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Some Cycle and Star Related Nordhaus-Gaddum Type Relations on Strong Efficient Dominating Sets

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Abstract. Let $G = (V, E)$ be a simple graph with p vertices and q edges. A subset S of $V(G)$ is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$ we have $|N_s[v] \cap S| = 1$ (resp. $|N_w[v] \cap S| = 1$), where $N_s(v) = \{u \in V(G) : uv \in E(G), deg(u) \geq deg(v)\}$. The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is strong efficient if there exists a strong efficient dominating set of G. In this paper, some cycle and star related Nordhaus-Gaddum type relations on strong efficient dominating sets and the number of strong efficient dominating sets are studied.

1. Introduction

Throughout this paper only finite, undirected and simple graphs are considered. Let $G = (V, E)$ be a graph with p vertices and q edges. The degree of any vertex u in G is the number of edges incident with u and is denoted by $deg(u)$. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex of degree 0 in G is called an *isolated vertex* and a vertex of degree 1 in G is called a *pendant vertex*. A subset S of $V(G)$ is called a *dominating set* of G if every vertex in $V(G) - S$ is adjacent to a vertex in S (see [5]). The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Sampathkumar et al. introduced the concepts of strong and weak domination in graphs (see [10]). A subset S of $V(G)$ is called a *strong dominating set* of G if for every $v \in V - S$ there exists a $u \in S$ such that u and v are adjacent and $deg(u) \geq deg(v)$. A subset S of $V(G)$ is called an efficient dominating set of G if

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for every $v \in V(G), |N[v] \cap S| = 1$ (see [1] and [4]).

The concept of strong (weak) efficient domination in graphs was introduced by Meena et al. (see [7]). A subset S of $V(G)$ is called a *strong (weak) efficient dominating set* of G if for every $v \in V(G)$ we have $|N_s[v] \cap S| = 1$ (resp. $|N_w[v] \cap S| = 1$). Here, $N_s(v)$ denotes the set of all vertices $u \in V(G)$ such that uv is an edge in G and where $deg(u) \geq deg(v)$. The minimum cardinality of a strong (weak) efficient dominating set is called *strong (weak) efficient domination number* and is denoted by $\gamma_{se}(G)$ (resp. $\gamma_{we}(G)$). A graph G is strong efficient if there exists a strong efficient dominating set of G. The number of strong efficient dominating sets of a graph G is donoted by $\#\gamma_{se}(G)$. Murugan et al. studied the Nordhaus-Gaddum type relations on strong efficient dominating sets in [8]. In this paper, some cycle and star related Nordhaus-Gaddum type relations on strong efficient dominating sets and the number of strong efficient dominating sets are studied. For all graph theoretic terminology and notations, Harary [3] is followed. The following definitions and results are necessary for the present study.

Results.([7, 8])

1.1: $\gamma_{se}(G) = 1$ *if and only if* G *has a full degree vertex.*

1.2: $\gamma_{se}(K_n) = 1, n \geq 1.$

1.3: $\gamma_{se}(K_{1,n}) = 1, n \geq 1.$

1.4: $\gamma_{se}(C_{3n}) = n, n \geq 1.$

1.5: *Since* C_{3n+1} *and* C_{3n+2} *do not have efficient dominating sets, they do not have strong efficient dominating sets.*

1.6: *If there exists exactly one maximum degree vertex, then any strong efficient dominating set must contain it.*

1.7: *For any path* $P_m, \gamma_{se}(P_m) = \begin{cases} n \ if \ m = 3n, n \in N, \\ n+1 \ if \ m = 3n+1, n \in N, \\ n+2 \ if \ m = 3n+2, n \in N. \end{cases}$

1.8: *A graph* G *does not admit a strong efficient dominating set if the distance between any two maximum degree vertices is exactly two.*

1.9: *Any strong efficient dominating set is independent.*

1.10: *The sub division graph* $S(G)$ *of a graph* G *is obtained from* G *by inserting a new vertex into every edge of* G.

1.11: $\gamma_{se}[S(C_{3n})] = 2n$ *for all* $n \in N$.

1.12: $\gamma_{se}[S(k_{1,n})] = n + 1$ *for all* $n \in N$.

1.13: *If an efficient graph* G *of order* n *is an r-regular, then* $\gamma = \frac{n}{r+1}$.

1.14: *Let G be a graph with a strong efficient dominating number $\gamma_{se}(G)$. The number of distinct strong efficient dominating sets of a graph G is denoted by $\#\gamma_{se}(G)$.*

1.15: $\#\gamma_{se}(P_m) = \begin{cases} 1 \; if \;\; m = 3n \; or \; m = 3n+2, \;\; n \in N, \\ 2 \; if \;\; m = 2 \; or \; m = 3n+1, \;\; n \in N. \end{cases}$

1.16: $\#\gamma_{se}(K_n) = n, n \in N.$

1.17: $\#\gamma_{se}(C_{3n}) = 3, n \in N.$

2. Main Results

In this section, line graph, jump graph, semi-total point graph, semi-total line graph, total graph, quasi-vertex total graph and complementary prism are defined. Cycle and start related Nordhaus-Gaddum type relations of strong efficient dominating sets and the number of strong efficient dominating sets are studied.

Definition 2.1.([12]) The *line graph* $L(G)$ of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in G.

The following theorem is established first.

Theorem 2.2. *$L(C_n)$ is strong efficient if and only if $n = 3m, m \in N$. Further*

$$\gamma_{se}(C_{3m}) + \gamma_{se}[L(C_{3m})] = 2m \; and \; \#\gamma_{se}(C_{3m}) + \#\gamma_{se}[L(C_{3m})] = 6.$$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices and $e_i = v_i v_{i+1}; 1 \leq i \leq n-1, e_n = v_n v_1$ be the edges of the cycle C_n. Obviously $L(C_n)$ is a C_n with vertices $e_1, e_2, ..., e_n$.

Therefore by Results 1.4 and 1.5, $L(C_n)$ is strong efficient if and only if $n = 3m$. Therefore $\gamma_{se}(C_{3m}) + \gamma_{se}[L(C_{3m})] = 2m$ and by Result 1.17, $\#\gamma_{se}(C_{3m}) + \#\gamma_{se}[L(C_{3m})] = 6$. □

Theorem 2.3. *$L(K_{1,n})$ is strong efficient for all $n \geq 1$. Further*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[L(K_{1,n})] = 2 \; and \; \#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[L(K_{1,n})] = n+1.$$

Proof. $L(K_{1,n})$ is strong efficient for all $n \geq 1$. Further $\gamma_{se}(K_{1,n}) + \gamma_{se}[L(K_{1,n})] = 2$ and $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[L(K_{1,n})] = n+1$. □

Now the concept of jump graph of a graph is defined.

Definition 2.4.([2]) The *jump graph* $J(G)$ of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are non-adjacent in G.

Theorem 2.5. *$J(C_n)$ is strong efficient if and only if $n = 3$ or $n = 4$. Moreover*

$$\gamma_{se}[J(C_n)] = \begin{cases} 3 \; if \;\; n = 3, \\ 2 \; if \;\; n = 4, \end{cases} \quad and \quad \#\gamma_{se}[J(C_n)] = \begin{cases} 1 \; if \;\; n = 3, \\ 4 \; if \;\; n = 4. \end{cases}$$

Proof. Let $v_1, v_2, ..., v_n$ be vertices of C_n, and $e_i = v_i v_{i+1}$ for all $1 \leq i \leq n-1$

and $e_n = v_n v_1$ be the edges. Suppose $n > 4$. For all i with $1 \leq i \leq n-1$, e_{1i} is adjacent in $J(C_n)$ with all vertices other than e_{i-1} and e_{i+1}, similarliy e_n is adjacent with all the vertices other than e_{n-1} and e_1. Thus $J(C_n)$ is regular of degree $n-3$. Suppose $J(C_n)$ is strong efficient, and let S be a strong efficient dominating set of $J(C_n)$. Suppose further that $e_1 \in S$. The vertex e_1 strongly dominates all vertices other than e_2 and e_n, which are adjacent. If $e_2 \in S$, then $|N_s[e_4] \cap S| = |\{e_1, e_2\}| = 2 > 1$, which is a contradicition. Therefore $e_2 \notin S$. If $e_n \in S$, then $|N_s[e_3] \cap S| = |\{e_1, e_n\}| = 2 > 1$; also a contradicition. Therefore $e_n \notin S$. This is true for any $e_i \in S, 1 \leq i \leq n$. Hence $J(C_n)$ is not strong efficient when $n > 4$.

Conversely suppose $n \leq 4$. Two cases are considered.

Case (i): Suppose $n = 3$. $J(C_3)$ is $3K_1$ which is obviously strong efficient with the unique strong efficient dominating set $\{e_1, e_2, e_3\}$.

Case (ii): Suppose $n = 4$. $J(C_4)$ is $2K_2$ for which $\{e_1, e_2\}$,$\{e_1, e_4\}$, $\{e_3, e_2\}$ and $\{e_3, e_4\}$ are strong efficient dominating sets.

This completes the proof of the theorem. □

Theorem 2.6. *$J(K_{1,n})$ is strong efficient for all $n \geq 1$. Moreover*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[J(K_{1,n})] = n+1 \text{ and } \#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[J(K_{1,n})] = 2.$$

Proof. Since $J(K_{1,n})$ is $\bar{K}_n$, we have $\gamma_{se}[J(K_{1,n})] = n$ and$\#\gamma_{se}[J(K_{1,n})] = 1$. Therefore $\gamma_{se}(K_{1,n}) + \gamma_{se}[J(K_{1,n})] = n+1$ and $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[J(K_{1,n})] = 2$. □

Definition 2.7. The *paraline graph* $PL(G)$ is a line graph of the subdivision graph of G.

Theorem 2.8. *$PL(C_n)$ is strong efficient if and only if $n = 3m, m \in N$. Further*

$$\gamma_{se}(C_{3m}) + \gamma_{se}[PL(C_{3m})] = 3m \text{ and } \#\gamma_{se}(C_{3m}) + \#\gamma_{se}[PL(C_{3m})] = 6.$$

Proof. Obviously $PL(C_n)$ is C_{2n} and hence from Results 1.4 and 1.5,$\gamma_{se}[PL(C_{3m})] = 2m$ and by Result 1.17, $\#\gamma_{se}[PL(C_{3m})] = 3$. Therefore $\gamma_{se}(C_{3m}) + \gamma_{se}[PL(C_{3m})] = 3m$ and $\#\gamma_{se}(C_{3m}) + \#\gamma_{se}[PL(C_{3m})] = 6$. □

Theorem 2.9. *$PL[K_{1,n}]$ is strong efficient for all $n \geq 1$. Further*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[PL(K_{1,n})] = n+1 \text{ and}$$

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[PL(K_{1,n})] = \begin{cases} 4 \; if \; n = 1, \\ n+1 \; if \; n > 1. \end{cases}$$

Proof. Let v and v_i for $1 \leq i \leq n$ be the vertices if $K_{1,n}$ and let vv_i for $1 \leq i \leq n$ be the edges. Let u_i be the vertice obtained by subdividing the edge vv_i of the star for $1 \leq i \leq n$. Let $e_i = vu_i$ and $e_{n+1} = u_i v_i$ for $1 \leq i \leq n$ be the edges of $PL[K_{1,n}]$.

Case (i): Suppose $n = 1$. $PL[K_{1,1}]$ is P_2 which is obviously strong efficient and hence $\gamma_{se}[PL(K_{1,1})] = 1$ and $\#\gamma_{se}[PL(K_{1,1})] = 2$.

Case (ii): Suppose that $n > 1$, that $\Delta PL[K_{1,n}] = deg(e_i) = n$ for $1 \leq i \leq n$ and that $deg(e_j) = 1; n+1 \leq j \leq 2n$. We see that e_1 is adjacent with the e_j^s for $2 \leq j \leq n+1$. Hence e_1 strongly dominates all of these vertices. Also, the vertices e_{n+j} for $2 \leq j \leq n$ are muthually non-adjacent. Therefore $\{e_1, e_{n+2}, e_{n+3}, \cdots, e_{2n}\}$ is a strong efficient dominating set of $PL[K_{1,n}]$. Similarly $\{e_2, e_{n+1}, e_{n+3}, e_{n+4}, \cdots, e_{2n}\}, \{e_3, e_{n+1}, e_{n+2}, e_{n+4}, e_{n+5}, \cdots, e_{2n}\}, \cdots,$ $\{e_n, e_{n+1}, e_{n+2}, \cdots, e_{2n-1}\}$ is also a strong efficient dominating set of $PL(K_{1,n})$. Therefore $\gamma_{se}[PL(K_{1,n})] = n$ and $\#\gamma_{se}(PL[K_{1,n}]) = \begin{cases} 2 \ if \ n = 1, \\ n \ if \ n > 1. \end{cases}$

Therefore $\gamma_{se}(K_{1,n}) + \gamma_{se}[PL(K_{1,n})] = n + 1$ and

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[PL(K_{1,n})] = \begin{cases} 4 \ if \ n = 1, \\ n+1 \ if \ n > 1. \end{cases} \qquad \square$$

Definition 2.10.([9]) The *semi-total point graph* $T_2(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if

(i) they are adjacent vertices of G or

(ii) one is a vertex of G and the other is an edge of G incident with it.

Theorem 2.11. *$T_2(C_n)$ is strong efficient if and only if $n = 3m$ for $m \in N$. Further*

$$\gamma_{se}(C_{3m}) + \gamma_{se}[T_2(C_{3m})] = 3m \text{ and } \#\gamma_{se}(C_{3m}) + \#\gamma_{se}[T_2(C_{3m})] = 6.$$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices and $e_i = v_iv_{i+1}$ for $1 \leq i \leq n-1$, as well as $e_n = v_nv_1$ be the edges of the cycle C_n. Let $n \neq 3m$. Suppose $T_2(C_n)$ is strong efficient. Let S be a strong efficient dominating set of $T_2(C_n)$.
Case (i): Let $n = 3m+1$. $\Delta[T_2(C_{3m+1})] = deg(v_i) = 4, deg(e_i) = 2; 1 \leq i \leq 3m+1$. Suppose $v_1 \in S$. We have that v_1 strongly dominates the vertices v_2, v_{3m+1}, e_1 and e_{3m+1}. Similarly v_{3i-2} strongly dominates $v_{3i-3}, v_{3i-1}, e_{3i-3}$ and $e_{3i-2}; 2 \leq i \leq m$. If $v_{3m} \in S$, then $|N_s[V_{3m+1}] \cap S| = |\{v_1, v_{3m}\}| = 2 > 1$. This is a contradicition. Therefore $v_{3m} \notin S$. Hence there is no vertex in S that strongly efficiently dominates V_{3m}. Hence $T_2(C_{3m+1})$ is not strong efficient.
Case (ii): Let $n = 3m+2$, and observe that $\Delta[T_2(C_{3m+2})] = deg(v_i) = 4$ and $deg(e_i) = 2$ for $1 \leq i \leq 3m+2$. Suppose $v_1 \in S$. The vertex v_1 strongly dominates the vertices v_2, v_{3m+2}, e_1 and e_{3m+2}. Similarly v_{3i-2} strongly dominates $v_{3i-3}, v_{3i-1}, e_{3i-3}$ and e_{3i-2} for $2 \leq i \leq m$. Moreover, v_{3m} and v_{3m+1} are adjacent.
Subcase (ii a): Suppose $v_{3m} \in S$. Then $|N_s[V_{3m-1}] \cap S| = |\{v_{3m}, v_{3m-2}\}| = 2 > 1$. This is also a contradicition. Therefore $v_{3m} \notin S$.
Subcase (ii b): Suppose $v_{3m+1} \in S$. Then $|N_s[V_{3m+2}] \cap S| = |\{v_1, v_{3m+1}\}| = 2 > 1$. This is also a contradicition. Therefore $v_{3m+1} \notin S$. Hence there is no vertex in S to strongly efficiently dominate V_{3m+1}. Therefore $T_2(C_n)$ is not strong efficient when $n = 3m+1$ or $3m+2$.

Conversely suppose $n = 3m$. Then $\Delta[T_2(C_{3m})] = deg(v_i) = 4$ and $deg(e_i) = 2$ for $1 \leq i \leq 3m$. Also, e_i^s are non-adjacent. For $1 \leq i \leq m$ the vertex v_{3i-2} strongly dominate all the vertices other than e_{3i-1}. The vertices e_{3i-1} for $1 \leq$

$i \leq m$ are strongly dominated by themselves. Hence $\{v_{3i-2}, e_{3i-1}; 1 \leq i \leq m\}$ is a strong efficient dominating set of $T_2(C_{3m})$. By symmetry, $\{v_{3i-1}, e_{3i}; 1 \leq i \leq m\}$ and $\{v_{3i}, e_{3i-2}; 1 \leq i \leq m\}$ are also strong efficient dominating sets of $T_2(C_n)$.

Therefore $\gamma_{se}[T_2(C_{3m})] = 2m$ and $\#\gamma_{se}[T_2(C_{3m})] = 3$.

Hence $\gamma_{se}(C_{3m}) + \gamma_{se}[T_2(C_{3m})] = 3m$ and $\#\gamma_{se}(C_{3m}) + \#\gamma_{se}[T_2(C_{3m})] = 6$. □

Now the following theorem is established.

Theorem 2.12. *$T_2(K_{1,n})$ is strong efficient for all $n \geq 1$. Further*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[T_2(K_{1,n})] = 2 \textit{ and}$$

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T_2(K_{1,n})] = \begin{cases} 5 \ if \ n = 1, \\ 2 \ if \ n > 1. \end{cases}$$

Proof. Let v and v_i for $1 \leq i \leq n$ be the vertices and $e_i = vv_i$ for $1 \leq i \leq n$ be the edges of the star $K_{1,n}$.

Case (i): Suppose $n = 1$. Then $T_2(K_{1,1})$ is the cycle c_3 for which $\{e_1\}, \{v\}, \{v_1\}$ are the strong efficient dominating sets. Hence $\gamma_{se}[T_2(K_{1,1})] = 1$ and $\#\gamma_{se}[T_2(K_{1,1})] = 3$.

Case (ii): Suppose $n > 1$. In $T_2(K_{1,n}), v$ is adjacent with all the v_i^s and $e_i^s; 1 \leq i \leq n$. Thus v is the unique full degree vertex. Therefore, by Result 1.1, $\gamma_{se}[T_2(K_{1,n})] = 1$ and $\#\gamma_{se}[T_2(K_{1,n})] = 1$.

Therefore $\gamma_{se}(K_{1,n}) + \gamma_{se}[T_2(K_{1,n})] = 2$ and

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T_2(K_{1,n})] = \begin{cases} 5 \ if \ n = 1, \\ 2 \ if \ n > 1. \end{cases}$$ □

Definition 2.13. ([9]) The *semi-total line graph* $T_1(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if

(i) they are adjacent edges of G or

(ii) one is a vertex of G and the other is an edge of G incident with it.

Theorem 2.14. *$T_1(C_n)$ is strong efficient if and only if $n = 3m, m \in N$. Further*

$$\gamma_{se}(C_{3m}) + \gamma_{se}[T_1(C_{3m})] = 3m \textit{ and } \#\gamma_{se}(C_{3m}) + \#\gamma_{se}[T_1(C3m)] = 6.$$

Proof. $T_1(C_n)$ is obtained from $T_2(C_n)$ by replacing v_i and e_i. Hence the result follows from Theorem 2.11. □

Theorem 2.15. *$T_1(K_{1,n})$ is strong efficient for all $n \geq 1$. Further*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[T_1(K_{1,n})] = n + 1, \textit{ if } n \geq 1 \textit{ and}$$

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T_1(K_{1,n})] = \begin{cases} 3 \ if \ n = 1, \\ n + 1 \ if \ n > 1. \end{cases}$$

Proof. Let v and v_i for $1 \leq i \leq n$ be the vertices and $e_i = vv_i$ for $1 \leq i \leq n$ be the edges of the star $K_{1,n}$.

$T_1(K_{1,1})$ is P_3 which is strong efficient and $\gamma_{se}[T_1(K_{1,1})] = 1$. Thus $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T_1(K_{1,n})] = 3$

Suppose $n \geq 2$. In $T_1(K_{1,n})$, we have $deg(v) = n, deg(v)_i = 1$, and $deg(e_i) = n + 1 = \Delta[T_1(K_{1,n})]$ for $1 \leq i \leq n$. Each e_i strongly uniquely dominates v and all v_j's for $j \neq i$.

Hence $\{e_i, v_j | j \neq i, 1 \leq j \leq n\}$ for $1 \leq i \leq n$, form strong efficient dominating sets of $T_1(K_{1,n})$. Therefore $T_1(K_{1,n})$ is strong efficient and $\gamma_{se}(K_{1,n}) = n$, if $n \geq 1$. $\#\gamma_{se}[T_1(K_{1,n})] = n$. Hence $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T_1(K_{1,n})] = n + 1$ if $n > 1$ □

Definition 2.16. The *total graph* $T(G)$ of a graph G is the graph with vertex set $V(G) \cup E(G)$ where two vertices are adjacent if and only if

(i) they are adjacent vertices of G or

(ii) they are adjacent edges of G or

(iii) one is a vertex of G and the other is an edge of G incident with it.

Theorem 2.17. *$T(C_n)$ is strong efficient if and only if $n = 5m, m \in N$. Further $\gamma_{se}[T(C_{5m})] = 2m$ and $\#\gamma_{se}[T(C_{5m})] = 5$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$ and $e_n = v_n v_1$ be the edges of the cycle C_n. In $T(C_n)$, v_i is adjacent with v_{i+1}, v_{i-1}, e_i and e_{i-1} for $2 \leq i \leq n-1$. We have that e_i is adjacent with e_{i-1}, e_{i+1}, v_i and v_{i+1} for $2 \leq i \leq n-1$, and that v_1 is adjacent with v_2, v_n, e_1 and e_n. The vertex v_n is adjacent with v_{n-1}, v_1, e_{n-1} and e_n. The vertex e_1 is adjacent with e_2, e_n, v_1 and v_2. The vertex e_n is adjacent with v_n, v_1, e_{n-1} and e_1. Hence $deg(v_i) = deg(e_i) = 4$ for $1 \leq i \leq n$. Therefore $T(C_n)$ is regular of degree 4. Suppose $n \neq 5m$. Suppose $T(C_n)$ is strong efficient. Let S be a strong efficient dominating set.

Case (i): Let $n = 5m+1$. Suppose $v_{5i-4}, e_{5i-2} \in S$ for $1 \leq i \leq m$. Then v_{5i-4}, e_{5i-2} for $1 \leq i \leq m$ strongly dominates all the vertices other than v_{5m} and e_{5m}. Also v_{5m} and e_{5m} are adjacent. If $v_{5m} \in S$, then $|N_s[V_{5m+1}] \cap S| = |\{v_1, v_{5m}\}| = 2 > 1$. This is a contradicition. Therefore $v_{5m} \notin S$. If $e_{5m} \in S$, then $|N_s[v_{5m+1}] \cap S| = |\{v_1, e_{5m}\}| = 2 > 1$. This is also a contradiction. Therefore $e_{5m} \notin S$. This is for any $v_i \in S$. Therefore $T(C_n)$ is not strong efficient when $n = 5m + 1$ for $m \in N$.

Case (ii): Let $n = 5m + 2$. Suppose $v_{5i-4}, e_{5i-2} \in S$ for $1 \leq i \leq m$. As before v_{5i-4}, e_{5i-2} for $1 \leq i \leq m$ strongly dominates all the vertices other than v_{5m+1}, v_{5m} and e_{5m}. Also v_{5m+1}, v_{5m} and e_{5m} are mutually adjacent. If $v_{5m+1} \in S$, then $|N_s[V_{5m+2}] \cap S| = |\{v_1, v_{5m+1}\}| = 2 > 1$. This is a contradicition. Therefore $v_{5m+1} \notin S$. If $v_{5m} \in S$, then $|N_s[V_{5m-1}] \cap S| = |\{v_{5m}, e_{5m-2}\}| = 2 > 1$. This is a contradicition. Therefore $v_{5m} \notin S$. If $e_{5m} \in S$, then $|N_s[e_{5m-1}] \cap S| = |\{e_{5m}, e_{5m-2}\}| = 2 > 1$. This is a contradicition. Therefore $e_{5m} \notin S$. If $e_{5m+1} \in S$, then $|N_s[e_{5m+2}] \cap S| = |\{v_1, e_{5m+1}\}| = 2 > 1$. This is also a contradicition. Therefore $e_{5m+1} \notin S$. Therefore $e_{5m} \notin S$. This is for any $v_i \in S$. Therefore $T(C_n)$ is not strong efficient when $n = 5m + 2, m \in N$.

Case (iii): Let $n = 5m + 3$. Suppose $v_{5i-4}, e_{5i-2} \in S$ for $1 \leq i \leq m$. Then v_{5i-4}, e_{5i-2} for $1 \leq i \leq m$ strongly dominates all the vertices other than e_{5m+2}. If $e_{5m+2} \in S$, then $|N_s[V_{5m+3}] \cap S| = |\{v_1, e_{5m+2}\}| = 2 > 1$. This is also a contradicition. Therefore $e_{5m+2} \notin S$. Therefore $e_{5m} \notin S$. This is for any $v_i \in S$. Therefore $T(C_n)$ is not strong efficient when $n = 5m + 3, m \in N$.
Case (iv): Let $n = 5m + 4$. Suppose $v_{5i-4}, e_{5i-2} \in S; 1 \leq i \leq m$. As before $v_{5i-4}, e_{5i-2}; 1 \leq i \leq m$ strongly dominates all the vertices other than v_{5m+3}, e_{5m+2} and e_{5m+3}. If $v_{5m+3} \in S$, then $|N_s[v_{5m+4}] \cap S| = |\{v_1, v_{5m+3}\}| = 2 > 1$. This is a contradicition. Therefore $v_{5m+3} \notin S$. If $e_{5m+2} \in S$, then $|N_s[e_{5m+1}] \cap S| = |\{v_{5m+1}, e_{5m+2}\}| = 2 > 1$. This is also a contradicition. Therefore $e_{5m+2} \notin S$. If $e_{5m+3} \in S$, then $|N_s[v_{5m+4}] \cap S| = |\{e_{5m+3}, v_1\}| = 2 > 1$. This is also a contradicition. Therefore $e_{5m+3} \notin S$. Therefore $e_{5m} \notin S$. This is for any $v_i \in S$. Therefore $T(C_n)$ is not strong efficient when $n = 5m + 4, m \in N$.
Case (v): Let $n = 4$. Suppose $v_1 \in S$. v_1 strongly dominates v_2, v_4, e_1 and e_4. If e_2 or v_3 belongs to S then v_2 is strongly dominated by two vertices v_1 and e_2 or v_1 and v_3 respectively. If e_3 belongs to S then v_4 is strongly dominated by two vertices v_1 and e_3. Therefore e_2, e_3 and v_3 do not belong to S. There is no vertex in S to strongly dominate these three vertices, a contradiction. This is true if any v_i or e_i belong to S. Hence $T(C_n)$ is not strong efficient when $n = 4$.
Case (vi): Let $n = 3$. Suppose $v_1 \in S$. v_1 strongly dominates all the vertices other than e_2. If e_2 belongs to S then all the vertices other than v_1 are strongly dominated by two vertices v_1 and e_2. Therefore $e_2 \notin S$. Hence there is no vertex in S to strongly dominate e_2, a contradiction. This is true if any v_i or e_i belong to S. Hence $T(C_n)$ is not strong efficient when $n = 3$.

Conversely suppose $n = 5m$. In $T(C_{5m}), v_{5i-4}$ strongly dominates the vertices $v_{5i-3}, e_{5i-4}, e_{5m}$ and $v_{5i-1}; 1 \leq i \leq m$. Similarly e_{5i-2} strongly dominates the vertices $e_{5i-3}, e_{5i-1}, v_{5i-2}$ and $v_{5i-1}; 1 \leq i \leq m$. Hence $\{v_{5i-4}, e_{5i+2}; 1 \leq i \leq m\}$ are also strong efficient dominating sets of $T(C_{5m})$. Therefore $\gamma_{se}[T(C_{5m})] = 2m$ and $\#\gamma_{se}[T(C_{5m})] = 5, m \in N$. □

Theorem 2.18. *$T(K_{1,n})$ is strong efficient for all $n \geq 1$. Further*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[T(K_{1,n})] = 2 \textit{ and}$$
$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T(K_{1,n})] = \begin{cases} 5 \; if \; n = 1, \\ 2 \; if \; n > 1. \end{cases}$$

Proof. Let v and v_i for $1 \leq i \leq n$ be the vertices and $e_i = vv_i$ be the edges of the star $K_{1,n}$.
Case (i): Suppose $n = 1$. $T(K_{1,1})$ is a cycle C_3 for which $\{e_1\}, \{v\}, \{v_1\}$ are the strong efficient dominating set.
Hence $\gamma_{se}[T(K_{1,1})] = 1$ and $\#\gamma_{se}[T(K_{1,1})] = 3$.
Case (ii): Suppose $n > 1$. In $T(K_{1,n})$, v is adjacent with all v_i^s and e_i^s for $1 \leq i \leq n$. Hence v is the unique full degree vertex, $deg(v_i) = 2, deg(e_i) = 1 + i$ for $1 \leq i \leq n$. By Result 1.1, $\gamma_{se}[T(K_{1,n})] = 1$ and $\#\gamma_{se}[T(K_{1,n})] = 1$.

Therefore $\gamma_{se}(K_{1,n}) + \gamma_{se}[T(K_{1,n})] = 2$ and

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[T(K_{1,n})] = \begin{cases} 5 \; if \; n = 1, \\ 2 \; if \; n > 1. \end{cases} \qquad \Box$$

Definition 2.19.([11]) The *quasi-total graph* $P(G)$ is the graph with vertex set $V(G) \cup E(G)$ where two vertices are adjacent if and only if

(i) they are non adjacent vertices of G or

(ii) they are adjacent edges of G or

(iii) one is a vertex of G and the other is an edge of G incident with it.

Theorem 2.20. *$P(C_n)$ is strong efficient if and only if $n = 3$. Further*

$\gamma_{se}[P(C_n)] = 2$ *and* $\#\gamma_{se}[P(C_n)] = 3$.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices and $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1, e_n = v_n v_1$ be the edges of the cycle C_n. Let $n > 3$. Suppose $P(C_n)$ is strong efficient. Let S be a strong efficient dominating set of $P(C_n)$.
Case (i): Let $n = 4. \Delta[P(C_4)] = deg(e_i) = 4, deg(v_i) = 3; 1 \leq i \leq 4$. Suppose $e_1 \in S$. The vertex e_1 strongly dominates all the vertices other than e_3, v_3 and v_4. If $e_3 \in S$, then $|N_s[e_2] \cap S| = |\{e_1, e_3\}| = 2 > 1$. This is a contradicition. Therefore $e_3 \notin S$. If $v_3 \in S$, then $|N_s[v_1] \cap S| = |\{e_1, v_3\}| = 2 > 1$. This is a contradicition. Therefore $v_3 \notin S$. If $v_4 \in S$, then $|N_s[v_2] \cap S| = |\{e_1, v_4\}| = 2 > 1$. This is also a contradicition. Therefore $v_4 \notin S$. Therefore $P(C_4)$ is not strong efficient.
Case (ii): Let $n = 5$. In $P(C_5), deg(v_i) = deg(e_i) = 4$ for $1 \leq i \leq 5$. Suppose $e_1 \in S$. The vertex e_1 strongly dominates v_2, e_2, v_1 and e_5. If either v_3 or e_3 belong to S, then e_2 is dominated by two elements $v_3, e_1 or e_3, e_1$ of S respectively, a contradiction. Therefore v_3 and e_3 do not belong to S. If $v_4 \in S$, then v_2 is dominated by two elements v_4 and e_1, a contradiction. Therefore v_4 doesnot belong to S. If either v_5 or e_4 belong to S, then e_5 is dominated by two elements v_5, e_1 or e_4,e_1 of S respectively, a contradicition. Therefore v_5 and e_4 do not belong to S. Hence $P(C_5)$ is not efficient.
Case (iii): Let $n > 5$. Then $\Delta[P(C_n)] = deg(v_i) = n - 1$, and $deg(e_i) = 4$ for $1 \leq i \leq 4$. The vertex v_i strongly dominates all the v_j^s other than v_{i-1} and v_{i+1}. Also v_{i-1} and v_{i+1} are adjacent. If $v_{i-1} \in S$, then $|N_s[v_{i-3}] \cap S| = |\{v_i, v_{i-1}\}| = 2 > 1$. This is a contradicition. Therefore $v_{i-1} \notin S$. If $v_{i+1} \in S$, then $|N_s[v_{i+3}] \cap S| = |\{v_i, v_{i+1}\}| = 2 > 1$. This is also a contradicition. Therefore $v_{i+1} \notin S$. Therefore $P(C_n)$ is not strong efficient when $n > 3$.

Conversely suppose $n = 3$. Obviously $\{e_1, v_3\}, \{e_2, v_1\}$ and $\{e_1, v_3\}$ are strong efficient dominating sets $P(C_3)$. Therefore $\gamma_{se}[P(C_3)] = 2$ and $\#\gamma_{se}[P(C_3)] = 3$. $\Box$

Theorem 2.21. *$P(K_{1,n})$ is strong efficient if and only if $n = 1$. Further*

$\gamma_{se}[P(K_{1,1})] = \#\gamma_{se}[P(K_{1,n})] = 1$.

*Proof.*Let $v, v_1, v_2, ..., v_n$ be the vertices and $e_i = vv_i$ for $1 \leq i \leq n$ be the edges of the star $K_{1,n}$. Suppose $n > 1$. Let $P(K_{1,n})$ be strong efficient and let S be a strong efficient dominating set of $P(K_{1,n})$.

In $P(K_{1,n})$ the vertex v_i is adjacent with all other v_j^s and e_i for $1 \leq i \leq n$. Therefore $deg(v_i) = n$. Also e_i is adjacent with all other e_j^s, v_i and v for $1 \leq i \leq n$. Therefore $deg(e_i) = n + 1$. Similarly v is adjacent with all other e_j^s for $1 \leq i \leq n$. Therefore $deg(v) = n$. Suppose $e_i \in S$. Then e_i strongly dominates all other e_j^s, v_i and $v; 1 \leq i \leq n$. Suppose $v_j \in S, j \neq i$, then $|N_s[v_i] \cap S| = |\{e_i, v_j\}| = 2 > 1$. This is a contradicition. Therefore $v_j \notin S$. Therefore $P(K_{1,n})$ is not strong efficient if $n > 1$.

Conversely suppose $n = 1$. Then $P(K_{1,1})$ is P_3 which is obviously strong efficient with the unique strong efficient dominating set $\{e_1\}$. Therefore $\gamma_{se}[P(K_{1,1})] = \#\gamma_{se}[P(K_{1,1})] = 1$. □

Definition 2.22.([11]) The *quasi vertex-total graph* $Q(G)$ is the graph with vertex set $V(G) \cup E(G)$ where two vertices are adjacent if and only if

(i) they are adjacent vertices of G or

(ii) they are nonadjacent vertices of G or

(iii) they are adjacent edges of G or

(iv) one is a vertex of G and the other is an edge of G incident with it.

Theorem 2.23. *$Q(C_n)$ is strong efficient if and only if $n = 3m + 2, m \in N$. Further*

$\gamma_{se}[Q(C_{3m+2})] = m + 1$ *and* $\#\gamma_{se}[Q(C_{3m+2})] = 3m + 2$.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices and $e_i = v_iv_{i+1}$ for $1 \leq i \leq n - 1, e_n = v_nv_1$ be the edges of the cycle C_n. Let $n \neq 3m + 2, m \in N$. Suppose $Q(C_n)$ is strong efficient. Let S be a strong efficient dominating set of $Q(C_n)$.
Case (i): Suppose $n = 3m, m \in N$. We have $\Delta[Q(C_{3m})] = deg(v_i) = 3m + 1$ and $deg(e_i) = 4; 1 \leq i \leq 3m$. Suppose $v_1 \in S$. Then v_1 strongly dominates all other v_j^s, e_{3m} and e_1. The remaining $(3m-2)e_j^s$ which are adjacent with v_j and v_{j+1} form a path of length $3m - 2$. Obviously $e_3, e_6, e_9, \cdots, e_{3m-3}$ strongly dominates all the e_j^s except e_{3m-1} If $e_{3m-1} \in S$then $|N_s[e_{3m}] \cap S| = |\{e_{3m-1}, v_1\}| = 2 > 1$. This is a contradicition. Therefore $e_{3m-1} \notin S$. Therefore $Q(C_n)$ is not strong efficient when $n = 3m$ for $m \in N$.
Case (ii): Suppose $n = 3m + 1$ for $m \in N$. $\Delta[Q(C_{3m+1})] = deg(v_i) = 3m + 2$ and $deg(e_i) = 4$ for $1 \leq i \leq 3m + 1$. Suppose $v_1 \in S$. Then v_1 strongly dominates all other v_j^s, e_{3m+1} and e_1. The remaining $(3m - 1)e_j^s$ which are adjacent with v_j and v_{j+1} form a path of length $3m - 1$. Obviously $e_3, e_6, e_9, \cdots, e_{3m-3}$ strongly dominates all the e_j^s except e_{3m} and e_{3m-1}. If $e_{3m} \in S$, then $|N_s[e_{3m+1}] \cap S| = |\{e_{3m}, v_1\}| = 2 > 1$. This is a contradicition. Therefore $e_{3m} \notin S$. If $e_{3m-1} \in S$, then $|N_s[e_{3m-2}] \cap S| = |\{e_{3m-1}, e_{3m-3}\}| = 2 > 1$. This is also a contradicition. Therefore $e_{3m-1} \notin S$. Therefore $Q(C_n)$ is not strong efficient if $n = 3m$ or $3m + 1, m \in N$.

Conversely suppose $n = 3m + 2$ for $m \in N$. Then $\Delta[Q(C_{3m+2})] = deg(v_i) = 3m + 3$ and $deg(e_i) = 4$ for $1 \leq i \leq 3m + 2$. Suppose $v_1 \in$

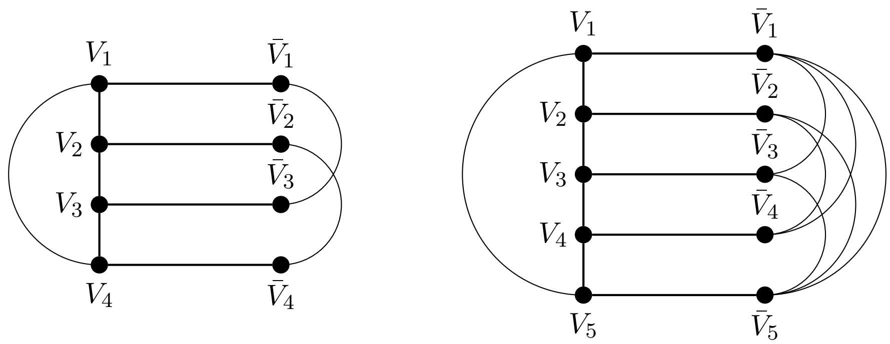

Figure 1: The graphs $C_4\bar{C}_4$ and $C_5\bar{C}_5$

S. We see that v_1 strongly dominates all other v_j^s, e_{3m+2} and e_1. The remaining $3m$ vertices e_j^s which are adjacent with v_j and v_{j+1} form a path of length $3m$. Obviously $e_3, e_6, e_9, \cdots, e_{3m}$ strongly dominates all the remaining e_j^s. Hence $\{v_1, e_3, e_6, e_9, \cdots, e_{3m}\}$ is a strong efficient dominating set. Similarly $\{v_2, e_4, e_7, e_{10}, \cdots, e_{3m+1}\}$,$\{v_3, e_5, e_8, e_{10}, \cdots, e_{3m+2}\}\{v_4, e_6, e_9, e_{12}, \cdots, e_{3m}, e_1\}$, $\{v_5, e_7, e_{10}, e_{13}, \cdots, e_{3m+1}, e_2\} \cdots$ and $\{v_{3m+2}, e_2, e_5, e_8, \cdots, e_{3m-1}\}$ are also strong efficient dominating sets.

Therefore $\gamma_{se}[Q(C_{3m+2})] = m + 1$ and $\#\gamma_{se}[Q(C_{3m+2})] = 3m + 2, m \in N$. □

Theorem 2.24. *$Q(K_{1,n})$ is strong efficient for all $n \geq 1$. Further*

$$\gamma_{se}(K_{1,n}) + \gamma_{se}[Q(K_{1,n})] = 2 \ \textit{and}$$
$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[Q(K_{1,n})] = \begin{cases} 5 \ if \ n = 1, \\ 2 \ if \ n > 1. \end{cases}$$

Proof. Let v and v_i for $1 \leq i \leq n$ be the vertices and $e_i = vv_i$ be the edges of the star $K_{1,n}$. If $n = 1$, then $Q(K_{1,1})$ is a cycle C_3. So $\gamma_{se}[Q(K_{1,1})] = 1$ and $\#\gamma_{se}[Q(K_{1,1})] = 3$. So let $n > 1$. In $Q(K_{1,n})$, v is adjacent with all v_i^s and e_i^s for $1 \leq i \leq n$. Then v is the unique full degree vertex. Therefore by Result 1.1, $\gamma_{se}[Q(K_{1,n})] = 1$ and $\#\gamma_{se}[Q(K_{1,n})] = 1$. Therefore $\gamma_{se}(K_{1,n}) + \gamma_{se}[Q(K_{1,n})] = 2$ and

$$\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[Q(K_{1,n})] = \begin{cases} 5 \ if \ n = 1, \\ 2 \ if \ n > 1. \end{cases}$$ □

Definition 2.25. ([6]) For a graph G, the complementary prism, donoted by $G\bar{G}$, is formed from a copy of G and a copy of $\bar{G}$ by adding a perfect matching between corresponding vertices.

Theorem 2.26. *$C_n\bar{C}_n$ is strong efficient if and only if $n = 3$. Moreover $\gamma_{se}[C_n\bar{C}_n] = 3$ and $\#\gamma_{se}[C_n\bar{C}_n] = 3$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n and $\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n$ be the vertices in the copy of $\bar{C}_n$. Let $n > 3$.

Case (i): Suppose $n = 4$ or $n = 5$. The graph $C_4\bar{C}_4$ and $C_5\bar{C}_5$ are shown in Fig 1. The subgraph induced by the maximum degree vertices is C_4, which is not strong efficient. Hence $C_4\bar{C}_4$ is not strong efficient. $C_5\bar{C}_5$ is the Peterson graph which is not efficient. Hence $C_5\bar{C}_5$ is not strong efficient.

Case (ii): Suppose $C_n\bar{C}_n$ is strong efficient. Let S be a strong efficient dominating set of $C_n\bar{C}_n$ Suppose first $n > 5$. Then $\Delta[C_n\bar{C}_n] = deg(\bar{v}_i) = n-2$ for $1 \leq i \leq n$ and $deg(v_i) = 3$ for $1 \leq i \leq n$. Moreover $\bar{v}_1$ and $\bar{v}_n$ are non-adjacent. Suppose $\bar{v}_1 \in S$ and observe that $\bar{v}_1$ strongly dominates v_1 and all $\bar{v}_i$ other than $\bar{v}_2$ and $\bar{v}_n$. Therefore $\bar{v}_2 \in S$. So $|N_s[\bar{v}_4] \cap S| = |\{\bar{v}_1, \bar{v}_2\}| = 2 > 1$. This is a contradicition. Therefore $\bar{v}_2 \notin S$. If $\bar{v}_n \in S$, then the vertices $\bar{v}_i$ for $3 \leq i \leq n-2$ are strongly dominated by two vertices $\bar{v}_i$ and $\bar{v}_n$, a contradicition. This is true if any $\bar{v}_i \in S$. Therefore $C_n\bar{C}_n$ is not strong efficient when $n > 3$.
Conversely let $n = 3$. $C_3\bar{C}_3$ is strong efficient with three strong efficient dominating sets $\{v_1, \bar{v}_2, \bar{v}_3\}$,$\{v_2, \bar{v}_1, \bar{v}_3\}$ and $\{v_3, \bar{v}_2, \bar{v}_1\}$. Therefore $\gamma_{se}[C_n\bar{C}_n] = 3$ and $\#\gamma_{se}[Cn\bar{C}_n] = 3$.

Theorem 2.27. *$K_{1,n}, \bar{K}_{1,n}$ is strong efficient if and only if $n = 1$. Moreover*

$\gamma_{se}[K_{1,n}\bar{K}_{1,n}] = 2$ *and* $\#\gamma_{se}[K_{1,n}\bar{K}_{1,n}] = 2$.

Proof. Let $v, v_i; 1 \leq i \leq n$ be the vertices of $K_{1,n}$ and $\bar{v}, \bar{v}_i; 1 \leq i \leq n$ be the vertices of the copy of $\bar{K}_{1,n}$ of $\bar{K}_{1,n}$. In $K_{1,n}\bar{K}_{1,n}$, v is adjacet with all v_i^s and $\bar{v}; 1 \leq i \leq n$. $\Delta[K_{1,n}\bar{K}_{1,n}] = deg(v) = n+1, deg(v_i) = 2, deg(\bar{v}_i) = n$ and $deg(\bar{v}) = 1; 1 \leq i \leq n$. Let $n > 1$. Suppose $K_{1,n}, \bar{K}_{1,n}$ is strong efficient. Any strong efficient dominating set must contain v. Let S be a strong efficient dominating set. v strongly dominates $\bar{v}$ and $v_i; 1 \leq i \leq n$. If $\bar{v}_i \in S$, then $|N_s[\bar{v}_i] \cap S| = |\{v, \bar{v}_i\}| = 2 > 1$. This is a contradicition. Therefore $\bar{v}_i \notin S$. Therefore $K_{1,n}\bar{K}_{1,n}$ is not strong efficient when $n > 1$.

Conversely let $n = 1$. $K_{1,1}\bar{K}_{1,1}$ is the path P_4 which is obviously strong efficient with strong efficient dominating set $\{v, \bar{v}_1\}$ and $\{\bar{v}, v_1\}$. Therefore $\gamma_{se}[K_{1,n}\bar{K}_{1,n}] = 2$ and $\#\gamma_{se}[K_{1,n}\bar{K}_{1,n}] = 2$. □

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References

[1] D. W. Bange, A. E. Barkauskas and P. J. Slater, *Efficient dominating sets in graphs*, Application of Discrete Mathematics, SIAM, Philadephia, (1988), 189–199.

[2] G. Chartrand, H. Hevia, E. B. Jarette and M. Schultz, *Subgraph distances in graphs defined by edge transfers*, Discrete Math., **170**(1997), 63–79.

[3] F. Harary, *Graph theory*, Addison-Wesley, 1969.

[4] F. Harary, T. W. Haynes and P. J. Slater, *Efficient and excess domination in graphs*, J. Combin. Math. Combin. Comput., **26**(1998), 83–95.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc, New York, 1998.

[6] T. W. Haynes, M. A. Henning, P. J. Slater and L. C. Van Der Merwe, *The complementary product of two graphs*, Bull. Inst. Combin. Appl., **51**(2007), 21–30.

[7] N. Meena, *Studies in graph theory-efficient domination and related topics*, Ph. D. Thesis, Manonmaniam Sundaranar University, 2013.

[8] K. Murugan and N. Meena, *Some Nordhaus-Gaddum type relation on strong efficient dominating sets*,J. New Results Sci., **5(11)**(2016), 4–16.

[9] E. Sampathkumar and S. B. Chikkodimath, *Semi-total graphs of a graph I*, J. Karnatak Univ. Sci., **18**(1973), 274–280.

[10] E. Sampathkumar and L. P. Latha, *Strong weak domination and domination balance in a graph*, Discrete Math., **161**(1996), 235–242.

[11] D. V. S. S. Sastry and B.S. P. Raju, *Graph equations for line graphs, total graphs, middle graphs and quasitotal graphs*, Discrete Math., **48**(1984), 113–119.

[12] H. Whitney, *Congruent graphs and the connectivity graphs*, Amer. J. Math., **54**(1932), 150–168.