KYUNGPOOK Math. J. 59(2019), 415-431
https://doi.org/10.5666/KMJ.2019.59.3.415
pISSN 1225-6951 eISSN 0454-8124

Spectral Properties of k-quasi-class $A(s,t)$ Operators

Salah Mecheri
Taibah University, College of Science Department of Mathematics, P. O. Box 20003 Al Madinah Al Munawarah, Saudi Arabia
e-mail : mecherisalah@hotmail.com

Naim Latif Braha*
Research Institute Ilirias, Rruga Janina, No-2, ferizaj, 70000, Kosovo
Department of Mathematics and Computer Sciences, University of Prishtina, Avenue Mother Teresa, No-4, Prishtine, 10000, Kosova
e-mail : nbraha@yahoo.com

Abstract. In this paper we introduce a new class of operators which will be called the class of k-quasi-class $A(s,t)$ operators. An operator $T \in B(H)$ is said to be k-quasi-class $A(s,t)$ if

$$T^{*k}((|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{t+s}} - |T^*|^{2t})T^k \geq 0,$$

where $s > 0, t > 0$ and k is a natural number. We show that an algebraically k-quasi-class $A(s,t)$ operator T is polaroid, has Bishop's property β and we prove that Weyl type theorems for k-quasi-class $A(s,t)$ operators. In particular, we prove that if T^* is algebraically k-quasi-class $A(s,t)$, then the generalized a-Weyl's theorem holds for T. Using these results we show that T^* satisfies generalized the Weyl's theorem if and only if T satisfies the generalized Weyl's theorem if and only if T satisfies Weyl's theorem. We also examine the hyperinvariant subspace problem for k-quasi-class $A(s,t)$ operators.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. An operator $T \in B(H)$ is said to be p-hyponormal, for $p \in (0,1]$, if $(T^*T)^p \geq (TT^*)^p$ [2]. A 1-hyponormal operator is simply called hyponormal and $\frac{1}{2}$-hyponormal is simply called semi-hyponormal. An invertible operator T is said to be log-hyponormal if $\log|T| \geq \log|T^*|$ [37]. An operator T is said to be paranormal if $||T^2x|| \geq ||Tx||^2$. It is known [22] that p-

* Corresponding Author.
Received March 13, 2016; revised October 11, 2018; accepted October 16, 2018.
2010 Mathematics Subject Classification: primary 47B47, 47A30, 47B20, secondary 47B10.
Key words and phrases: ω-hyponormal operators, class $A(s,t)$, Weyl's theorem.

hyponormal and log-hyponormal operators are paranormal. An operator T belongs to class $A(k)$ for $k>0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$. When $k=1$ we say that T belongs to class A. Furuta et al. [19] showed that every class A operator is paranormal. As a further generalization of class $A(k)$, Fujii et al. [20] introduced the class $A(s,t)$: T belongs to the class $A(s,t)$ for $s>0$ and $t>0$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{t+s}} \geq |T^*|^{2t}$. Class $AI(s,t)$ is the class of all invertible class $A(s,t)$ operators for $s>0$ and $t>0$. Fujii et al [20] showed several properties of class $A(s,t)$ and class $AI(s,t)$ as extensions of the properties of class $A(k)$ shown in [20]. They also showed that T is log-hyponormal if and only if T belongs to class $AI(s,t)$ for all $s,t>0$. It is known [40] that class $A(k,1)$ equals class $A(k)$.

Let T be an operator with polar decomposition $T=U|T|$, where $|T|=(T^*T)^{\frac{1}{2}}$. For $s,t>0$, the generalized Aluthge transformation $\tilde{T}_{s,t}$ of T, is $\tilde{T}_{s,t}=|T|^sU|T|^t$. If $s=t=\frac{1}{2}$, then $\tilde{T}_{s,t}$ is called the Aluthge transformation of T denoted by $\tilde{T}$ [2]. The following equalities are relations between T and its transformation $\tilde{T}_{s,t}$

$$\tilde{T}_{s,t}|T|^s = |T|^sU|T|^t|T|^s = |T|^sT,$$
$$U|T|^t\tilde{T}_{s,t} = U|T|^t|T|^sU|T|^t = TU|T|^t.$$

Aluthge and Wang [3] introduced ω-hyponormal operators defined as follows: An operator T is said to be ω-hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T^*}|$. In order to generalize the class of ω-hyponormal operators, Ito [22] introduced the class $\omega A(s,t)$. An operator T belongs to class $\omega A(s,t)$ for $s,t>0$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{s+t}} \geq |T^*|^{2t} \tag{1.1}$$

and

$$|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}. \tag{1.2}$$

Ito [22] showed that $\omega A(s,t)$ operators can be characterized via generalized Aluthge transformation as follows:

An operator T belongs to the class $\omega A(s,t)$ for $s,t>0$ if and only if

$$|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |\tilde{T^*}|^{\frac{2s}{s+t}}. \tag{1.3}$$

An operator T is ω-hyponormal if and only if T belongs to the class $\omega A(\frac{1}{2},\frac{1}{2})$ [39]. An operator T is said to be of class $\omega F(s,t,q)$ for $s,t>0$ and $q\geq 1$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{q}} \geq |T^*|^{\frac{2(s+t)}{q}}$$

and

$$|T|^{2(s+t)(1-\frac{1}{q})} \geq (|T|^s|T^*|^{2t}|T|^s)^{1-\frac{1}{q}}.$$

Because q and $(1-q^{-1})^{-1}$ with $q>1$ are conjugate exponents, it is clear that the class $\omega A(s,t)$ equals the class $\omega F(s,t,\frac{s+t}{t})$. We have the following inclusion of classes from [39] for all $s>0, t>0$:

$$A(s,t) \supseteq \omega A(s,t) \supseteq AI(s,t).$$

In order to generalize the class $A(s,t)$ we introduce the class of k-quasi-class $A(s,t)$ operators defined as follows:

Definition 1.1. An operator $T \in B(H)$ is said to be *k-quasi-class* $A(s,t)$, or in $k_{QC}A(s,t)$ if $T^{*k}((|T^*|^t|T|^{2s}|T^*|^t)^{\frac{1}{t+s}} - |T^*|^{2t})T^k \geq 0$, where $s > 0, t > 0$ and k is a natural number.

We have

$$A(s,t) \supseteq \omega A(s,t) \supseteq AI(s,t) \supseteq k_{QC}A(s,t)$$

for all $s > 0, t > 0$. The connection between k-quasi-class $A(s,t)$ operators and the classes of operators mentioned above is given in Figure 1.

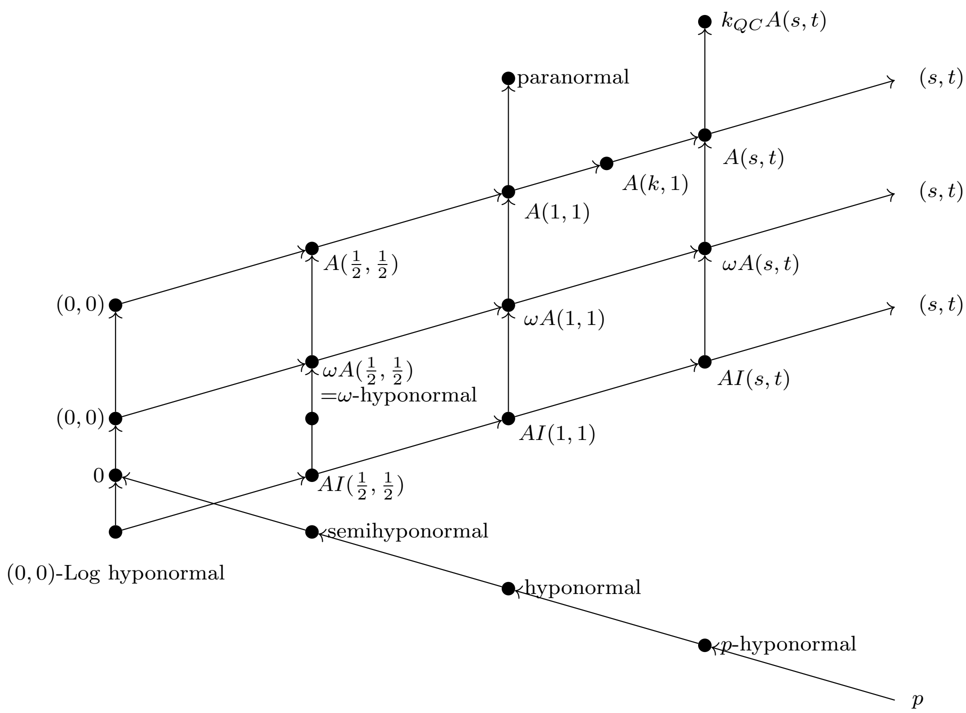

Figure 1: Relationship of classes of operators

If $T \in B(\mathcal{H})$, we shall write $\ker(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim\ker(T) < \infty$ and $\beta(T) = \dim \mathcal{H}/R(T) < \infty$. Moreover if

$i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup \text{acc}\ \sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso $K = K \backslash \text{acc}\ K$, then we let

$$\pi_{00}(T) = \{\lambda \in \text{iso}\ \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T).$$

In [38], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators [14], algebraically hyponormal operators [21], p-hyponormal operators [13] and quasi-$*$-class A [36].

More generally, M. Berkani investigated the generalized Weyl's theorem which extends Weyl's theorem, and proved that the generalized Weyl's theorem holds for hyponormal operators ([6, 7, 8]). In a recent paper [25] the author showed that the generalized Weyl's theorem holds for (p, k)-quasi-hyponormal operators.

M. Berkani also investigated B-Fredholm theory as follows (see [4, 6, 7, 8]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]}) = 0$. The following results is due to M. Berkani and M. Sarih [8].

Proposition 1.1. *Let $T \in B(\mathcal{H})$.*

(1) *If $R(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m \geq n$. Moreover,* $\text{ind}\ T_{[m]} = \text{ind}\ T_{[n]} (= \text{ind}\ T)$.

(2) *An operator T is B-Fredholm (B-Weyl) if and only if there exist T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $T = T|\mathcal{M} \oplus T|\mathcal{N}$ where $T|\mathcal{M}$ is Fredholm (Weyl) and $T|\mathcal{N}$ is nilpotent.*

The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [7]. Recently in [6] M. Berkani and A. Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T.

We define $T \in SF_+^-$ if $R(T)$ is closed, $\dim\ker(T) < \infty$ and $\operatorname{ind} T \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim\ker(T-\lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \{\lambda \mid T - \lambda \notin SF_+^-\} \subset \sigma_W(T)$. We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

V. Rakočević [33, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

We define $T \in SBF_+^-$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim\ker(T_{[n]}) = \dim\ker(T) \cap R(T^n) < \infty$) and $0 \geq \operatorname{ind} T_{[n]} (= \operatorname{ind} T)$ [8]. We define $\sigma_{SBF_+^-}(T) = \{\lambda \mid T - \lambda \notin SBF_+^-\} \subset \sigma_{SF_+^-}(T)$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \dim\ker(T - \lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T).$$

M. Berkani and J.J. Koliha [7] proved that if generalized a-Weyl's theorem holds for T, then a-Weyl's theorem holds for T.

An operator $T \in B(H)$ is said to have the single-valued extension property (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to H$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, which verifies $(T - z)f(z) = x$, and it is called the local resolvent set of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspace of T , $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. An operator $T \in B(H)$ is said to have the property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

$$\text{Property } (\beta) \Rightarrow \text{ Dunford's property}(C) \Rightarrow \text{ SVEP}.$$

SVEP is possessed by many important classes of operators such as hyponormal operators and decomposable operators. The interested reader is referred to [26, 27, 28, 29, 30] for more details. A closed subspace of H is said to be hyperinvariant for T if it is invariant under every operator in the commutant $\{T\}^{'}$ of T. Knowledge of the hyperinvariant subspaces of T gives information on the structure of the commutant of T. The question of whether every operator on H has an hyperinvariant subspace is one of the most difficult problems in operator theory. Some partial solutions on this problem were given in the literature. Perhaps the most elegant results on the hyperinvariant subspace problem are the affirmative answers for non-scalar normal operators and for non-zero compact operators . It is known that every operator which commutes with a (nonzero) compact operator has a (proper closed) hyperinvariant subspace [31].

In this paper, we show that an algebraically k-quasi-class $A(s,t)$ operator T is polaroid and we prove that Weyl type theorems hold for k-quasi-class $A(s,t)$ operators. Especially we prove that if T^* is an algebraically k-quasi-class $A(s,t)$, then generalized a-Weyl's theorem holds for T. Using these results we show that T^* satisfies generalized Weyl's theorem if and only if T satisfies generalized Weyl's theorem if and only if T satisfies Weyl's theorem. We also examine the hyperinvariant subspace problem for k-quasi-class $A(s,t)$ operators.

2. Main Results

Proposition 2.1. *Let T be k-quasi-class $A(s,t)$ operator for $t=\frac{1}{2}$ and $s>0$. Then the restriction $T|_{\mathcal{M}}$ to an invariant subspace $\mathcal{M}$ of T is also k-quasi-class $A(s,t)$.*

Proof. Let P be the projection of H onto M. Thus we can represent T as the following matrix with respect to the decomposition $M\oplus M^{\perp}$,

$$T=\begin{pmatrix} A & B \\ O & C \end{pmatrix}.$$

Put $A=T|_M$ and we have

$$\begin{pmatrix} A & O \\ O & O \end{pmatrix}=TP=PTP.$$

Since T is k-quasi-class $A(s,t)$, we have

$$PT^{*k}\left(\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{1}{t+s}}-|T^*|^{2t}\right)T^kP\geq 0.$$

We remark

$$\begin{aligned} PT^{*k}|T^*|^{2t}T^kP &= PT^{*k}P|T^*|^{2t}PT^kP \\ &= \begin{pmatrix} A^{*k}(|A^*|^{2t}+|B^*|^{2t})A^k & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} A^{*k}|A^*|^{2t}A^k & O \\ O & O \end{pmatrix} + \begin{pmatrix} A^{*k}|B^*|^{2t}A^k & O \\ O & O \end{pmatrix} \\ &\geq \begin{pmatrix} A^{*k}|A^*|^{2t}A^k & O \\ O & O \end{pmatrix} \end{aligned} \tag{2.1}$$

and by Hansen inequality, we have

$$\begin{aligned} PT^{*k}\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{1}{t+s}}T^kP &= PT^{*k}P\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{1}{t+s}}PT^kP \\ &\leq PT^{*k}\left(P|T^*|^t|T|^{2s}|T^*|^tP\right)^{\frac{1}{t+s}}T^kP \\ &= PT^{*k}\left(P|T^*|^tP|T|^{2s}P|T^*|^tP\right)^{\frac{1}{t+s}}T^kP \\ &\leq PT^{*k}\left(P|T^*|^tP(PTT^*P)^sP|T^*|^tP\right)^{\frac{1}{t+s}}T^kP \\ &= \begin{pmatrix} A^{*k}(|A^*|^t|A|^{2s}|A^*|^t)^{\frac{1}{t+s}}A^k & O \\ O & O \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \begin{pmatrix} A^{*k}(|A^*|^t|A|^{2s}|A^*|^t)^{\frac{1}{t+s}}A^k & O \\ O & O \end{pmatrix} &\geq PT^{*k}\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{1}{t+s}}T^kP \\ &\geq PT^{*k}|T^*|^{2t}T^kP \\ &\geq \begin{pmatrix} A^{*k}|A^*|^{2t}A^k & O \\ O & O \end{pmatrix}. \end{aligned}$$

Hence, A is k-quasi-class $A(s,t)$ operator on M. □

Lemma 2.1.([39]) *Let A, B and C be positive operators, $0 < p$ and $0 < r \leq 1$. If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $B \geq C$, then $(C^{\frac{r}{2}}A^pC^{\frac{r}{2}})^{\frac{r}{p+r}} \geq C^r$.*

Lemma 2.2. *Let T be k-quasi-class $A(s,t)$ operator for $t = \frac{1}{2}$ and $s > 0$. Let $R(T^k)$ be not dense. Decompose*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \qquad on \qquad \mathcal{H} = \overline{R(T^k)} \oplus \ker(T^{*k})$$

where $\overline{R(T^k)}$ is closure of $R(T^k)$. Then T_1 is in $A(s,t)$, $T_3^k = 0$, and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Suppose $T \in B(H)$ is $k-$quasi-class $A(s,t)$ for $t = \frac{1}{2}$ and $s > 0$. If $R(T^k)$ is not dense and T has representation

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = [R(T^k)] \oplus [\ker(T^{*k})]$, then $T_1 = TP = PTP$, where P is the projection onto $R(T^k)$.
By Hansen's inequality, we have

$$|T_1|^{2s} = (P|T|^2P)^2 \geq P|T|^{2s}P \tag{2.2}$$

and

$$|T^*|^2 = TT^* \geq TPT^* = |T_1|^2 \tag{2.3}$$

T is $k-$quasi-class $A(s,t)$ for $t = \frac{1}{2}$ and $s > 0$ if and only if

$$P(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}}P \geq P|T^*|^{2t}P \tag{2.4}$$

By Hansen's inequality, (2.3) becomes, $(P|T^*|^t|T|^{2s}|T^*|^tP)^{\frac{t}{s+t}} \geq P|T^*|^{2t}P$ which implies,

$$(P|T^*|^tP|T|^{2s}P|T^*|^tP)^{\frac{t}{s+t}} \geq P|T^*|^{2t}P$$

Since $t = \frac{1}{2}$, the above inequality can be written as

$$((P|T^*|P)^t|T|^{2s}(P|T^*|P)^t)^{\frac{t}{s+t}} \geq (P|T^*|P)^{2t} \tag{2.5}$$

Then from (2.2) and applying Lemma 2.2, (2.4) becomes

$$((P|T_1^*|P)^t|T|^{2s}(P|T_1^*|P)^t))^{\frac{t}{s+t}} \geq (P|T_1^*|P)^{2t}$$

Since $|T_1^*|^tP = P|T_1^*|^t = |T_1^*|^t$, the above inequality changes to

$$(|T_1^*|^t|T|^{2s}|T_1^*|^t)^{\frac{t}{s+t}} \geq |T_1^*|^{2t}$$

Then by (2.1) we have the following:

$$(|T_1^*|^t|T_1|^{2s}|T_1^*|^t)^{\frac{t}{s+t}} \geq |T_1^*|^{2t}.$$

Thus, T_1 is in $A(s,t)$ for $t = \frac{1}{2}$ and $s > 0$.

Let $x \in \ker(T^{*k})$. Then by simple calculations we have $T^kx = 0$ and so $T_3^kx = 0$. Further, we have

$$\langle T_3^kx_2, y_2\rangle = \langle T^k(I-P)x, (I-P)y\rangle = \langle (I-P)x, T^{*k}(I-P)y\rangle = 0,$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H$. Thus $T_3^{*k} = 0$. Since $\sigma(T_3) = \{0\}$, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$. □

Lemma 2.3. *Let T be k-quasi-class $A(s,t)$ for $t = \frac{1}{2}$ and $0 < s < 1$. Then T has Bishop's property β. Hence T has SVEP.*

Proof. Suppose $T \in B(H)$ be $k-$quasi-class $A(s,t)$ for $t = \frac{1}{2}$ and $s \in (0,1]$. If $R(T^k)$ is dense, then T is class $A(s,t)$ for $t = \frac{1}{2}$ and $s \in (0,1]$, so T has Bishop's property β [35]. Suppose that $R(T^k)$ is not dense. From Lemma 2.2, we write the matrix representation of T on $H = [R(T^k)] \oplus [kerT^{*k}]$ as follows.

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

where $T_1 = T \mid R(T^k)$ is in $A(s,t)$ for $t = \frac{1}{2}$ and $s > 0$, and T_3 is nilpotent.
Let $f_n(\mu)$ be analytic on $\mathcal{U} \subseteq \mathbb{C}$ with $(T - \mu)f_n(\mu) \to 0$ uniformly on every compact subset of $\mathcal{U}$. Then

$$\begin{pmatrix} T_1 - \mu & T_2 \\ 0 & T_3 - \mu \end{pmatrix} \begin{pmatrix} f_{n_1}(\mu) \\ f_{n_2}(\mu) \end{pmatrix} = \begin{pmatrix} (T_1 - \mu)f_{n_1}(\mu) + T_2 f_{n_2}(\mu) \\ (T_3 - \mu)f_{n_2}(\mu) \end{pmatrix} \to 0$$

Since T_3 is nilpotent, T_3 satisfies Bishop property β. Thus, $f_{n_2}(\mu) \to 0$ uniformly on every compact subset of $\mathcal{U}$. Thus, $(T_1 - \mu)f_{n_1} \to 0$. Since T_1 is in $A(s,t)$ with $t = \frac{1}{2}$ and $s \in (0,1]$, T_1 satisfies Bishop property (β) [35]. Thus, $f_{n1}(\mu) \to 0$ and so T has Bishop's property β. □

Corollary 2.1. *Let $T \in B(H)$ be a k-quasi-class $A(s,t)$ with thick spectra, then T has a nontrivial invariant subspace*

Proof. Since T has Bishop's property, it suffices to apply [18]. □

An operator T is said to be polaroid if points in $iso(\sigma(T))$ are poles of the resolvent of T. Recall that if T is algebraically polaroid, i.e., $p(T)$ is polaroid for some non-constant polynomial p, then T is polaroid [15]. We will show that an algebraically k-quasi-class $A(s,t)$ operator is polaroid. For this we need the following lemmas.

Lemma 2.4. *Assume that T is quasinilpotent k-quasi-class $A(s,t)$. Then T is nilpotent.*

Proof. Assume that T is k-quasi-class $A(s,t)$. We consider two cases:

Case 1: Suppose T^k has dense range. It follows that T is class $A(s,t)$. Since a class $A(s,t)$ operator is normaloid [20, 39], every quasinilpotent class $A(s,t)$ operator is the zero operator. Hence T is nilpotent.

Case 2: Suppose T^k does not have dense range. Then by Lemma 2.1, T can be represented on $\mathcal{H} = \overline{R(T^k)} \oplus \ker T^{*k}$ as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

So T_1 is in $A(s,t)$, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since T is quasinilpotent, $\sigma(T) = \{0\}$. But $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since T_1 is in $A(s,t)$, $T_1 = 0$, so T is nilpotent. □

Lemma 2.5. *Let $T \in B(H)$ be an algebraically k-quasi-class $A(s,t)$ operator, and $\sigma(T) = \{\mu_0\}$, then $T - \mu_0$ is nilpotent.*

Proof. Assume $p(T)$ is k-quasi-class $A(s,t)$ for some nonconstant polynomial $p(z)$. Since $\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\}$, the operator $p(T) - p(\mu_0)$ is nilpotent by Lemma 2.4. Let

$$p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1} \cdots (z - \mu_t)^{k_t},$$

where $\mu_j \neq \mu_s$ for $j \neq s$. Then

$$0 = \{p(T) - p(\mu_0)\}^m = a^m(T - \mu_0)^m k_0(T - \mu_1)^{mk_1} \cdots (T - \mu_t)^{mk_t}$$

and hence $(T - \mu_0)^{mk_0} = 0$. □

In the following theorem we will prove that an algebraically k-quasi-class $A(s,t)$ operator is polaroid.

Theorem 2.1. *Let T be an algebraically k-quasi-class $A(s,t)$ operator. Then T is polaroid.*

Proof. If T is an algebraically k-quasi-class $A(s,t)$ operator. Then $p(T)$ is a k-quasi-class $A(s,t)$ operator for some nonconstant polynomial p. Let $\mu \in iso(\sigma(T))$ and let E_μ be the Riesz idempotent associated to μ defined by

$$E := \frac{1}{2\pi i} \int_{\partial D} (\mu I - T)^{-1} d\mu,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. Then T can be represented as follows

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\mu\}$ and $\sigma(T_2) = \sigma(T_2) \setminus \{\mu\}$. Since T_1 is an algebraically k-quasi-class $A(s,t)$ operator by Proposition 2.1 and $\sigma(T_1) = \mu$, it follows from Lemma 2.4 that $T_1 - \lambda I$ is nilpotent. Therefore $T_1 - \mu$ has finite ascent and descent. On the other hand, since $T_2 - \mu I$ is invertible, it has finite ascent and descent. Therefore $T - \mu I$ has finite ascent and descent. Therefore μ is a pole of the resolvent of T. Now if $\mu \in iso(\sigma(T))$, then $\mu \in \pi(T)$. Thus $iso(\sigma(T)) \in \pi(T)$, where $\pi(T)$ denote the set of poles of the resolvent of T. Hence T is polaroid. □

Corollary 2.2. *A k-quasi-class $A(s,t)$ operator is isoloid.*

If a Banach space operator T has SVEP (everywhere), the single-valued extension property, then T and T^* satisfy Browder's (equivalently, generalized Browder's) theorem and a-Browder's (equivalently, generalized a- Browder's) theorem. A sufficient condition for an operator T satisfying Browder's (generalized Browder's) theorem to satisfy Weyl's (resp., generalized Weyl's) theorem is that T is polaroid. Observe that if $T \in B(H)$ has SVEP, then $\sigma(T) = \overline{\sigma_a(T^*)}$. Hence, if T has SVEP and is polaroid, then T^* satisfies generalized a-Weyl's (so also, a-Weyl's) theorem [5].

Theorem 2.2. *Let $T \in B(H)$.*

(i) *If T^* is an algebraically k-quasi-class $A(s,t)$ operator, then generalized a-Weyl's theorem holds for T.*

(ii) *If T is an algebraically k-quasi-class $A(s,t)$ operator, then generalized a-Weyl's theorem holds for T^*.*

Proof. (i) It is well known that T is polaroid if and only if T^* is polaroid [5, Theorem 2.11]. Now since an algebraically k-quasi-class $A(s,t)$ operator is polaroid by Lemma 2.2 and has SVEP by Theorem 2.1, [5, Theorem 3.10] gives us the result of the theorem. For (ii) we can also apply [5, Theorem 3.10]. □

Corollary 2.3. *If T is algebraically k-quasi-class $A(s,t)$, then the following statements are equivalent.*

(i) *generalized Weyl's theorem holds for T^*.*

(ii) *generalized Weyl's theorem holds for T.*

(iii) *Weyl's theorem holds for T.*

(iv) *Weyl's theorem holds for T^*.*

Proof. If T is algebraically k-quasi-class $A(s,t)$, then T is polaroid. Now, observe for polaroid operators T satisfying generalized Weyl's theorem,

$$E(T) = \pi(T) = \pi(T^*) = E(T^*),$$

where $\pi(T)$ is the set of poles of the resolvent of T. Hence, for a polaroid operator T, T^* satisfies generalized Weyl's theorem if and only if T satisfies generalized Weyl's theorem if and only if T satisfies Weyl's theorem if and only if T^* satisfies Weyl's theorem. □

Remark 2.1.

(1) Recall [5] that if T is polaroid, then T satisfies generalized Weyl's theorem (resp. generalized a-Weyl's) theorem if and only if T satisfies Weyl's theorem (resp. a-Weyl's theorem). Hence if T is an algebraically k-quasi-class $A(s,t)$ operator, the above equivalences hold.

(2) Let $f(z)$ be an analytic function on $\sigma(T)$. If T is polaroid, then $f(T)$ is polaroid too [5].

(i) If T^* is algebraically k-quasi-paranormal, then $f(T)$ satisfies generalized a-Weyl's theorem. Indeed, since T^* is polaroid, the result holds by [5, Theorem 3.12]

(ii) If T is algebraically k-quasi-class $A(s,t)$, then $f(T^*)$ satisfies generalized a-Weyl's theorem. Indeed, since T is polaroid, the result holds by [5, Theorem 3.12].

Since $d_{A,B}$ is polaroid and has SVEP, we get the following theorems for $d_{A,B}$

Theorem 2.3. *If A, B^* are k-quasi-class $A(s,t)$ operators, then the following statements are equivalent.*

(i) *generalized Weyl's theorem holds for $d^*_{A,B}$.*

(ii) *generalized Weyl's theorem holds for $d_{A,B}$.*

(iii) *Weyl's theorem holds for $d_{A,B}$.*

Theorem 2.4. *Let $A, B \in B(H)$.*

(i) *If A, B^* are k-quasi-class $A(s,t)$ operators, then generalized a-Weyl's theorem holds for $d_{A,B}$.*

(ii) *If A, B^* are k-quasi-class $A(s,t)$ operators, then generalized a-Weyl's theorem holds for $d^*_{A,B}$.*

Theorem 2.5. *If A, B^* are k-quasi-class $A(s,t)$ operators, then*

(i) *Weyl's theorem holds for $d_{A,B}$ if and only if generalized Weyl's theorem holds for $d_{A,B}$.*

(ii) *a-Weyl's theorem holds for $d_{A,B}$ if and only if generalized a-Weyl's theorem holds for $d_{A,B}$.*

Corollary 2.4. *Let $A, B \in B(H)$.*

(i) *If A, B^* are k-quasi-class $A(s,t)$ operators, then Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and generalized a-Weyl's theorem hold for $d_{A,B}$ and are equivalent.*

(ii) *If A, B^* are k-quasi-class $A(s,t)$ operators, then Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and generalized a-Weyl's theorem hold for $d^*_{A,B}$ and are equivalent.*

Let $f \in Hol(\sigma(T))$, where $Hol(\sigma(T))$ is the space of all functions that are analytic in an open neighborhoods of $\sigma(T)$. If $d_{A,B}$ is polaroid, then $f(d_{A,B})$ is also polaroid [5]. Thus we have

Theorem 2.6. *Let $A, B^* \in B(H)$*

(i) *If A, B^* are k-quasi-class $A(s,t)$ operators, then $f(d_{A,B})$ satisfies Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and generalized a-Weyl's theorem.*

(ii) *If A, B^* are k-quasi-class $A(s,t)$ operators, then $f(d^*_{A,B})$ satisfies Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and generalized a-Weyl's theorem.*

Corollary 2.16 may be extended as follows.

Theorem 2.7. *Let $A, B \in B(H)$.*

(i) *If A, B^* are k-quasi-class $A(s,t)$ operators, then Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and generalized a-Weyl's theorem hold for $f(d_{A,B})$ and are equivalent.*

(ii) *If A, B^* are k-quasi-class $A(s,t)$ operators, then Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and generalized a-Weyl's theorem hold for $f(d^*_{A,B})$ and are equivalent.*

Remark 2.2.

(1) Since a quasi-class A operator is k-quasi-class $A(s,t)$, hence all Weyl's theorems (generalized or not) hold for algebraically quasi-class A operators and are equivalent by Theorem 2.8, Corollary 2.9 and Remark 2.10. This subsumes and extends [1, Theorem 2.4 and Theorem 3.3].

(2) Since a class $A(s,t)$ operator is k-quasi-class $A(s,t)$, hence Theorem 2.8, Corollary 2.9 and Remark 2.10 generalize the results on Weyl type theorems for class $A(s,t)$ operators proved in [11, 41].

(3) Our results on Weyl type theorems for $d_{A,B}$ generalize a recent result on generalized Weyl's theorem for $d_{A,B}$ when A and B^* are class A operators [24]. Also, since a quasi-class A operators is a k-quasi-class $A(s,t)$, hence all Weyl's theorems (generalized or not) hold for algebraically quasi-class A and algebraically w-hyponormal operators and are equivalent by Theorem 2.15, Theorem 2.17 and Theorem 2.18. This subsumes and extends [12, 16, 17]

3. Hyperinvariant Subspaces

The purpose of this section is to make a beginning on the hyperinvariant subspace problem for another class of operators closely related to the normal operators namely, the class of k-quasi-paranormal operators.

Theorem 3.1. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(H_1 \oplus H_2).$$

If T has bishop's property β and there exists a non zero $x \in H_1 \oplus H_2$ such that $\sigma_T(x) \subsetneqq \sigma(T)$. Then T has a nontrivial hyperinvariant subspace.

Proof. Assume that

$$\mathcal{M} = \{y \in H \oplus H : \sigma_T(y) \subseteq \sigma_T(x)\},$$

that is, $\mathcal{M} = H_T(\sigma_T(x))$. Since T has Bishop's property β, hence T has Dunford's property (C). It follows from [10] that $\mathcal{M}$ is a T-hyperinvariant subspace. Since $x \in \mathcal{M}$, we have $\mathcal{M} \neq \{0\}$. Now, set $\mathcal{M} = H_1 \oplus H_2$. Since T has the single extension property, we get $\sigma(T) = \bigcup\{\sigma_T(y) : y \in H_1 \oplus H_2\} \subseteq \sigma_T(x) \subsetneqq \sigma(T)$ from [23]. This is a contradiction. Hence $\mathcal{M}$ is a nontrivial T-hyperinvariant subspace. □

Since a k-quasi-class $A(s,t)$ operator has property β by applying Lemma 2.1 and Theorem 3.1, we get the following corollary.

Corollary 3.1. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(H \oplus H).$$

be a k-quasi-class $A(s,t)$ operator. If there exists a nonzero $x \in H \oplus H$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.

It is remarkable that simply knowing when solutions to $AX - XB = Y$ exist gives striking results on many topics, including similarity, commutativity, hyperinvariant subspaces, spectral operators and differential equations for details (see [9, 32]). Some of these are discussed below.

Lemma 3.1.([34, Rosemblum Theorem]) *Let $S, T \in B(H)$. If the spectrum of S and T are disjoint, then the operator equation $TX - XS = R$ has a unique solution for every operator X.*

It is known that an invariant subspace for an operator T may not be hyperinvariant. However, a sufficient condition that an invariant subspace be hyperinvariant can be derived from the Rosemblum theorem. If a subspace $\mathcal{M}$ of the Hilbert space H is invariant under T, then with respect to the orthogonal decomposition $H = \mathcal{M} \oplus \mathcal{M}^\perp$, then the operator T can be decomposed as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Thus we have the following theorem.

Theorem 3.2. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

If the spectrum of T_1 and T_2 are disjoint and $ST = TS$, then $S_4 = 0$.

Proof. It is clear that if $ST = TS$, then $T_3 S_4 = S_4 T_1$. Hence it follows from the Rosemblum theorem that $S_4 = 0$. □

Thus $\mathcal{M}$ is a nontrivial hyperinvariant subspace for T if the spectrum of T_1 and the spectrum of T_3 are disjoint. In particular we have.

Corollary 3.2. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

be k-quasi-class $A(s,t)$. If $0 \notin \sigma(T_1)$, then T has a non trivial hyperinvariant subspace.

References

[1] J. An and Y. M. Han, *Weyl's theorem for algebraically quasi-class A operators*, Integral Equations Operator Theory, **62(1)**(2008), 1–10.

[2] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations operator theory, **13(3)**(1990), 307–315.

[3] A. Aluthge and D. Wang, *ω-hyponormal operators*, Integral Equations operator theory, **36(1)**(2000), 1–10.

[4] P. Aiena, *Fredholm and local spectral theory, with applications to multipliers*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.

[5] P. Aiena, E. Aponte and E. Balzan, *Weyl type theorems for left and right Polaroid operators*, Integral Equations operator theory, **66(1)**(2010), 1–20.

[6] M. Berkani and A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Aust. Math. Soc., **76**(2004), 291–302.

[7] M. Berkani and J. J. Koliha, *Weyl type theorems for bounded linear operators*, Acta. Sci. Math. (Szeged), **69**(2003), 359–376.

[8] M. Berkani and M. Sarih, *On semi B-Fredholm operators*, Glasgow Math. J., **43**(2001), 457–465.

[9] R. Bhatia and P. Rosenthal, *How and why to solve the operator equation $AX - XB =$* Y, Bull. London Math. Soc., **29**(1997), 1–21.

[10] I. Colojoara and C. Foias, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

[11] X. Cao, *Topological uniform descent and Weyl type theorem*, Linear Algebra Appl., **420**(2007), 175–182.

[12] M. Cho, S .V. Djordjevic, B. P. Duggal and T. Yamazaki, *On an elementary operators with w-hyponormal operator entries*, Linear Algebra Appl., **433**(2010), 2070–2079.

[13] M. Cho, M. Ito and S. Oshiro, *Weyl's theorem holds for p-hyponormal operators,* Glasgow Math. J., **39**(1997), 217–220.

[14] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan. Math. J., **13**(1966), 285–288.

[15] B. P. Duggal, *P*olaroid operators and generalized Browder-Weyl theorems, Math. Proc. R. Ir. Acad., **108**(2008), 149–164.

[16] B. P. Duggal, *Weyl's theorem for a generalized derivation and an elementary operator*, Mat. Vesnik, **54**(2002), 71–81.

[17] B. P. Duggal, *An elementary operator with* log*-hyponormal, p-hyponormal entries*, Linear Algebra Appl., **428**(2008), 1109–1116.

[18] J. Eschmeier, *Invariant subspaces for subscalar operators*, Arch. Math, **52**(1989), 562–570.

[19] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math., **1**(1998), 389–403.

[20] M. Fujii, D. Jung, S. H. Lee, M. Y. Lee and R. Nakamoto, *Some classes of operators related to paranormal and log -hyponormal operators*, Math. Japon., **51**(2000), 395–402.

[21] Y. M. Han and W. Y. Lee, *Weyl's theorem holds for algebraically hyponormal operators*, Proc. Amer. Math. Soc., **128**(2000), 2291–2296.

[22] M. Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Math., **35**(1999), 149–165.

[23] K. B. Laursen and M. M. Neumann, *An introduction to local spectral theory*, London Mathematical society Monographs, Oxford, 2000.

[24] F. Lombarkia, *Generalized Weyl's theorem for an elementary operator*, Bull. Math. Anal. Appl., **3**(2011), 123–133.

[25] S. Mecheri, *Generalized Weyl's theorem for some classes of operators*, Kyungpook Math. J., **46(4)**(2006), 553–563.

[26] S. Mecheri, *Bishop's property (β) and Riesz idempotent for k-quasi-paranormal operators*, Banach J. Math. Anal., **6**(2012), 147–154.

[27] S. Mecheri, *On k-quasi-M-hyponormal operators*, Math. Inequal. Appl., **16**(2013), 895–902.

[28] S. Mecheri, *Bishop's property (β), SVEP and Dunford property* (C), Electron. J. Linear Algebra, **23**(2012), 523–529.

[29] S. Mecheri, *Isolated points of spectrum of k-quasi-$*$-class A operators*, Studia Math., **208**(2012), 87–96.

[30] S. Mecheri, *Bishop's property* (β)*, Hypercyclicity and hyperinvariant subspaces*, Oper. Matrices, **8**(2014), 555–562.

[31] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, 1973.

[32] H. Radjavi and P. Rosenthal, *Hyperinvariant subspaces for spectral and n-normal operators*, Acta Sci. Math. (Szeged), **32**(1971), 121–126.

[33] V. Rakočević, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl., **34**(1989), 915-919.

[34] M. A. Rosenblum, *O*n the operator equation $BX-XA=Q$, Duke Math. J., **23**(1956), 263–269.

[35] M. H. M. Rashid and H. Zguitti, *Weyl type theorems and class* $A(s,t)$ *operators*, Math. Inequal. Appl., **14(3)**(2011), 581–594.

[36] J. L. Shen, F. Zuo and C. S. Yang, *On Operators Satisfying* $T^*|T^2|T \geq T^*|T^{*2}|T$, Acta Math. Sin. (Engl. Ser.), **26**(2010), 2109–2116.

[37] K. Tanahashi, *Putnam's Inequality for* log-*hyponormal operators*, Integral Equations Operator Theory, **48**(2004), 103–114.

[38] H. Weyl, *Über beschränkte quadratische Formen, deren Differenz vollsteig ist*, Rend. Circ. Mat. Palermo, **27**(1909), 373–392.

[39] M. Yanagida, *Powers of class* $wA(s,t)$ *operators associated with generalized Aluthge transformation*, J. Inequal. Appl., **7**(2002), 143–168.

[40] T. Yamazaki, *On powers of class* $A(k)$ *operators including* p*-hyponormal and* log-*hyponormal operators*, Math. Inequal. Appl., **3**(2000), 97–104.

[41] H. Zguitti, *A note on generalized Weyl's theorem*, J. Math. Anal. Appl, **316**(2006), 373–381.