker [5, Theorem 1];

(3) $\gamma=0$ and $\lambda=\delta=1$, then we obtain the results which was given by Srivastava et al. [16, Theorem 2].

For one-fold symmetric bi-univalent functions, Theorem 2.1 reduce to the following corollary:

Corollary 2.1. *Let* $f\in\mathcal{WS}_{\Sigma}(\lambda,\gamma,\delta;\alpha)$ $(0<\alpha\leq1,\ \lambda\geq0,\ 0\leq\gamma\leq1,\ \delta\in\mathbb{C}\setminus\{0\})$ *be given by (1.1). Then*

$$|a_2|\leq\frac{2\alpha\left|\delta\right|}{\sqrt{\left|2\alpha\delta\left(2\gamma\left(5\lambda+1\right)+2\lambda+1\right)+(1-\alpha)\left(\gamma\left(5\lambda+1\right)+\lambda+1\right)^2\right|}}$$

and

$$|a_3|\leq\frac{4\alpha^2\left|\delta\right|^2}{\left(\gamma\left(5\lambda+1\right)+\lambda+1\right)^2}+\frac{2\alpha\left|\delta\right|}{\left(2\gamma\left(5\lambda+1\right)+2\lambda+1\right)},$$

Remark 2.4. In Corollary 2.1, if we choose

(1) $\gamma = 0$ and $\delta = 1$, then we obtain the results which was proven by Frasin and Aouf [6, Theorem 2.2];

(2) $\gamma = 0$ and $\lambda = \delta = 1$, then we obtain the results which was given by Srivastava et al. [16, Theorem 1].

3. Coefficient Estimates for the Functions Class $\mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$

Definition 3.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the class $\mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ if it satisfies the following conditions:

(3.1)
$$Re\left\{1 + \frac{1}{\delta}\left[\lambda\gamma\left(zf''(z) - 2\right) + (\gamma(\lambda+1) + \lambda) f'(z) + (1-\lambda)(1-\gamma)\frac{f(z)}{z} - 1\right]\right\} > \beta,$$

and

(3.2)
$$Re\left\{1 + \frac{1}{\delta}\left[\lambda\gamma\left(wg''(w) - 2\right) + (\gamma(\lambda+1) + \lambda) g'(w) + (1-\lambda)(1-\gamma)\frac{g(w)}{w} - 1\right]\right\} > \beta,$$

$$(z, w \in U, 0 \leq \beta < 1,\ \lambda \geq 0,\ 0 \leq \gamma \leq 1,\ \delta \in \mathbb{C}\setminus\{0\},\ m \in \mathbb{N}),$$

where the function $g = f^{-1}$ is given by (1.4).

Remark 3.1. It should be remarked that the class $\mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ is a generalization of well-known classes consider earlier. These classes are:

(1) For $\gamma = 0$, the class $\mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ reduce to the class $\mathcal{B}^*_{\Sigma_m}(\tau, \lambda; \beta)$ which was introduced recently by Srivastava et al. [13];

(2) For $\gamma = 0$ and $\delta = 1$, the class $\mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ reduce to the class $\mathcal{A}^{\lambda}_{\Sigma,m}(\beta)$ which was investigated recently by Eker [5];

(3) For $\gamma = 0$ and $\lambda = \delta = 1$, the class $\mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ reduce to the class $\mathcal{H}_{\Sigma,m}(\beta)$ which was given by Srivastava et al. [16].

Remark 3.2. For one-fold symmetric bi-univalent functions, we denote the class $\mathcal{WS}^*_{\Sigma_1}(\lambda, \gamma, \delta; \beta) = \mathcal{WS}^*_{\Sigma}(\lambda, \gamma, \delta; \beta)$. Special cases of this class illustrated below:

(1) For $\gamma = 0$ and $\delta = 1$, the class $\mathcal{WS}^*_{\Sigma}(\lambda, \gamma, \delta; \beta)$ reduce to the class $\mathcal{B}_{\Sigma}(\beta, \lambda)$ which was investigated recently by Frasin and Aouf [6];

(2) For $\gamma = 0$ and $\lambda = \delta = 1$, the class $\mathcal{WS}^*_{\Sigma}(\lambda, \gamma, \delta; \beta)$ reduce to the class $\mathcal{H}_{\Sigma}(\beta)$ which was given by Srivastava et al. [15].

Theorem 3.1. *Let $f \in \mathcal{WS}^*_{\Sigma_m}(\lambda, \gamma, \delta; \beta)$ $(0 \leq \beta < 1,\ \lambda \geq 0,\ 0 \leq \gamma \leq 1,\ \delta \in \mathbb{C}\setminus\{0\},$ $m \in \mathbb{N})$ be given by (1.3). Then*

$$|a_{m+1}| \leq 2\sqrt{\frac{|\delta|(1-\beta)}{(m+1)\left[\lambda\gamma\left(4m(m+1)+2\right) + 2m(\lambda+\gamma) + 1\right]}} \tag{3.3}$$

and

$$|a_{2m+1}| \leq \frac{2|\delta|^2(1-\beta)^2(m+1)}{\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1} + \frac{2|\delta|(1-\beta)}{\lambda\gamma(4m(m+1)+2)+2m(\lambda+\gamma)+1}. \tag{3.4}$$

Proof. It follows from conditions (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$1+\frac{1}{\delta}\left[\lambda\gamma(zf''(z)-2)+(\gamma(\lambda+1)+\lambda)f'(z)+(1-\lambda)(1-\gamma)\frac{f(z)}{z}-1\right] = \beta+(1-\beta)p(z) \tag{3.5}$$

and

$$1+\frac{1}{\delta}\left[\lambda\gamma(wg''(w)-2)+(\gamma(\lambda+1)+\lambda)g'(w)+(1-\lambda)(1-\gamma)\frac{g(w)}{w}-1\right] = \beta+(1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$\frac{\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1}{\delta}a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$\frac{\lambda\gamma(4m(m+1)+2)+2m(\lambda+\gamma)+1}{\delta}a_{2m+1} = (1-\beta)p_{2m}, \tag{3.8}$$

$$-\frac{\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1}{\delta}a_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$\frac{\lambda\gamma(4m(m+1)+2)+2m(\lambda+\gamma)+1}{\delta}\left((m+1)a_{m+1}^2-a_{2m+1}\right) = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m = -q_m \tag{3.11}$$

and

$$\frac{2\left[\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1\right]^2}{\delta^2}a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$(m+1)\,\frac{\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1}{\delta}a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}).$$

Therefore, we have

$$a_{m+1}^2=\frac{\delta(1-\beta)(p_{2m}+q_{2m})}{(m+1)\left[\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1\right]}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq 2\sqrt{\frac{|\delta|\,(1-\beta)}{(m+1)\left[\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1\right]}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).
In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$\begin{aligned}&\frac{2\left[\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1\right]}{\delta}a_{2m+1}\\&-(m+1)\frac{\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1}{\delta}a_{m+1}^2=(1-\beta)\left(p_{2m}-q_{2m}\right),\end{aligned}$$

or equivalently

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^2+\frac{\delta(1-\beta)\left(p_{2m}-q_{2m}\right)}{2\left[\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1\right]}.$$

Upon substituting the value of a_{m+1}^2 from (3.12), it follows that

$$\begin{aligned}a_{2m+1}=&\frac{\delta^2\,(1-\beta)^2\,(m+1)(p_m^2+q_m^2)}{4\left[\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1\right]}\\&+\frac{\delta\,(1-\beta)\,(p_{2m}-q_{2m})}{2\left[\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1\right]}.\end{aligned}$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$\begin{aligned}|a_{2m+1}|\leq&\frac{2\,|\delta|^2\,(1-\beta)^2\,(m+1)}{\lambda\gamma\left((m+1)^2+1\right)+m(\lambda+\gamma)+1}\\&+\frac{2\,|\delta|\,(1-\beta)}{\lambda\gamma\,(4m(m+1)+2)+2m(\lambda+\gamma)+1}.\end{aligned}$$

which completes the proof of Theorem 3.1. □

Remark 3.3. In Theorem 3.1, if we choose

(1) $\gamma = 0$, then we obtain the results which was proven by Srivastava et al. [13, Theorem 3.1];

(2) $\gamma = 0$ and $\delta = 1$, then we obtain the results which was obtained by Eker [5, Theorem 2];

(3) $\gamma = 0$ and $\lambda = \delta = 1$, then we obtain the results which was given by Srivastava et al. [16, Theorem 3].

For one-fold symmetric bi-univalent functions, Theorem 3.1 reduce to the following corollary:

Corollary 3.1. Let $f \in \mathcal{WS}^*_\Sigma(\lambda, \gamma, \delta; \beta)$ $(0 \leq \beta < 1,\ \lambda \geq 0,\ 0 \leq \gamma \leq 1,\ \delta \in \mathbb{C} \setminus \{0\})$ be given by (1.1). Then

$$|a_2| \leq \sqrt{\frac{2\,|\delta|\,(1-\beta)}{2\gamma\,(5\lambda+1)+2\lambda+1}}$$

and

$$|a_3| \leq \frac{4\,|\delta|^2\,(1-\beta)^2}{\gamma\,(5\lambda+1)+\lambda+1} + \frac{2\,|\delta|\,(1-\beta)}{2\gamma\,(5\lambda+1)+2\lambda+1}.$$

Remark 3.4. In Corollary 3.1, if we choose

(1) $\gamma = 0$ and $\delta = 1$, then we obtain the results which was proven by Frasin and Aouf [6, Theorem 3.2];

(2) $\gamma = 0$ and $\lambda = \delta = 1$, then we obtain the results which was given by Srivastava et al. [15, Theorem 2].

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