KYUNGPOOK Math. J. 59(2019), 481-491
https://doi.org/10.5666/KMJ.2019.59.3.481
pISSN 1225-6951 eISSN 0454-8124

Some Coefficient Inequalities Related to the Hankel Determinant for a Certain Class of Close-to-convex Functions

Yong Sun*
School of Science, Hunan Institute of Engineering, Xiangtan, 411104, Hunan, People's Republic of China
e-mail : yongsun2008@foxmail.com

Zhi-Gang Wang
Mathematics and Computing Science, Hunan First Normal University, Changsha, 410205, Hunan, People's Republic of China
e-mail : wangmath@163.com

Abstract. In the present paper, we investigate the upper bounds on third order Hankel determinants for certain class of close-to-convex functions in the unit disk. Furthermore, we obtain estimates of the Zalcman coefficient functional for this class.

1. Introduction

Let $\mathcal{A}$ be the class of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions.

A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{D}).$$

* Corresponding Author.
Received November 14, 2017; revised February 1, 2019; accepted March 04, 2019.
2010 Mathematics Subject Classification: 30C45, 30C50, 58E20, 30C80.
Key words and phrases: coefficient inequality, Hankel determinant, Zalcman's conjecture, close-to-convex functions.
This work was supported by the *Natural Science Foundation of Hunan Province* under Grant no. 2016JJ2036, and the *Foundation of Educational Committee of Hunan Province* under Grant no. 15C1089.

We denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions of order α. In particular, $\mathcal{S}^* =: \mathcal{S}^*(0)$.

Recall that a function $f \in \mathcal{A}$ is close-to-convex in $\mathbb{D}$ if it is univalent and the range $f(\mathbb{D})$ is a close-to-convex domain, i.e., the complement of $f(\mathbb{D})$ can be written as the union of nonintersecting half-lines. A normalized analytic function f in $\mathbb{D}$ is close-to-convex in $\mathbb{D}$ if there exists a function $g \in \mathcal{S}^*$, such that the following inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \qquad \big(z \in \mathbb{D}\big) \tag{1.2}$$

holds. Denote $\mathcal{C}$ by the class of close-to-convex functions. We refer to [8, 16, 17, 28] for discussion and basic results on close-to-convex functions.

In [11], Gao and Zhou investigated the following class of close-to-convex functions.

Definition 1.1. Suppose that $f \in \mathcal{A}$ is analytic in $\mathbb{D}$ of the form (1.1). We say that $f \in \mathcal{K}_s$, if there exists a function $g \in \mathcal{S}^*(1/2)$, such that

$$\Re\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) < 0 \qquad \big(z \in \mathbb{D}\big). \tag{1.3}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2) \qquad \big(z \in \mathbb{D}\big) \tag{1.4}$$

and

$$G(z) = -\frac{g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \qquad \big(z \in \mathbb{D}\big). \tag{1.5}$$

Then $G(-z) = G(z)$, so $G(z)$ is an odd starlike function. It is well-known that

$$|B_{2n-1}| \leq 1 \qquad (n = 2, 3, \cdots). \tag{1.6}$$

Substituting the series expressions of $g(z)$, $G(z)$ in (1.4) and (1.5), and using (1.6), then the following result holds.

Theorem A.([11]) *Let $g \in \mathcal{S}^*(1/2)$. Then for $n \geq 2$,*

$$|B_{2n-1}| = \big|2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1} b_n^2\big| \leq 1. \tag{1.7}$$

The estimates are sharp, with the extremal function given by $g(z) = z/(1-z)$.

Theorem B.([11]) *Let $f \in \mathcal{K}_s$ be of the form* (1.1). *Then*

$$|a_n| \leq 1 \qquad (n = 2, 3, \cdots). \tag{1.8}$$

The estimates are sharp, with the extremal function given by $f(z) = z/(1-z)$.

Noonan and Thomas [24] studied the Hankel determinant $H_{q,n}(f)$ defined as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \qquad (q, n \in \mathbb{N}).$$

Problems involving Hankel determinants $H_{q,n}(f)$ in geometric function theory originate from the work of such authors as Hadamard, Polya and Edrei (see [7, 9]), who used them in study of singularities of meromorphic functions. For example, Hankel determinants can be used in showing that a function of bounded characteristic in $\mathbb{D}$, i.e., a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [5]. Pommerenke [25] proved that the Hankel determinants of univalent functions satisfy the inequality $|H_{q,n}(f)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, where $\beta > 1/4000$ and K depends only on q. Furthermore, Hayman [12] proved a stronger result for areally mean univalent functions, i.e., he showed that $H_{2,n}(f) < An^{1/2}$, where A is an absolute constant.

We note that $H_{2,1}(f)$ is the well-known Fekete-Szegő functional, see [10, 16, 17]. The sharp upper bounds on $H_{2,2}(f)$ were obtained in the articles [2, 14, 15, 18] for various classes of functions.

By the definition, $H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

Note that for $f \in \mathcal{A}$, $a_1 = 1$ so that

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2),$$

by the triangle inequality, we have

$$|H_{3,1}(f)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|.$$

Obviously, the case of the upper bound of $H_{3,1}(f)$ is much more difficult than the cases of $H_{2,1}(f)$ and $H_{2,2}(f)$. Recently, Prajapat *et al.*[26] studied the upper bounds on the Hankel determinants for the class of close-to-convex functions.

Theorem C. *Let* $f \in \mathcal{C}$ *be of the form* (1.1). *Then*

$$\left|a_2a_3 - a_4\right| \le 3, \quad \left|a_2a_4 - a_3^2\right| \le \frac{85}{36} \quad \text{and} \quad \left|H_{3,1}(f)\right| \le \frac{289}{12}.$$

For further information about the upper bounds of the third Hankel determinants for some classes of univalent functions, see e.g. [1, 3, 6, 27, 29].

In 1960, Lawrence Zalcman posed a conjecture that the coefficients of $\mathcal{S}$ satisfy the sharp inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2 \qquad (n \in \mathbb{N}),$$

with equality only for the Koebe function $k(z) = z/(1-z)^2$ and its rotations. We call $J_n(f) = a_n^2 - a_{2n-1}$ the Zalcman functional for $f \in \mathcal{S}$. Clearly, for $f \in \mathcal{S}$, we have $|J_2(f)| = |H_{2,1}(f)|$. The Zalcman conjecture was proved for certain special subclasses of $\mathcal{S}$ in [4, 19, 22, 23].

In the present investigation, our purpose is to develop similar results on the Hankel determinants in the context the close-to-convex functions $f \in \mathcal{K}_s$. Furthermore, the upper bounds to the Zalcman functional for this class are obtained.

2. Preliminary Results

Denote by $\mathcal{P}$ the class of Carathéodory functions p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{and} \quad \Re\big(p(z)\big) > 0 \quad (z \in \mathbb{D}). \tag{2.1}$$

The following results are well known for functions belonging to the class $\mathcal{P}$.

Lemma 2.1.([8]) *If $p \in \mathcal{P}$ is of the form* (2.1)*, then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N}). \tag{2.2}$$

The inequality (2.2) *is sharp and the equality holds for the function*

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n.$$

Lemma 2.2.([13]) *If $p \in \mathcal{P}$ is of the form* (2.1)*, then the sharp estimate* (2.3) *is valid.*

$$|c_n - \mu c_k c_{n-k}| \leq 2 \qquad (n,\ k \in \mathbb{N},\ n > k;\ 0 \leq \mu \leq 1). \tag{2.3}$$

Lemma 2.3.([20, 21]) *If $p \in \mathcal{P}$ is of the form* (2.1)*, then there exist x, z such that $|x| \leq 1$ and $|z| \leq 1$,*

$$2c_2 = c_1^2 + (4 - c_1^2)x, \tag{2.4}$$

and

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z. \tag{2.5}$$

3. The Upper Bounds of the Hankel Determinant

In this section, we first give an upper bound of the functional $\left|a_2a_3 - a_4\right|$ for functions $f \in \mathcal{K}_s$.

Theorem 3.1. *Let $f \in \mathcal{K}_s$ be of the form* (1.1). *Then*

$$\left|a_2a_3 - a_4\right| \leq \frac{1}{2}.$$

Proof. Let g be given by (1.4), and

$$p(z) = \frac{z^2 f'(z)}{-g(z)g(-z)} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{D}). \tag{3.1}$$

Then, we have $\Re\big(p(z)\big) > 0$, and

$$z^2 f'(z) = -g(z)g(-z)p(z). \tag{3.2}$$

Substituting the expansions of $f(z)$, $g(z)$ and $p(z)$ in (3.2), and equating the coefficients, we obtain

$$\begin{cases} a_2 &= \frac{1}{2}c_1, \\ a_3 &= \frac{1}{3}\big(c_2 + 2b_3 - b_2^2\big), \\ a_4 &= \frac{1}{4}\big[c_3 + \big(2b_3 - b_2^2\big)c_1\big]. \end{cases} \tag{3.3}$$

Hence, by using the above values of a_2, a_3 and a_4 from (3.3), and the relations of (2.4) and (2.5) we obtain, for some x and z such that $|x| \leq 1$ and $|z| \leq 1$,

(3.4)
$$\begin{aligned} \left|a_2a_3 - a_4\right| &= \frac{1}{12}\left| -(2b_3 - b_2^2)c_1 + 2c_1c_2 - 3c_3\right| \\ &= \frac{1}{48}\left|c_1^3 - 4(2b_3 - b_2^2)c_1 + (4 - c_1^2)\big[-2c_1x + 3c_1x^2 - 6(1 - |x|^2)z\big]\right|. \end{aligned}$$

By Lemma 2.1, we have $|c_1| \leq 2$. By setting $c := c_1$, we may assume without loss of generality that $c \in [0, 2]$. Thus, by applying the triangle inequality in (3.4) with $\mu = |x|$, we obtain

$$\left|a_2a_3 - a_4\right| \leq \frac{1}{48}\left\{c^3 + 4c + (4 - c^2)\big[3(c-2)\mu^2 + 2c\mu + 6\big]\right\} =: F(c, \mu).$$

Let

$$\varphi(\mu) = 3(c-2)\mu^2 + 2c\mu + 6 \qquad (c \in [0, 2];\ \mu \in [0, 1]).$$

In particular, for the case of $c = 2$, we have

$$\varphi(\mu) = 4\mu + 6 \leq \varphi(1) = 10.$$

For the case of $0 \leq c < 2$, then $\varphi(\mu)$ is a quadratic function of $\mu \in [0,1]$, and we can get

$$\varphi(\mu) = 3(c-2)\left(\mu - \frac{c}{3(2-c)}\right)^2 + \frac{c^2-18c+36}{3(2-c)}.$$

If $\mu_0 = \frac{c}{3(2-c)} \leq 1$, that is, $0 \leq c \leq \frac{3}{2}$, we obtain

$$\varphi(\mu) \leq \varphi(\mu_0) = \frac{c^2-18c+36}{3(2-c)}.$$

If $\mu_0 = \frac{c}{3(2-c)} \geq 1$, that is, $\frac{3}{2} \leq c < 2$, we get

$$\varphi(\mu) \leq \varphi(1) = 5c.$$

Thus, we have

$$F(c,\mu) \leq G(c) = \begin{cases} G_1(c) = \frac{1}{36}(c^3 - 4c^2 + 3c + 18) & (0 \leq c \leq 3/2), \\ G_2(c) = \frac{1}{12}(-c^3 + 6c) & (3/2 \leq c \leq 2). \end{cases}$$

For $G_1(c)$, we have

$$G_1'(c) = \frac{1}{36}(3c^2 - 8c + 3) \quad \text{and} \quad G_1''(c) = \frac{1}{18}(3c-4).$$

Let

$$C_0 = \frac{4-\sqrt{7}}{3} \in \left[0, \frac{3}{2}\right],$$

then, we obtain

$$G_1'(C_0) = 0 \quad \text{and} \quad G_1''(C_0) < 0.$$

For $G_2(c)$, we have

$$G_2'(c) = \frac{1}{4}(2-c^2) < 0, \qquad \left(\frac{3}{2} \leq c \leq 2\right).$$

Obviously, $G_2(c)$ is an decreasing function of c on $[3/2, 2]$ and, hence,

$$G_2(c) \leq G_2\left(\frac{3}{2}\right) = \frac{15}{32}.$$

Since $G(c)$ is a continuous function of c on the closed interval $[0,2]$, it follows that

$$\left|a_2a_3 - a_4\right| \leq G(c) \leq \max\left\{G_1(0),\ G_1(C_0),\ G_2\left(\frac{3}{2}\right)\right\} = \frac{1}{2}. \qquad \square$$

Now, we are ready to give an upper bound of $\left|a_2a_4 - a_3^2\right|$ for functions $f \in \mathcal{K}_s$.

Theorem 3.2. *Let $f \in \mathcal{K}_s$ be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq 1.$$

Proof. Using the values of a_2, a_3 and a_4 from (3.3), and using (2.4) and (2.5) for some x and z such that $|x| \leq 1$ and $|z| \leq 1$, we get

$$\begin{aligned} a_2a_4 - a_3^2 =& \frac{1}{288}\left\{c_1^4 + (4-c_1^2)\left[2c_1^2x - (32+c_1^2)x^2 + 18(1-|x|^2)c_1z\right]\right\} \\ &- \frac{2}{9}\left(2b_3 - b_2^2\right)\left(c_2 - \frac{9}{16}c_1^2\right) - \frac{1}{9}\left(2b_3 - b_2^2\right)^2. \end{aligned}$$

By Lemma 2.1, we may assume that $|c_1| = c \in [0,2]$. By applying Theorem **A**, Lemma 2.1, Lemma 2.2 and the triangle inequality in above relation with $\mu = |x|$, we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{288}\left\{c^4 + (4-c^2)\left[(c^2 - 18c + 32)\mu^2 + 2c^2\mu + 18c\right]\right\} + \frac{5}{9}.$$

Let

$$\psi(\mu) = (c^2 - 18c + 32)\mu^2 + 2c^2\mu + 18c, \qquad S(c,\mu) = \frac{1}{288}\left[c^4 + (4-c^2)\psi(\mu)\right].$$

Therefore,

$$\psi'(\mu) = 2(c-2)(c-16)\mu + 2c^2 \geq 0,$$

which implies that $\psi(\mu)$ is an increasing function of μ on $[0,1]$. Hence, we have

$$\psi(\mu) \leq \psi(1) = 3c^2 + 32,$$

which yields that

$$S(c,\mu) \leq S(c,1) = \frac{1}{144}\left(-c^4 - 10c^2 + 64\right) \leq \frac{4}{9}, \qquad (0 \leq c \leq 2).$$

Thus, we obtain the bound of $|a_2a_4 - a_3^2|$. □

Let $f \in \mathcal{K}_s$. Then using the above results in theorem **B**, Theorem 3.1 and Theorem 3.2, together with the known inequality $|a_2^2 - a_3| \leq 1$ (see [16]), we obtain the upper bound of the third Hankel determinant for close-to-convex functions $f \in \mathcal{K}_s$.

Theorem 3.3. *Let $f \in \mathcal{K}_s$ be of the form* (1.1). *Then*

$$\left|H_{3,1}(f)\right| \leq \frac{5}{2}.$$

Remark 3.1. In Theorem 3.1, Theorem3.2 and Theorem 3.3, we have obtained the upper bounds for the Hankel determinant. However, these results are far from sharp.

4. The Upper Bounds of the Zalcman Functional

In this section, we consider the Zalcman functional for functions $f \in \mathcal{K}_s$.

Theorem 4.1. *Let $f \in \mathcal{K}_s$ be of the form* (1.1). *Then*

$$\left|a_2^2 - a_3\right| \leq 1, \qquad \left|a_3^2 - a_5\right| \leq \frac{34}{45},$$

and

$$\left|a_n^2 - a_{2n-1}\right| \leq \begin{cases} 2 - \frac{4(n-1)}{n^2} & (n = 2k \geq 4), \\ 2 - \frac{4}{n} & (n = 2k+1 \geq 5). \end{cases}$$

Proof. Let $g(z)$, $G(z)$ and $p(z)$ be given by (1.4), (1.5) and (3.1), respectively. Then, we have

$$zf'(z) = p(z)G(z) \qquad (z \in \mathbb{D}).$$

Comparing the coefficients of two sides of this equation, we obtain

$$a_n = \begin{cases} \frac{1}{2k}\left(c_{2k-1}B_1 + c_{2k-3}B_3 + \cdots + c_1 B_{2k-1}\right) & (n = 2k), \\ \frac{1}{2k+1}\left(c_{2k}B_1 + c_{2k-2}B_3 + \cdots + c_0 B_{2k+1}\right) & (n = 2k+1), \end{cases}$$

where $k \in \mathbb{N}$ and $B_1 = c_0 = 1$.

For the case of $n = 2k$, we have

$$\begin{aligned}
&\left|a_n^2 - a_{2n-1}\right| = \left|a_{4k-1} - a_{2k}^2\right| \\
&= \Bigg|\frac{1}{4k-1}\left(c_{4k-2}B_1 + c_{4k-4}B_3 + \cdots + c_{2k}B_{2k-1} + c_{2k-2}B_{2k+1} + \cdots + c_0 B_{4k-1}\right) \\
&\quad - \frac{1}{4k^2}\left(c_{2k-1}B_1 + c_{2k-3}B_3 + \cdots + c_1 B_{2k-1}\right)^2\Bigg| \\
&= \Bigg|\frac{1}{4k-1}\left(c_{4k-2} - \frac{4k-1}{4k^2}c_{2k-1}^2\right) + \frac{B_3}{4k-1}\left(c_{4k-4} - \frac{4k-1}{2k^2}c_{2k-1}c_{2k-3}\right) \\
&\quad + \cdots + \frac{B_{2k-1}}{4k-1}\left(c_{2k} - \frac{4k-1}{2k^2}c_{2k-1}c_1\right) \\
&\quad + \frac{1}{4k-1}\left(c_{2k-2}B_{2k+1} + \cdots + c_2 B_{4k-3} + c_0 B_{4k-1}\right) \\
&\quad - \frac{1}{4k^2}\left(c_{2k-3}B_3 + \cdots + c_1 B_{2k-1}\right)^2\Bigg|.
\end{aligned}$$

If $k = 1$, using Theorem **B** and Lemma 2.2, we have

$$\left|a_2^2 - a_3\right| = \left|\frac{1}{3}\left(c_2 - \frac{3}{4}c_1^2\right) + \frac{1}{3}B_3\right| \leq \frac{1}{3}\left|c_2 - \frac{3}{4}c_1^2\right| + \frac{1}{3}\left|B_3\right| \leq 1.$$

If $k \geq 2$, we note that

$$\frac{4k-1}{4k^2} \leq 1 \qquad \text{and} \qquad \frac{4k-1}{2k^2} \leq 1 \qquad (k \geq 2),$$

by Theorem **B**, Lemma 2.1, Lemma 2.2 and the triangle inequality, we obtain

$$\left|a_n^2 - a_{2n-1}\right| \leq \frac{2k}{4k-1} + \frac{2(k-1)+1}{4k-1} + \frac{[2(k-1)]^2}{4k^2} = 2 - \frac{4(n-1)}{n^2}.$$

For the case of $n = 2k+1$, we have

$$\begin{aligned}
&\left|a_n^2 - a_{2n-1}\right| = \left|a_{4k+1} - a_{2k+1}^2\right| \\
&= \left|\frac{1}{4k+1}\left(c_{4k}B_1 + c_{4k-2}B_3 + \cdots + c_{2k+2}B_{2k-1}\right.\right. \\
&\qquad\qquad \left. + c_{2k}B_{2k+1} + c_{2k-2}B_{2k+3} + \cdots + c_0B_{4k+1}\right) \\
&\qquad \left. - \frac{1}{(2k+1)^2}\left(c_{2k}B_1 + c_{2k-2}B_3 + \cdots + c_2B_{2k-1} + c_0B_{2k+1}\right)^2\right| \\
&= \left|\frac{1}{4k+1}\left(c_{4k} - \frac{4k+1}{(2k+1)^2}c_{2k}^2\right) + \frac{B_3}{4k+1}\left(c_{4k-2} - \frac{2(4k+1)}{(2k+1)^2}c_{2k}c_{2k-2}\right)\right. \\
&\qquad + \cdots + \frac{B_{2k-1}}{4k+1}\left(c_{2k+2} - \frac{2(4k+1)}{(2k+1)^2}c_{2k}c_2\right) \\
&\qquad + \left(\frac{1}{4k+1} - \frac{2}{(2k+1)^2}\right)c_{2k}B_{2k+1} \\
&\qquad + \frac{1}{4k+1}\left(c_{2k-2}B_{2k+3} + \cdots + c_2B_{4k-1} + c_0B_{4k+1}\right) \\
&\qquad \left. - \frac{1}{(2k+1)^2}\left(c_{2k-2}B_3 + \cdots + c_2B_{2k-1} + c_0B_{2k+1}\right)^2\right|.
\end{aligned}$$

If $k = 1$, using Theorem **B**, Lemma 2.1 and Lemma 2.2, we have

$$\left|a_3^2 - a_5\right| \leq \frac{1}{5}\left|c_4 - \frac{5}{9}c_2^2\right| + \left|\frac{1}{5} - \frac{2}{9}\right|\left|c_2B_3\right| + \frac{1}{5}\left|B_5\right| + \frac{1}{9}\left|B_3^2\right| \leq \frac{34}{45}.$$

If $k \geq 2$, we note that

$$\frac{1}{4k+1} - \frac{2}{(2k+1)^2} \geq 0, \qquad \frac{4k+1}{(2k+1)^2} \leq 1 \quad \text{and} \quad \frac{2(4k+1)}{(2k+1)^2} \leq 1 \quad (k \geq 2),$$

by Theorem **B**, Lemma 2.1, Lemma 2.2 and the triangle inequality, we obtain

$$\left|a_n^2 - a_{2n-1}\right| \leq \frac{2k}{4k+1} + \left(\frac{2}{4k+1} - \frac{4}{(2k+1)^2}\right) + \frac{2k-1}{4k+1} + \frac{(2k-1)^2}{(2k+1)^2} = 2 - \frac{4}{n}.$$

This completes the proof. □

References

[1] K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl., **6**(2007), 1–7.

[2] D. Bansal, *Upper bound of second Hankel determinant for a new class of analytic functions*, Appl. Math. Lett., **26**(2013), 103–107.

[3] D. Bansal, S. Maharana and J. K. Prajapat, *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc., **52**(2015), 1139–1148.

[4] J. E. Brown and A. Tsao, *On the Zalcman conjecture for starlike and typically real functions*, Math. Z., **191**(1986), 467–474.

[5] D. G. Cantor, *Power series with integral coefficients*, Bull. Amer. Math. Soc., **69**(1963), 362–366.

[6] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim, *Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha*, J. Math. Inequal., **11**(2017), 429–439.

[7] P. Dienes, *The Taylor series: an introduction to the theory of functions of a complex variable*, Dover Publications, New York, 1957.

[8] P. L. Duren, *Univalent functions*, Springer Verlag. New York, 1983.

[9] A. Edrei, *Sur les déterminants récurrents et les singularités d'une fonction donnée par son développement de Taylor*, Compos. Math., **7**(1940), 20–88.

[10] M. Fekete and G. Szegő, *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc., **8**(1933), 85–89.

[11] C. Gao and S. Zhou, *On a class of analytic functions related to the starlike functions*, Kyungpook Math. J., **45**(2005), 123–130.

[12] W. K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc., **18**(1968), 77–94.

[13] T. Hayami and S. Owa, *Generalized Hankel determinant for certain classes*, Int. J. Math. Anal., **4**(2010), 2573–2585.

[14] A. Janteng, S. A. Halim and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math., **7**(2006), Article 50, 5 pp.

[15] A. Janteng, S. A. Halim and M. Darus, *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal., **1**(2007), 619–625.

[16] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc., **101**(1987), 89–95.

[17] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions II*, Arch. Math., **49**(1987), 420–433.

[18] S. K. Lee, V. Ravichandran and S. Subramaniam, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl., **2013**(2013), Article 281, 17 pp.

[19] L. Li and S. Ponnusamy, *Generalized Zalcman conjecture for convex functions of order* $-1/2$, J. Anal., **22**(2014), 77–87.

[20] R. J. Libera and E. J. Złotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc., **85**(1982), 225–230.

[21] R. J. Libera and E. J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivatives in P*, Proc. Amer. Math. Soc., **87**(1983), 251–257.

[22] W. C. Ma, *The Zalcman conjecture for close-to-convex functions*, Proc. Amer. Math. Soc., **104**(1988), 741–744.

[23] W. Ma, *Generalized Zalcman conjecture for starlike and typically real functions*, J. Math. Anal. Appl., **234**(1999), 328–339.

[24] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223**(1976), 337–346.

[25] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc., **41**(1966), 111–122.

[26] J. K. Prajapat, D. Bansal, A. Singh and A. K. Mishra, *Bounds on third Hankel determinant for close-to-convex functions*, Acta Univ. Sapientiae Math., **7**(2015), 210–219.

[27] M. Raza and S. N. Malik, *Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli*, J. Inequal. Appl., **2013** (2013), Article 412, 8 pp.

[28] T. J. Suffridge, *Some special classes of conformal mappings*, Handbook of complex analysis: Geometric Function Theory **2**, 309–338, Elsevier Sci. B. V., Amsterdam, 2005.

[29] P. Zaprawa, *Third Hankel determinants for subclasses of univalent functions*, Mediterr. J. Math., **14(1)**(2017), Art. 19, 10 pp.