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Strong Convergence Theorems for Common Points of a Finite Family of Accretive Operators

Jae Ug Jeong* and Soo Hwan Kim
Department of Mathematics, Dongeui University, Pusan 47340, South Korea
e-mail : jujeong@deu.ac.kr *and* sh-kim@deu.ac.kr

Abstract. In this paper, we propose a new iterative algorithm generated by a finite family of accretive operators in a q-uniformly smooth Banach space. We prove the strong convergence of the proposed iterative algorithm. The results presented in this paper are interesting extensions and improvements of known results of Qin et al. [Fixed Point Theory Appl. 2014(2014): 166], Kim and Xu [Nonlinear Anal. 61(2005), 51-60] and Benavides et al. [Math. Nachr. 248(2003), 62-71].

1. Introduction

In the real world, many important problems, have reformulations which require finding zero points of some nonlinear operators. This is true of such problems as inverse problems, transportation problems, evolution equations, complementarity problems, mini-max problems, variational inequalities and optimization problems (see [7, 10, 11, 15, 19, 27] and the references therein). In the past several decades, many authors have studied the existence and convergence of different schemes for finding zero points for maximal monotone operators (see [20, 23, 24] and the references therein).

Let E be a real Banach space with dual space E^*, and C be a nonempty closed convex subset of E. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T, *i.e.*, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \phi\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if for $t > 0$ and $x, y \in D(A)$, $\|x - y\| \leq \|x - y + t(u - v)\|$ for every $u \in Ax$, $v \in Ay$. It follows from Kato [13] that A is accretive if and only if

* Corresponding Author.

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for $x, y \in D(A)$, there exists $j_q(x-y)$ such that $\langle u-v, j_q(x-y)\rangle \geq 0$. An accretive operator is said to be m-accretive if $R(I+rA)=E$ for all $r>0$. In a real Hilbert space H, an operator A is m-accretive if and only if A is maximal monotone.

One of the major problems in the theory of accretive operators is as follows: Find a point $z \in E$ such that

$$0 \in Az,$$

where A is an operator from E into E^*. A point $z \in E$ is called a zero point of A. The set of zero points of the operator A is denoted by $A^{-1}(0)$. If A is m-accretive, then the solution set $A^{-1}(0)$ is closed and convex. For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$ for each $r>0$, which is called the resolvent of A.

A well-known method for solving zeros of maximal monotone operator is a proximal point algorithm. Let A be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 = x \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $\{\lambda_n\} \subset (0,\infty)$ and $J_{\lambda_n} = (I+\lambda_n A)^{-1}$. Rockafellar [24] proved that the sequence $\{x_n\}$ defined by (1.1) converges weakly to an element of $A^{-1}(0)$.

Recently, many authors have investigated the strong convergence of a proximal point algorithm. Strong convergence theorems for zero points of accretive operators were established in [12, 14, 22, 29] and the references therein.

In 2003, Benavides et al. [4] proposed one iterative scheme for approaching a zero of an m-accretive operator in a uniformly convex Banach space as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 0. \end{cases} \tag{1.2}$$

Recently, Kim and Xu [16] presented the following iteration scheme in a uniformly smooth Banach space:

$$\begin{cases} x_0, u \in E, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 0. \end{cases} \tag{1.3}$$

And they proved the sequence $\{x_n\}$ converges strongly to a zero of an m-accretive operator.

In 2014, Qin et al. [21] studied the problem of modifying the proximal point algorithm to have strong convergence for the sum of two accretive operators in a real q-uniformly smooth Banach space. To be more precise, they considered the following iterative process:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n J_{\lambda_n}(x_n - \lambda_n A x_n + e_n) + \gamma_n f_n, \quad \forall n \geq 0. \end{cases} \tag{1.4}$$

They proved that the sequence $\{x_n\}$ generated in the above iterative process converges strongly to $x = \text{Proj}_{(A+B)^{-1}(0)}f(x)$.

Inspired and motivated by the corresponding convergence results of (1.2)–(1.4), we are concerned with the problem of finding a common zero point of a family of accretive operators based on the proximal point algorithm. A strong convergence theorem is established in a real q-uniformly smooth Banach space.The results obtained in this paper improve and extend the results of Qin et al. [21], Kim and Xu [16], Benavides et al. [4] and some other results in this direction.

2. Preliminaries

Let C be a subset of a real Banach space E. Let E^* be a dual space of E and $q > 1$ be a real number. We recall that the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called a normalized duality mapping and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. We know that J_q is single-valued if E is smooth, which is denoted by j_q. If E is a Hilbert space, then $J = I$, the identity mapping.

Let $S(E) = \{x \in E : \|x\| = 1\}$. A Banach space E is called uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, where $\rho_E : [0,\infty) \to [0,\infty)$ is the modulus of smoothness of E which is defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(E), \|y\| \leq t\}.$$

A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth. It is shown in [8] that there is no Banach space which is q-uniformly smooth with $q > 2$. Hilbert spaces, L^p(or l^p) spaces and Sobolev space W^p_m, where $p \geq 2$, are 2-uniformly smooth.

Definition 2.1. A mapping $T : C \to E$ is said to be

(1) *η-strongly accretive* if for all $x, y \in C$, there exists $\eta > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq \eta\|x - y\|^q;$$

(2) *μ-inverse strongly accretive* if for all $x, y \in C$, there exists $\mu > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq \mu\|Tx - Ty\|^q.$$

(3) *L-Lipschitzian* if for all $x, y \in C$, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

(4) κ-*contractive* if for all $x, y \in C$, there exists a constant $\kappa \in (0,1)$ such that

$$\|Tx - Ty\| \leq \kappa \|x - y\|.$$

Let D be a nonempty subset of C. Let $Q : C \to D$ be a mapping. Then Q is said to be *sunny* if $Q(tx + (1-t)Qx) = Qx$ whenever $tx + (1-t)Qx \in C$ for $x \in C$ and $t \geq 0$. Q is called a retraction if $Qx = x$ for all $x \in D$. Next we give the following lemmas which play an important role in this article.

Lemma 2.1.([18]) *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q} b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.2. *Let $q > 1$. For any two nonnegative real numbers a and b, we have*

$$(a+b)^q \leq 2^q(a^q + b^q).$$

Lemma 2.3.([28]) *Let E be a real q-uniformly smooth Banach space. Then the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + K_q\|y\|^q, \quad \forall x, y \in E,$$

where K_q is some fixed positive constant.

Lemma 2.4.([3]) *Let E be a Banach space and A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$ and $x \in E$, we have*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x\right),$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

Lemma 2.5.([25]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for every $n \geq 1$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6.([20]) *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $f : C \to C$ be a contractive mapping and let $T : C \to C$ be a nonexpansive mapping. For each $t \in (0,1)$, let x_t be the unique solution of the equation $x = tf(x) + (1-t)Tx$. Then $\{x_t\}$ converges strongly to a fixed point $\overline{x} = Q_{Fix(T)}f(\overline{x})$.*

Lemma 2.7.([2]) *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $A : C \to E$ be a single-valued operator and let $M : E \to 2^E$ be an m-accretive operator. Then $Fix(J_r(I - rA)) = (A + M)^{-1}(0)$, where $J_r = (I + rM)^{-1}$.*

Lemma 2.8.([17]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$, where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a number sequence. Assume that $\sum_{n=0}^{\infty} t_n = \infty$, $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

Theorem 3.1. *Let E be a real q-uniformly smooth Banach space with the constant K_q and C be a nonempty closed convex subset of E. Let $N \geq 1$ be some positive integer. For each $i = 1, 2, \cdots, N$, let $A_i : C \to E$ be an η_i-strongly accretive and L_i-Lipschitzian mapping. Let $M : D(M)(\subseteq C) \to 2^E$ be an m-accretive operator and $f : C \to C$ be a κ-contraction. Assume that $(\sum_{i=1}^N \lambda_i A_i + M)^{-1}(0) \neq \phi$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

(3.1)
$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n J_{r_n}((I - r_n \sum_{i=1}^N \lambda_i A_i)x_n + e_n) + \gamma_n x_n + \delta_n g_n, \forall n \geq 0, \end{cases}$$

where $J_{r_n} = (I + r_n M)^{-1}$, $\{e_n\}$ is a sequence in E, $\{g_n\}$ is a bounded sequence in E and $\{r_n\}$ is a positive real number sequence. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$ satisfying the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $\sum_{i=1}^N \lambda_i = 1$;

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;

(4) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\sum_{n=0}^{\infty} \delta_n < \infty$;

(5) $r_n \geq \mu > 0$ *for each* $n \geq 0$, $r_n \leq [\frac{q\eta}{(2^{N-1}L)^q K_q}]^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where $\eta = \min\{\eta_1, \eta_2, \cdots, \eta_N\}$ and $L = \max\{L_1, L_2, \cdots, L_N\}$. Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = Q_{(\sum_{i=1}^N \lambda_i A_i + M)^{-1}(0)} f(\overline{x})$, which is the unique solution to the following variational inequality:

$$\langle f(\overline{x}) - \overline{x}, j_q(p - \overline{x})\rangle \leq 0, \quad \forall p \in (\sum_{i=1}^N \lambda_i A_i + M)^{-1}(0),$$

where $Q_{(\sum_{i=1}^N \lambda_i A_i + M)^{-1}(0)}$ is the unique sunny nonexpansive retraction of C onto $(\sum_{i=1}^N \lambda_i A_i + M)^{-1}(0)$.

Proof. First, we show that $\sum_{i=1}^{N}\lambda_i A_i$ is an $\frac{\eta}{(2^{N-1}L)^q}$-inverse strongly accretive mapping.

For all $x, y \in C$, from Lemma 2.2, we have

$$\begin{aligned}
\|(\sum_{i=1}^{N}\lambda_i A_i)x-(\sum_{i=1}^{N}\lambda_i A_i)y\|^q &= \|(\lambda_1 A_1 x-\lambda_1 A_1 y)+[(\sum_{i=2}^{N}\lambda_i A_i)x-(\sum_{i=2}^{N}\lambda_i A_i)y]\|^q \\
&\le 2^q(\|\lambda_1 A_1 x-\lambda_1 A_1 y\|^q+\|(\sum_{i=2}^{N}\lambda_i A_i)x-(\sum_{i=2}^{N}\lambda_i A_i)y\|^q) \\
&\;\;\vdots \\
&\le 2^{(N-1)q}\sum_{i=1}^{N}\lambda_i\|A_i x-A_i y\|^q. \qquad (3.2)
\end{aligned}$$

Since $A_i : C \to E$ is η_i-strongly accretive and L_i-Lipschitzian continuous, it follows from (3.2) that

$$\begin{aligned}
\langle(\sum_{i=1}^{N}\lambda_i A_i)x-(\sum_{i=1}^{N}\lambda_i A_i)y, j_q(x-y)\rangle &= \sum_{i=1}^{N}\lambda_i\langle A_i x-A_i y, j_q(x-y)\rangle \\
&\ge \sum_{i=1}^{N}\lambda_i\eta_i\|x-y\|^q \\
&\ge \sum_{i=1}^{N}\lambda_i\frac{\eta_i}{L_i^q}\|A_i x-A_i y\|^q \\
&\ge \sum_{i=1}^{N}\lambda_i\frac{\eta}{L^q}\|A_i x-A_i y\|^q \\
&\ge \frac{\eta}{(2^{N-1}L)^q}\|(\sum_{i=1}^{N}\lambda_i A_i)x-(\sum_{i=1}^{N}\lambda_i A_i)y\|^q, \qquad (3.3) \\
&\quad \forall x, y \in C.
\end{aligned}$$

This proves that $\sum_{i=1}^{N}\lambda_i A_i$ is $\frac{\eta}{(2^{N-1}L)^q}$-inverse strongly accretive.

Next, we show that $I - r_n\sum_{i=1}^{N}\lambda_i A_i$ is a nonexpansive mapping.

In view of Lemma 2.3 and (3.3), we find that

$$\begin{aligned}
&\|(I - r_n \sum_{i=1}^{N} \lambda_i A_i)x - (I - r_n \sum_{i=1}^{N} \lambda_i A_i)y\|^q \\
&\le \|x - y\|^q - q r_n \langle (\sum_{i=1}^{N} \lambda_i A_i)x - (\sum_{i=1}^{N} \lambda_i A_i)y, j_q(x - y)\rangle \\
&\quad + K_q r_n^q \|(\sum_{i=1}^{N} \lambda_i A_i)x - (\sum_{i=1}^{N} \lambda_i A_i)y\|^q \\
&\le \|x - y\|^q - (\frac{q\eta}{(2^{N-1}L)^q} - K_q r_n^{q-1}) r_n \|(\sum_{i=1}^{N} \lambda_i A_i)x - (\sum_{i=1}^{N} \lambda_i A_i)y\|^q.
\end{aligned}$$

From condition (5), we find that $I - r_n \sum_{i=1}^{N} \lambda_i A_i$ is a nonexpansive mapping.

Fixing $p \in (\sum_{i=1}^{N} \lambda_i A_i + M)^{-1}(0)$, we find from Lemma 2.7 that

$$\begin{aligned}
\|x_1 - p\| &= \|\alpha_0 f(x_0) + \beta_0 J_{r_0}((I - r_0 \sum_{i=1}^{N} \lambda_i A_i)x_0 + e_0) + \gamma_0 x_0 + \delta_0 g_0 - p\| \\
&\le \alpha_0 \|f(x_0) - p\| + \beta_0 \|J_{r_0}((I - r_0 \sum_{i=1}^{N} \lambda_i A_i)x_0 + e_0) - p\| \\
&\quad + \gamma_0 \|x_0 - p\| + \delta_0 \|g_0 - p\| \\
&\le \alpha_0 \kappa \|x_0 - p\| + \alpha_0 \|f(p) - p\| + \beta_0 \|x_0 - p\| + \beta_0 \|e_0\| \\
&\quad + \gamma_0 \|x_0 - p\| + \delta_0 \|g_0 - p\| \\
&\le [1 - \alpha_0(1 - \kappa)]\|x_0 - p\| + \alpha_0 \|f(p) - p\| + \|e_0\| + \delta_0 \|g_0 - p\|. \qquad (3.4)
\end{aligned}$$

Next, we prove that

$$\|x_n - p\| \le M_1 + \sum_{i=0}^{n-1} \|e_i\| + \sum_{i=0}^{n-1} \delta_i \|g_i - p\|, \qquad (3.5)$$

where $M_1 = \max\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\kappa}\} < \infty$.

In view of (3.4), we find that (3.5) holds for $n = 1$. We assume that the result holds for some m. Notice that

$$\begin{aligned}
&\|x_{m+1} - p\| \\
&= \|\alpha_m f(x_m) + \beta_m J_{r_m}((I - r_m \sum_{i=1}^{N} \lambda_i A_i)x_m + e_m) + \gamma_m x_m + \delta_m g_m - p\|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_m\|f(x_m)-p\| + \beta_m\|J_{r_m}((I-r_m\sum_{i=1}^N \lambda_i A_i)x_m + e_m) - p\| \\
&\quad + \gamma_m\|x_m-p\| + \delta_m\|g_m-p\| \\
&\le \alpha_m\kappa\|x_m-p\| + \alpha_m\|f(p)-p\| + \beta_m\|x_m-p\| + \beta_m\|e_m\| \\
&\quad + \gamma_m\|x_m-p\| + \delta_m\|g_m-p\| \\
&\le [1-\alpha_m(1-\kappa)]\|x_m-p\| + \alpha_m(1-\kappa)\frac{\|f(p)-p\|}{1-\kappa} \\
&\quad + \|e_m\| + \delta_m\|g_m-p\| \\
&\le M_1 + \sum_{i=0}^m \|e_i\| + \sum_{i=0}^m \delta_i\|g_i-p\|.
\end{aligned}$$

This shows that (3.5) holds.

In view of the restriction (4), we find that the sequence $\{x_n\}$ is bounded.

Put $y_n = (I - r_n\sum_{i=1}^N \lambda_i A_i)x_n + e_n$. Note that

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&= \|(I-r_{n+1}\sum_{i=1}^N \lambda_i A_i)x_{n+1} + e_{n+1} - (I-r_n\sum_{i=1}^N \lambda_i A_i)x_n - e_n\| \\
&\le \|(I-r_{n+1}\sum_{i=1}^N \lambda_i A_i)x_{n+1} - (I-r_{n+1}\sum_{i=1}^N \lambda_i A_i)x_n\| \\
&\quad + \|(I-r_{n+1}\sum_{i=1}^N \lambda_i A_i)x_n - (I-r_n\sum_{i=1}^N \lambda_i A_i)x_n\| + \|e_{n+1}\| + \|e_n\| \\
&\le \|x_{n+1}-x_n\| + |r_{n+1}-r_n|\|(\sum_{i=1}^N \lambda_i A_i)x_n\| + \|e_{n+1}\| + \|e_n\|. \qquad (3.6)
\end{aligned}$$

Put $z_n = \frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n}$. Now we compute $\|z_{n+1}-z_n\|$. Note that

$$\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n} \\
&= \frac{\alpha_{n+1}}{1-\gamma_{n+1}}(f(x_{n+1}) - J_{r_{n+1}}(y_{n+1})) + J_{r_{n+1}}(y_{n+1}) \\
&\quad + \frac{\delta_{n+1}}{1-\gamma_{n+1}}(g_{n+1} - J_{r_{n+1}}(y_{n+1})) - \frac{\alpha_n}{1-\gamma_n}(f(x_n) - J_{r_n}(y_n)) \\
&\quad - J_{r_n}(y_n) - \frac{\delta_n}{1-\gamma_n}(g_n - J_{r_n}(y_n)).
\end{aligned}$$

This yields

$$
\begin{aligned}
\|z_{n+1} - z_n\| \le & \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\|f(x_{n+1}) - J_{r_{n+1}}(y_{n+1})\| + \|J_{r_{n+1}}(y_{n+1}) - J_{r_n}(y_n)\| \\
& + \frac{\alpha_n}{1-\gamma_n}\|f(x_n) - J_{r_n}(y_n)\| + \frac{\delta_{n+1}}{1-\gamma_{n+1}}\|g_{n+1} - J_{r_{n+1}}(y_{n+1})\| \\
& + \frac{\delta_n}{1-\gamma_n}\|g_n - J_{r_n}(y_n)\|. \qquad (3.7)
\end{aligned}
$$

Next, we estimate $\|J_{r_{n+1}}(y_{n+1}) - J_{r_n}(y_n)\|$.
In view of Lemma 2.4 and (3.6), we find that

$$
\begin{aligned}
& \|J_{r_{n+1}}(y_{n+1}) - J_{r_n}(y_n)\| \\
& = \|J_{r_n}[\frac{r_n}{r_{n+1}}y_{n+1} + (1 - \frac{r_n}{r_{n+1}})J_{r_{n+1}}(y_{n+1})] - J_{r_n}(y_n)\| \\
& \le \|\frac{r_n}{r_{n+1}}(y_{n+1} - y_n) + (1 - \frac{r_n}{r_{n+1}})(J_{r_{n+1}}(y_{n+1}) - y_n)\| \\
& \le \|y_{n+1} - y_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|J_{r_{n+1}}y_{n+1} - y_{n+1}\| \\
& \le \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\left(\|(\sum_{i=1}^N \lambda_i A_i)x_n\| + \frac{\|J_{r_{n+1}}y_{n+1} - y_{n+1}\|}{r_{n+1}} \right) \\
& \quad + \|e_{n+1}\| + \|e_n\| \\
& \le \|x_{n+1} - x_n\| + |r_{n+1} - r_n|M_2 + \|e_{n+1}\| + \|e_n\|, \qquad (3.8)
\end{aligned}
$$

where M_2 is an appropriate constant such that $M_2 \ge \sup_{n\ge 0}\{\|(\sum_{i=1}^N \lambda_i A_i)x_n\| + \frac{\|J_{r_{n+1}}y_{n+1} - y_{n+1}\|}{\mu}\}$. Substituting (3.8) into (3.7), we find that

$$
\begin{aligned}
\|z_{n+1} - z_n\| \le & \frac{\alpha_n}{1-\gamma_{n+1}}\|f(x_{n+1}) - J_{r_{n+1}}(y_{n+1})\| + \|x_{n+1} - x_n\| + |r_{n+1} - r_n|M_2 \\
& + \|e_{n+1}\| + \|e_n\| + \frac{\alpha_n}{1-\gamma_n}\|f(x_n) - J_{r_n}(y_n)\| \\
& + \frac{\delta_{n+1}}{1-\gamma_{n+1}}\|g_{n+1} - J_{r_{n+1}}(y_{n+1})\| + \frac{\delta_n}{1-\gamma_n}\|g_n - J_{r_n}(y_n)\|.
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le & \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\|f(x_{n+1}) - J_{r_{n+1}}(y_{n+1})\| + |r_{n+1} - r_n|M_2 \\
& + \|e_{n+1}\| + \|e_n\| + \frac{\alpha_n}{1-\gamma_n}\|f(x_n) - J_{r_n}(y_n)\| \\
& + \frac{\delta_{n+1}}{1-\gamma_{n+1}}\|g_{n+1} - J_{r_{n+1}}(y_{n+1})\| \\
& + \frac{\delta_n}{1-\gamma_n}\|g_n - J_{r_n}(y_n)\|.
\end{aligned}
$$

In view of the restrictions (2), (3), (4) and (5), we find that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

It follows from Lemma 2.5 that $\lim_{n\to\infty}\|z_n-x_n\|=0$. From the restriction (3), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.9}$$

Notice that

$$\begin{aligned}\|x_n-J_{r_n}y_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-J_{r_n}y_n\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-J_{r_n}y_n\|\\ &\quad+\gamma_n\|x_n-J_{r_n}y_n\|+\delta_n\|g_n-J_{r_n}y_n\|.\end{aligned}$$

It follows that

$$(1-\gamma_n)\|x_n-J_{r_n}y_n\|\le\|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-J_{r_n}y_n\|+\delta_n\|g_n-J_{r_n}y_n\|.$$

In view of the restrictions (2), (3), (4) and (3.9), we find that

$$\lim_{n\to\infty}\|x_n-J_{r_n}y_n\|=0. \tag{3.10}$$

Notice that

$$\begin{aligned}\|x_n-J_{r_n}(I-r_n\sum_{i=1}^N\lambda_iA_i)x_n\| &\le \|x_n-J_{r_n}y_n\|\\ &\quad+\|J_{r_n}y_n-J_{r_n}(I-r_n\sum_{i=1}^N\lambda_iA_i)x_n\|\\ &\le \|x_n-J_{r_n}y_n\|+\|e_n\|.\end{aligned}$$

Since $\sum_{n=0}^\infty\|e_n\|<\infty$, we see from (3.10) that

$$\lim_{n\to\infty}\|x_n-J_{r_n}(I-r_n\sum_{i=1}^N\lambda_iA_i)x_n\|=0. \tag{3.11}$$

Since M is an accretive mapping, we have

$$\begin{aligned}&\langle M(J_r(I-r\sum_{i=1}^N\lambda_iA_i)x_n)-M(J_{r_n}(I-r_n\sum_{i=1}^N\lambda_iA_i)x_n),\\ &j_q(J_r(I-r\sum_{i=1}^N\lambda_iA_i)x_n-J_{r_n}(I-r_n\sum_{i=1}^N\lambda_iA_i)x_n)\rangle\ge 0.\end{aligned}$$

From $M = \frac{J_r^{-1}-I}{r} = \frac{J_{r_n}^{-1}-I}{r_n}$, we obtain

$$\Big\langle (\frac{1}{r}-\frac{1}{r_n})x_n + (\frac{1}{r_n}-\frac{1}{r})J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n, \\ j_q(J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n)\Big\rangle \\ + \frac{1}{r}\Big\langle J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n - J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n, \\ j_q(J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n)\Big\rangle \geq 0.$$

So, we get

$$\Big\langle J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n, \\ j_q(J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n)\Big\rangle \\ \leq \frac{r_n-r}{r_n}\Big\langle x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n, \\ j_q(J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n)\Big\rangle.$$

It follows that

$$\|J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n\|^q \\ \leq \frac{r_n-r}{r_n}\Big\langle x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n, \\ j_q(J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n)\Big\rangle \\ \leq \|x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n\| \\ \|J_r(I-r\sum_{i=1}^N \lambda_i A_i)x_n - J_{r_n}(I-r_n\sum_{i=1}^N \lambda_i A_i)x_n\|^{q-1}.$$

Hence we get from (3.11) that

$$\|J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n - J_{r_n}(I - r_n\sum_{i=1}^{N}\lambda_i A_i)x_n\| \le \|x_n - J_{r_n}(I - r_n\sum_{i=1}^{N}\lambda_i A_i)x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that

$$\lim_{n\to\infty}\|J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n - x_n\| = 0. \tag{3.12}$$

Since $J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)$ is nonexpansive and f is contractive, we find that the mapping $tf + (1-t)J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)$ is contractive, where $t \in (0,1)$. Let z_t be the unique fixed point of the $tf + (1-t)J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)$, that is,

$$z_t = tf(z_t) + (1-t)J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)z_t$$

for all $t \in (0,1)$. Put $\overline{x} = \lim_{t\to 0} z_t$. Then we have from Lemma 2.6 that $\overline{x} = Q_{(\sum_{i=1}^{N}\lambda_i A_i + M)^{-1}(0)}f(\overline{x})$, where $Q_{(\sum_{i=1}^{N}\lambda_i A_i + M)^{-1}(0)}$ is the unique sunny nonexpansive retraction from C onto $(\sum_{i=1}^{N}\lambda_i A_i + M)^{-1}(0)$.

Next, we claim that $\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x})\rangle \le 0$.

Since

$$\|z_t - x_n\| = \|(1-t)(J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)z_t - x_n) + t(f(z_t) - x_n)\|,$$

we see that for any $t \in (0,1)$,

$$\begin{aligned}
\|z_t - x_n\|^q &= (1-t)\langle J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)z_t - x_n, j_q(z_t - x_n)\rangle \\
&\quad + t\langle f(z_t) - x_n, j_q(z_t - x_n)\rangle \\
&= (1-t)(\langle J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)z_t - J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n, j_q(z_t - x_n)\rangle \\
&\quad + \langle J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n - x_n, j_q(z_t - x_n)\rangle) \\
&\quad + t\langle f(z_t) - z_t, j_q(z_t - x_n)\rangle + t\langle\langle z_t - x_n, j_q(z_t - x_n)\rangle
\end{aligned}$$

$$\leq (1-t)(\|z_t - x_n\|^q + \|J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n - x_n\|\|z_t - x_n\|^{q-1})$$
$$+ t\langle f(z_t) - z_t, j_q(z_t - x_n)\rangle + t\|z_t - x_n\|^q$$
$$\leq \|z_t - x_n\|^q + \|J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n - x_n\|\|z_t - x_n\|^{q-1}$$
$$+ t\langle f(z_t)) - z_t, j_q(z_t - x_n)\rangle.$$

It follows that

$$\langle z_t - f(z_t), j_q(z_t - x_n)\rangle \leq \frac{1}{t}\|J_r(I - r\sum_{i=1}^{N}\lambda_i A_i)x_n - x_n\|\|z_t - x_n\|^{q-1}.$$

By virtue of (3.12), we find that

$$\limsup_{n\to\infty}\langle z_t - f(z_t), j_q(z_t - x_n)\rangle \leq 0. \tag{3.13}$$

Since $z_t \to \overline{x}$ as $t \to 0$ and the duality mapping j_q is single-valued and strong-weak* uniformly continuous on bounded subsets of E, we see that

$$|\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x})\rangle - \langle z_t - f(z_t), j_q(z_t - x_n)\rangle|$$
$$\leq |\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x}) - j_q(x_n - z_t)\rangle| + |\langle f(\overline{x}) - \overline{x} + z_t - f(z_t), j_q(x_n - z_t)\rangle|$$
$$\leq |\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x}) - j_q(x_n - z_t)\rangle| + \|f(\overline{x}) - \overline{x} + z_t - f(z_t)\|\|x_n - z_t\|^{q-1}$$
$$\to 0 \quad \text{as} \quad t \to 0.$$

Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $t \in (0, \delta)$, the following inequality holds:

$$\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x})\rangle \leq \langle z_t - f(z_t), j_q(z_t - x_n)\rangle + \varepsilon.$$

This implies that

$$\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x})\rangle \leq \limsup_{n\to\infty}\langle z_t - f(z_t), j_q(z_t - x_n)\rangle + \varepsilon.$$

Using (3.13), we see that

$$\limsup_{n\to\infty}\langle f(\overline{x}) - \overline{x}, j_q(x_n - \overline{x})\rangle \leq 0. \tag{3.14}$$

Finally, we prove that $x_n \to \overline{x}$ as $n \to \infty$.

By virtue of (3.1), Lemma 2.1 and 2.7, we obtain that

$$
\begin{aligned}
&\|x_{n+1} - \overline{x}\|^q \\
&= \|\alpha_n f(x_n) + \beta_n J_{r_n}((I - r_n \sum_{i=1}^{N} \lambda_i A_i)x_n + e_n) + \gamma_n x_n + \delta_n g_n - \overline{x}\|^q \\
&\le \alpha_n \langle f(x_n) - f(\overline{x}), j_q(x_{n+1} - \overline{x})\rangle + \alpha_n \langle f(\overline{x}) - \overline{x}, j_q(x_{n+1} - \overline{x})\rangle \\
&\quad + \beta_n \|J_{r_n}((I - r_n \sum_{i=1}^{N} \lambda_i A_i)x_n + e_n) - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} \\
&\quad + \gamma_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} + \delta_n \|g_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} \\
&\le \alpha_n \kappa \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} + \alpha_n \langle f(\overline{x}) - \overline{x}, j_q(x_{n+1} - \overline{x})\rangle \\
&\quad + \beta_n \|J_{r_n}((I - r_n \sum_{i=1}^{N} \lambda_i A_i)x_n + e_n) - J_{r_n}(I - r_n \sum_{i=1}^{N} \lambda_i A_i)\overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} \\
&\quad + \gamma_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} + \delta_n \|g_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} \\
&\le \alpha_n \kappa \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} + \alpha_n \langle f(\overline{x}) - \overline{x}, j_q(x_{n+1} - \overline{x})\rangle \\
&\quad + \beta_n (\|x_n - \overline{x}\| + \|e_n\|) \|x_{n+1} - \overline{x}\|^{q-1} \\
&\quad + \gamma_n \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} + \delta_n \|g_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} \\
&\le (1 - \alpha_n(1-\kappa)) \|x_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} + \alpha_n \langle f(\overline{x}) - \overline{x}, j_q(x_{n+1} - \overline{x})\rangle \\
&\quad + \|e_n\| \|x_{n+1} - \overline{x}\|^{q-1} + \delta_n \|g_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1} \\
&\le (1 - \alpha_n(1-\kappa)) (\frac{1}{q} \|x_n - \overline{x}\|^q + \frac{q-1}{q} \|x_{n+1} - \overline{x}\|^q) \\
&\quad + \alpha_n \langle f(\overline{x}) - \overline{x}, j_q(x_{n+1} - \overline{x})\rangle \\
&\quad + \|e_n\| \|x_{n+1} - \overline{x}\|^{q-1} + \delta_n \|g_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1}.
\end{aligned}
$$

It follows that

$$
\|x_{n+1} - \overline{x}\|^q \le (1 - \alpha_n(1-\kappa)) \|x_n - \overline{x}\|^q + q\alpha_n \langle f(\overline{x}) - \overline{x}, j_q(x_{n+1} - \overline{x})\rangle + \nu_n,
$$

where $\nu_n = q(\|e_n\| \|x_{n+1} - \overline{x}\|^{q-1} + \delta_n \|g_n - \overline{x}\| \|x_{n+1} - \overline{x}\|^{q-1})$. In view of the restriction (4), we find that $\sum_{n=0}^{\infty} \nu_n < \infty$. Using the restriction (2), (3.14) and Lemma 2.8, we see that $\{x_n\}$ converges strongly to $\overline{x}$. This completes the proof. □

Remark 3.1.

(i) Theorem 3.1 is still valid in the framework of the spaces which are Hilbert spaces and L^p-spaces, where $p \ge 2$.

(ii) Theorem 3.1 improves and extends the main result of Qin et al. [21], Kim and Xu [16] and Benavides et al. [4] in the following aspects:

- From a single inverse strongly accretive mapping to the class of strongly accretive and Lipschitzian mappings:

- From the problem of finding an element of $(A+M)^{-1}(0)$ or $A^{-1}(0)$ to problem of finding an element of $(\sum_{i=1}^{N}\lambda_i A_i + M)^{-1}(0)$:
- From a forward-backward splitting algorithm with computational errors to a generalized forward-backward splitting algorithm algorithm with computational errors.

From Theorem 3.1, we obtain the following results:

Corollary 3.1. *Let E be a real q-uniformly smooth Banach space with constant K_q and C be a nonempty closed convex subset of E. Let $A : C \to E$ be an η-strongly accretive and L-Lipschitzian mapping. Let $M : D(M)(\subseteq C) \to 2^E$ be an m-accretive operator and $f : C \to C$ be a κ-contraction. Assume that $(A+M)^{-1}(0) \neq \phi$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n J_{r_n}((I - r_n A)x_n + e_n) + \gamma_n x_n + \delta_n g_n, \forall n \geq 0, \end{cases}$$

where $J_{r_n} = (I + r_n M)^{-1}$, $\{e_n\}$ is a sequence in E, $\{g_n\}$ is a bounded sequence in E and $\{r_n\}$ is a positive real number sequence. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;

(4) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\sum_{n=0}^{\infty} \delta_n < \infty$;

(5) $r_n > \mu > 0$ *for each* $n \geq 0$, $r_n \leq [\frac{q\eta}{L^q K_q}]^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = Q_{(A+M)^{-1}(0)} f(\overline{x})$, which is the unique solution to the following variational inequality:

$$\langle f(\overline{x}) - \overline{x}, j_q(p - \overline{x})\rangle \leq 0, \quad \forall p \in (A+M)^{-1}(0).$$

Proof. Taking $A_1 = A_2 = \cdots = A_n = A$ in Theorem 3.1, then $\sum_{i=1}^{N} \lambda_i A_i = A$. So, we can get the desired conclusion easily. □

Corollary 3.2. *Let E be a real q-uniformly smooth Banach space with the constant K_q and C be a nonempty closed convex subset of E. Let $A : C \to E$ be an η-strongly accretive and L-Lipschitzian mapping. Let $M : D(M)(\subseteq C) \to 2^E$ be an m-accretive operator. Assume that $(A+M)^{-1}(0) \neq \phi$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + \beta_n J_{r_n}((I - r_n A)x_n + e_n) + \gamma_n x_n, \quad \forall n \geq 0, \end{cases}$$

where u is a fixed element in C, $J_{r_n} = (I + r_n M)^{-1}$, $\{e_n\}$ is a sequence in E and $\{r_n\}$ is a positive real number sequence. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n = 1$;

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$;

(4) $\sum_{n=0}^{\infty} \|e_n\| < \infty$;

(5) $r_n > \mu > 0$ *for each* $n \ge 0$, $r_n \le [\frac{q\eta}{L^q K_q}]^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = Q_{(A+M)^{-1}(0)}u$, which is the unique solution to the following variational inequality:

$$\langle u - \overline{x}, j_q(p - \overline{x})\rangle \le 0, \quad \forall p \in (A + M)^{-1}(0).$$

Proof. Taking the mapping f maps any element in C into a fixed element u and $\delta_n = 0$ in Theorem 3.1, we can get the desired conclusion easily. □

4. Applications

In this section, we consider some applications of Theorem 3.1 in the framework of Hilbert spaces.

(I) Application to Theorem 3.1 for k-strict pseudocontractive mappings.

Let C be a nonempty closed convex subset of a Hilbert space H.

Definition 4.1.([6]) A mapping $T : C \to C$ is said to be a *ν-strict pseudocontractive mapping* if there exists $\nu \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \nu\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Let $T : C \to C$ be ν-strict pseudocontractive. Define a mapping $A = I - T : C \to C$. It is easy to see that A is a $\frac{1-\nu}{2}$-inverse strongly monotone mapping and $\mathrm{Fix}(T) = A^{-1}(0)$.

Lemma 4.1.([1]) *Let C be a nonempty closed convex subset of a Hilbert space H. Given an integer $N \ge 1$, assume that $\{T_i\}_{i=1}^N : H \to H$ is a finite family of ν_i-strict pseudocontractive mappings. Suppose that $\{\lambda_i\}_{i=1}^N$ is a positive real sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i T_i$ is a ν-strict pseudocontractive mapping with $\nu = \max\{\nu_i : 1 \le i \le N\}$ and $Fix(\sum_{i=1}^N \lambda_i T_i) = \cap_{i=1}^N Fix(T_i)$.*

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H. For each $i = 1, 2, \cdots, N$, $T_i : C \to C$ is ν_i-strict pseudocontractive mapping. Let $f : C \to C$*

be a κ-contraction. Assume that $\cap_{i=1}^N Fix(T_i) \neq \phi$. Let $\{x_n\}$ be a sequence generated in the following iterative process:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n[I - r_n(I - \sum_{i=1}^N \lambda_i T_i)]x_n + \gamma_n x_n + \delta_n g_n, \quad \forall n \geq 0, \end{cases}$$

where $\{g_n\}$ is a bounded sequence in E and $\{r_n\}$ is a positive real number sequence. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$ satisfying the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $\sum_{i=1}^N \lambda_i = 1$*;*

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{i=0}^\infty \alpha_n = \infty$*;*

(3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$*;*

(4) $\sum_{n=0}^\infty \delta_n < \infty$*;*

(5) $r_n > \mu > 0$ *for each* $n \geq 0$*,* $r_n \leq 1 - \nu$ *and* $\sum_{n=1}^\infty |r_n - r_{n-1}| < \infty$*, where* $\nu = \max\{\nu_i : 1 \leq i \leq N\}$*.*

Then the sequence $\{x_n\}$ converges strongly to $\overline{x}$, where $\overline{x} \in \cap_{i=1}^N Fix(T_i)$ is the unique solution to the following variational inequality

$$\langle f(\overline{x}) - \overline{x}, p - \overline{x}\rangle \leq 0, \quad \forall p \in \cap_{i=1}^N Fix(T_i).$$

Proof. From Lemma 4.1, we know that $\sum_{i=1}^N \lambda_i T_i$ is a ν-strict pseudocontractive mapping with $\nu = \max\{\nu_i : 1 \leq i \leq N\}$ and $\mathrm{Fix}(\sum_{i=1}^N \lambda_i T_i) = \cap_{i=1}^N \mathrm{Fix}(T_i)$. It follows that $I - \sum_{i=1}^N \lambda_i T_i$ is a $\frac{1-\nu}{2}$-inverse strongly monotone mapping. The conclusion of Theorem 4.1 can be obtained from Theorem 3.1 immediately. □

(II) Application to equilibrium problems.

Let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is a set of real numbers. The equilibrium problem for the function F is to find a point $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

The set of solutions of (4.2) is denoted by $\mathrm{EP}(F)$. For solving the equilibrium problem, we assume that F satisfies the following conditions (see [5]):

(A1) $F(x, y) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) F is upper-hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) $F(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$.

Lemma 4.2.([5]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r>0$ and $x\in H$. Then there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \forall y\in C.$$

Lemma 4.3.([9]) *Assume that $F: C\times C\to\mathbb{R}$ satisfies (A1)-(A4). For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_r(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\forall y\in C\}$$

for all $x\in H$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for all $x,y\in H$,*

$$\|T_rx-T_ry\|^2\leq\langle T_rx-T_ry,x-y\rangle;$$

(3) *$Fix(T_r)=EP(F),\quad\forall r>0$;*

(4) *$EP(F)$ is a closed and convex set.*

Lemma 4.4.([26]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let A_F be a multi-valued mapping of H into itself defined by*

$$A_Fx=\begin{cases}\{z\in H: F(x,y)\geq\langle y-x,z\rangle,\quad\forall y\in C\}, & x\in C,\\ \phi, & x\notin C.\end{cases}$$

Then $EP(F)=A_F^{-1}(0)$ and A_F is a maximal monotone operator with $D(A_F)\subseteq C$. Further, for any $x\in H$ and $r>0$, T_r coincides with the resolvent of A_F, i.e., $T_rx=(I+rA_F)^{-1}x$.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $f: C\to C$ be a κ-contraction. Assume $A_F^{-1}(0)\neq\phi$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases}x_0\in C,\\ x_{n+1}=\alpha_nf(x_n)+\beta_nT_{r_n}(x_n+e_n)+\gamma_nx_n+\delta_ng_n,\forall n\geq 0,\end{cases}\tag{4.3}$$

where $\{e_n\}$ is a sequence in H, $\{g_n\}$ is a bounded sequence in H. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ satisfying the following restrictions:

(1) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$;

(4) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ *and* $\sum_{n=0}^{\infty} \delta_n < \infty$;

(5) $r_n > \mu > 0$ *for each* $n \ge 0$ *and* $\sum_{n=1}^{\infty} \|r_n - r_{n_1}\| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $\overline{x} = P_{A_F^{-1}(0)} f(\overline{x})$, *which is the unique solution to the following variational inequality:*

$$\langle f(\overline{x}) - \overline{x}, p - \overline{x} \rangle \le 0, \quad \forall p \in A_F^{-1}(0).$$

Proof. Taking $A_i = 0$ for $i = 1, 2, \cdots, N$ and $J_{r_n} = T_{r_n}$, iterative scheme (4.3) reduces to (3.1) in a Hilbert space and the desired conclusion follows immediately from Lemma 4.4 and Theorem 3.1. This completes the proof. □

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