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A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation

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Abstract

In this paper, a new extension of Lindley distribution has been introduced. Certain characterizations based on truncated moments, hazard and reverse hazard function, conditional expectation of the proposed distribution are presented. Besides, these characterizations, other statistical/mathematical properties of the proposed model are also discussed. The estimation of the parameters is performed through different classical methods of estimation. Bayes estimation is computed under gamma informative prior under the squared error loss function. The performances of all estimation methods are studied via Monte Carlo simulations in mean square error sense. The potential of the proposed model is analyzed through two data sets. A modified goodness-of-fit test using the Nikulin-Rao-Robson statistic test is investigated via two examples and is observed that the new extension might be used as an alternative lifetime model.

Keywords: different characterizations, validation test, different method of estimation and applications

1. Introduction, motivation and physical interpretation

Statistical literature contains a large number of probability distributions for modeling real lifetime data, the most popular probability distributions are gamma (Ga), lognormal (Log-N), Weibull (W), and exponentiated exponential (Exp-E) distributions. However, these probability models suffer from some drawbacks. First, none of them exhibit bathtub shapes for their hazard rate functions (hrfs), the four models exhibit only monotonically decreasing, monotonically increasing or constant hrfs and this is a major weakness since most real life systems exhibit bathtub shapes for their hrfs. Second, at least three of these distributions exhibit constant hazard rates and this is an unrealistic feature since few real life systems have constant hrf. This work introduces a new three parameter lifetime distribution as an alternative to the Ga, Log-N, W, and the Exp-E probability models that does not have the above mentioned drawbacks. The new model, called the Topp Leone Generated Lindley (TLGLi), is constructed based on the Topp Leone Generated (TLG) family introduced by Rezaei *et*

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al. (2017). The TLGLi can have unimodal or monotonically decreasing or monotonically increasing or constant or bathtub hrf (Figure 2).

The cumulative distribution function (cdf) and the probability density function (pdf) of a random variable (r.v.) X with the Li distribution (Lindley, 1958) are given by

$$G(x;\lambda)\Big|\binom{(x\geq 0)}{(\lambda\in\mathbb{R}^{(+)})} = 1 - \left[\frac{(1+x)\lambda+1}{1+\lambda}\right]\mathrm{e}^{-\lambda x},$$

and

$$g(x;\lambda)\Big|\binom{(x\geq 0)}{(\lambda\in\mathbb{R}^{(+)})} = (1+x)\mathrm{e}^{-\lambda x}\left[\frac{\lambda^2}{1+\lambda}\right],$$

respectively. The scale parameter λ is a positive real number ($\mathbb{R}^{(+)}$) and can result in a unimodal or monotone decreasing density. The Li model has a thin tail since the distribution decreases exponentially for large values of x. The Li distribution is one way to describe the lifetime of a process or device having increasing failure rate. It can be used in a wide variety of fields such as biology, engineering and medicine. The cdf of the TLGLi model can be expressed as

$$F_{\beta,\theta,\lambda}(x)\Big|_{(\beta,\,\theta,\,\lambda\in\mathbb{R}^{(+)})}^{(x\geq 0)} = \left\{\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta}, \tag{1.1}$$

and the corresponding pdf is

$$f_{\beta,\theta,\lambda}(x)\Big|_{(\beta,\,\theta,\,\lambda\in\mathbb{R}^{(+)})}^{(x\geq 0)} = 2\beta\theta\frac{\lambda^2}{1+\lambda}(1+x)\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta\beta-1}\frac{1-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta}}{\left[2-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta}\right]^{1-\beta}}. \tag{1.2}$$

Equations (1.1) and (1.2) are established based on the TLG distribution.

Suppose $X_1, X_2, \ldots, X_\beta$ are independent r.v.'s distributed according to

$$F_{\mathrm{GLi}}(x;\theta,\lambda)\Big|_{(\theta,\,\lambda\in\mathbb{R}^{(+)})}^{(x\geq 0)} = \left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta}\right]$$

and represent the failure times of the components of a series system, assumed to be independent. Then the probability that the system will fail before time x is given by

$$\begin{aligned}
\Pr(\max(X_1, X_2, \ldots, X_\beta) \leq x) &= \Pr(X_1 \leq x)\cdots\Pr\left(X_\beta \leq x\right) \\
&= F_{\mathrm{GLi}}(x;\theta,\lambda)\times\cdots\times F_{\mathrm{GLi}}(x;\theta,\lambda) \\
&= \left[F_{\mathrm{GLi}}(x;\theta,\lambda)\right]^{\beta}\ \Big|_{(\theta\in\mathbb{R}^{(+)},\,\lambda\in\mathbb{R}^{(+)})}^{(x\geq 0)} \\
&= \left\{\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta}.
\end{aligned}$$

So, Equation (1.1) is the distribution of the failure of a series system with independent components if $\beta \in \mathbb{R}^{(+)}$. We shall refer to the new distribution in (1.1) as the TLGLi model. For $\theta = 1$, TLGLi reduces to the TLLi model. The cdf (1.1) can be expressed as

$$F_{\beta,\theta,\lambda}(x)\Big|_{(\beta,\,\theta,\,\lambda\in\mathbb{R}^{(+)})}^{(x\geq 0)} = \sum_{\zeta=0}^{\infty} \boldsymbol{\upsilon}_{\boldsymbol{\zeta}} \mathbf{H}_{(\theta\boldsymbol{\zeta}+\theta\beta),\lambda}(x), \tag{1.3}$$

where

$$\boldsymbol{\upsilon}_{\boldsymbol{\zeta}} = (-1)^{\zeta} 2^{\beta-\zeta}\binom{\beta}{\zeta},$$

and

$$H_{(\theta\boldsymbol{\zeta}+\theta\beta),\lambda}(x) = [G(x;\lambda)]^{(\theta\boldsymbol{\zeta}+\theta\beta)} = \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{(\theta\boldsymbol{\zeta}+\theta\beta)},$$

is the cdf of the Exp-Li distribution with power parameter $(\theta\boldsymbol{\zeta}+\theta\beta)$. The corresponding density function is obtained by differentiating (1.3)

$$f_{\beta,\theta,\lambda}(x)\Bigg|\begin{pmatrix}(x\geq 0)\\(\beta,\,\theta,\,\lambda\in\mathbb{R}^{(+)})\end{pmatrix} = \sum_{\zeta=0}^{\infty} \boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\, \mathbf{h}_{(\theta\boldsymbol{\zeta}+\theta\beta),\lambda}(x), \tag{1.4}$$

where

$$\mathbf{h}_{(\theta\boldsymbol{\zeta}+\theta\beta),\lambda}(x) = (\theta\boldsymbol{\zeta}+\theta\beta)\,\underbrace{\frac{\lambda^2}{1+\lambda}(1+x)\mathrm{e}^{-\lambda x}}_{g(x;\lambda)} \times \underbrace{\left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\,\lambda+1}{1+\lambda}\right)^{(\theta\boldsymbol{\zeta}+\theta\beta)-1}}_{G(x;\lambda)^{[(\theta\zeta+\theta\beta)]-1}},$$

is the Exp-Li density with power parameter $(\theta\boldsymbol{\zeta}+\theta\beta)$.Thus, several of its structural properties can be obtained from Equation (1.4) and those of the Exp-Li distribution.

We provide some plots of the pdf and hrf of the TLGLi model to show its flexibility. Figure 1 displays some plots of the TLGLi density for selected values of β, θ, and λ. These plots reveal that the new density can be right-skewed with different flexible shapes. The hrf plots of the TLGLi distribution given in Figure 2 can be unimodal, decreasing, bathtub, increasing and constant shapes.

In the literature, certain generalizations on the Li distribution are proposed and studied (Ghitany *et al.*, 2008a, b; Deniz and Ojeda, 2011; Ghitany *et al.*, 2011; Nadarajah *et al.*, 2011; Alizadeh *et al.*, 2016, 2017). This article shows how different estimators of the new distribution perform for different sample sizes and different parameter values and to promote a guideline for choosing the best estimation method for the new model, which we think would be of interest to applied statisticians. The unknown parameters of the new distribution are estimated using maximum likelihood, least squares, weighted least squares, Cramer-Von-Mises and Bayesian methods. The obtained estimators are compared using Monte Carlo simulations; in addition, it is observed that Bayesian estimators are more efficient compared to other estimators. The TLGLi distribution shows its suitability in modeling strength and relief times data sets.

The rest of the paper is outlined as follows: Certain characterizations of the TLGLi model are proposed in Section 2. In Section 3, we discuss some properties of this distribution. In Section 4, we describe four methods of estimation. In Section 5, the usefulness of the new distribution is illustrated by means of two real data sets. A modified goodness-of-fit test using a Nikulin-Rao-Robson (NRR) statistic test is presented in Section 6. In Section 7, a simulation study is carried out to compare the performance of the four methods of estimation. Section 8 offers some concluding remarks.

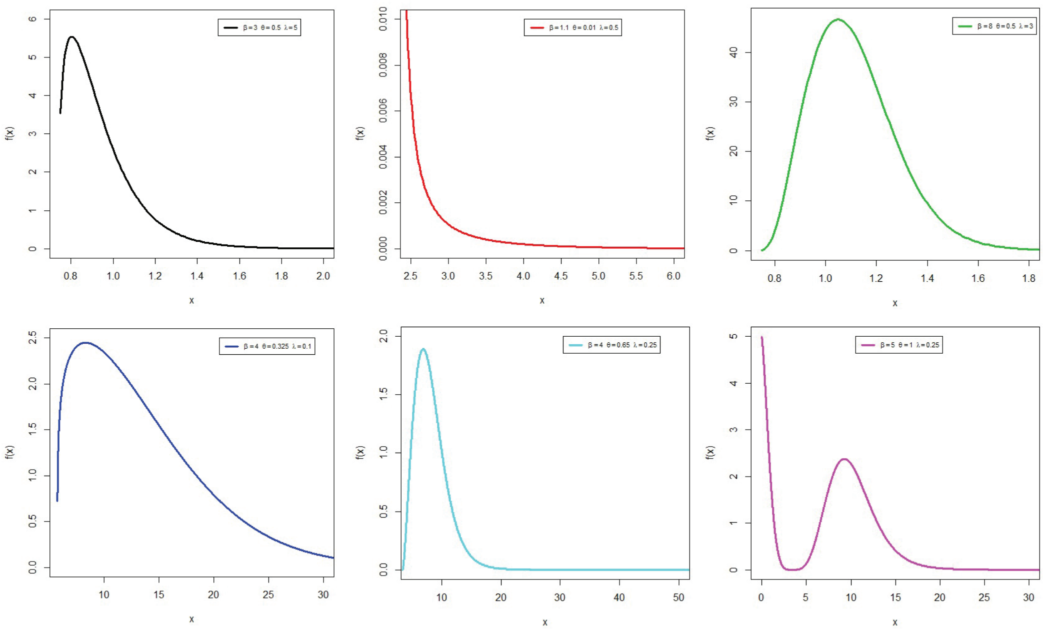

Figure 1: *Plots of the probability density function of the Topp Leone Generated Lindley for selected parameter values.*

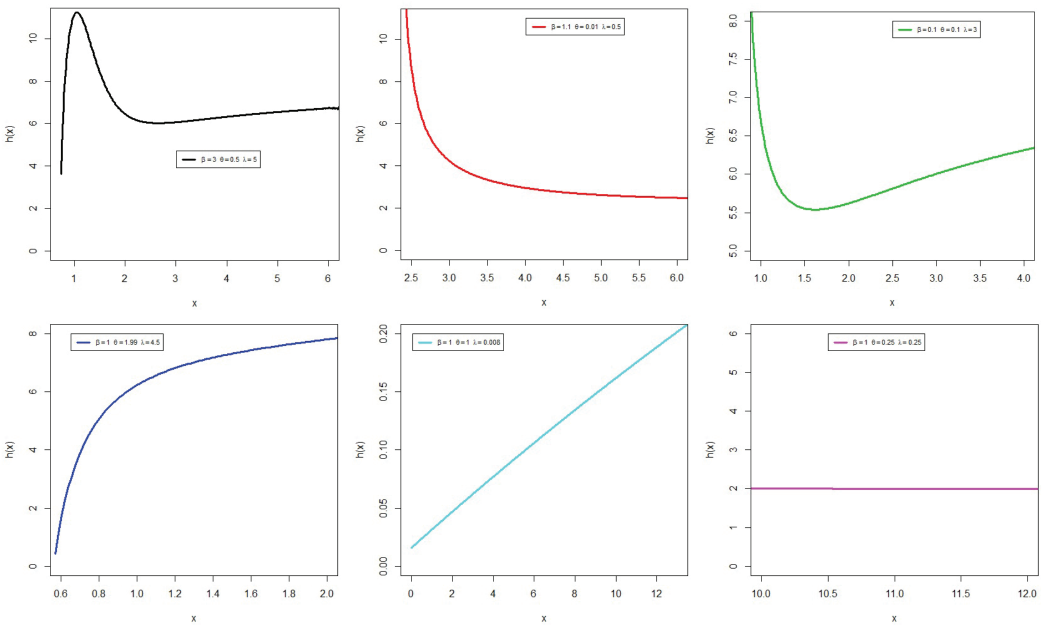

Figure 2: *Plots of the hazard rate function of the Topp Leone Generated Lindley for selected parameter values.*

2. Characterizations

In this Section, we present a number of characterizations of the TLGLi distribution in the following directions: (i) in terms of the ratio of two truncated moments; (ii) in terms of the hazard function; (iii) based on the reverse hazard function; and (iv) in terms of the conditional expectation of certain function of the random variable. These characterizations are presented in four subsections.

2.1. Characterizations in terms of two truncated moments

Certain characterizations of TLGLi distribution based on a simple relationship between two truncated moments are presented. The first characterization employs a theorem due to Glänzel (1987), see Hamedani *et al.* (2018a; 2018b, Theorem 1) and in the Appendix. However, the results holds also when the interval H is not closed since the condition of the Theorem is on the interior of H.

Proposition 1. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let*

$$q_1(x) = \frac{\left[2 - \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta}\right]^{1-\beta}}{1 - \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta}}$$

and

$$q_2(x) = q_1(x)\left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\Bigg|_{(x\geq 0)}.$$

The random variable X has pdf (1.2) only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2}\left\{1 + \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\}\Bigg|_{(x\geq 0)}.$$

Proof: Suppose the random variable X has pdf (1.2), then

$$(1 - F(x))\,\mathbf{E}\left[q_1(x)|X \geq x\right] = 2\left\{1 - \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\}\Bigg|_{(x\geq 0)}$$

and

$$(1 - F(x))\,\mathbf{E}\left[q_2(x)|X \geq x\right] = \left\{1 - \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{2\theta\beta}\right\}\Bigg|_{(x\geq 0)}.$$

Further,

$$\xi(x)q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left\{1 - \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\} > 0\,\Bigg|_{(x\geq 0)}.$$

Conversely, if ξ has the above form, then

$$s'(x) = \frac{q_1(x)\xi'(x)}{\xi(x)q_1(x) - q_2(x)} = \frac{\beta\theta\lambda^2(1+x)\,e^{-\lambda x}\left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta-1}}{(1+\lambda)\left\{1 - \left(1 - \mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\}}$$

and

$$s(x) = -\log\left\{1-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\}\Bigg|_{(x>0)}.$$

Now, according to Theorem 1, X has pdf (1.2). □

Corollary 1. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $q_1(x)$ be as given in Proposition 1. The random variable X has pdf (1.2) only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the following differential equation*

$$\frac{q_1(x)\xi'(x)}{\xi(x)q_1(x)-q_2(x)} = \frac{\beta\theta\lambda^2(1+x)e^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta-1}}{(1+\lambda)\left\{1-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\}}\Bigg|_{(x>0)}.$$

Corollary 2. *The general solution of the differential equation in Corollary 1 is*

$$\xi(x) = \left\{1-\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta}\right\}^{-1} \times\left[-\int\frac{\beta\theta\lambda^2}{(1+\lambda)}(1+x)\,\mathrm{e}^{-\lambda x}\left(1-\mathrm{e}^{-\lambda x}\frac{(1+x)\lambda+1}{1+\lambda}\right)^{\theta\beta-1}(q_1(x))^{-1}\,q_2(x)dx+D\right],$$

where D is a constant. One set of functions satisfying the above differential equation is given in Proposition 1 with $D = 1/2$. Of course, there are other triplets of function (q_1, q_2, ξ) which satisfy conditions of Theorem 1.

2.2. Characterization based on hazard function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following differential equation

$$\frac{f'(x)}{f(x)} = -h_F(x) + \frac{h_F'(x)}{h_F(x)}.$$

For many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present a non-trivial characterization of TLGLi distribution, for $\lambda = \theta = \beta = 1$, based on the hazard function.

Proposition 2. *Let $X : \Omega \to (0,\infty)$ be a continuous random variable. The random variable X has pdf (1.2), for $\lambda = \theta = \beta = 1$, only if its hazard function $h_F(x)$ satisfies the following first order differential equation*

$$\frac{h_F(x)}{(1+x)(2+x)} + h_F'(x) = 0\Big|_{(x>0)}.$$

Proof: Is straightforward and hence omitted. □

2.3. Characterization based on reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in \text{support of } F.$$

In this subsection we present a characterization of TLGLi distribution, for $\lambda = \theta = 1$, based on the reverse hazard function.

Proposition 3. *Let $X : \Omega \to (0, \infty)$ be a continuous random variable. The random variable X has pdf (1.2) for $\lambda = \theta = 1$ only if its reverse hazard function $r_F(x)$ satisfies the following first order differential equation*

$$2r_F(x) + r_F'(x) = 2\beta e^{-2x} \frac{d}{dx} \left\{ \frac{(2+x)(1+x)}{-(2+x)^2 e^{-2x} + 4} \right\}\Bigg|_{(x>0)}.$$

Proof: Is straightforward and hence omitted. □

2.4. Characterization based on the conditional expectation of certain function of the random variable

Here, we employ a function $\boldsymbol{\psi}$ of X and present characterize the distribution of X in terms of the truncated moment of $\boldsymbol{\psi}(X)$. The following proposition appeared in Hamedani's previous work (Hamedani, 2013), so we just state it here and use it to characterize TLGLi distribution, for $\lambda = \theta = \beta = 1$.

Proposition 4. *Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F. Let $\boldsymbol{\psi}(x)$ be a differentiable function on (e, f) with $\lim_{x\to e^+} \boldsymbol{\psi}(x) = 1$. Then for $\delta \neq 1$,*

$$\mathbf{E}\left[\boldsymbol{\psi}(X) | X \geq x\right] = \delta \boldsymbol{\psi}(x)\Big|_{[x\in(e,f)]}$$

only if

$$\boldsymbol{\psi}(x) = (1 - F(x))^{\frac{1}{\delta}-1}\Big|_{[x\in(e,f)]}.$$

Remark 1. For $(e, f) = (0, \infty)$, $\lambda = \theta = \beta = 1, \boldsymbol{\psi}(x) = (1 + (1/2)x)e^{-x}$ and $\delta = 2/3$, Proposition 4 provides a characterization of TLGLi distribution for $\lambda = \theta = \beta = 1$.

3. Mathematical properties

The r^{th} ordinary moment of X is given by

$$\begin{aligned} \mu_r' = \mathbf{E}(X^r) &= \sum_{\boldsymbol{\zeta}=0}^{\infty} \boldsymbol{\upsilon}_{\boldsymbol{\zeta}} \int_{-\infty}^{\infty} x^r \, \mathbf{h}_{(\theta\boldsymbol{\zeta}+\theta\beta),\lambda}(x) dx \\ &= (\theta\boldsymbol{\zeta} + \theta\beta) \frac{\lambda^2}{1+\lambda} \sum_{\boldsymbol{\zeta}=0}^{\infty} \boldsymbol{\upsilon}_{\boldsymbol{\zeta}} \mathbb{k}\left((\theta\boldsymbol{\zeta} + \theta\beta), \lambda, r, \lambda\right), \end{aligned} \tag{3.1}$$

where

$$\mathbb{k}\,(a,b,r,\delta) = \int_0^\infty x^r\,(1+x)\left(1-\frac{1+b+bx}{1+b}\mathrm{e}^{-bx}\right)^{a-1}\mathrm{e}^{-\delta x}dx$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{j+1}\zeta_{i,j,m}^{(a,r)}\,\Gamma\,(1+r+m)\,,$$

and

$$\zeta_{i,j,m}^{(a,r)} = \frac{(-1)^i b^j}{(1+b)^i\,(bi+\delta)^{1+r+m}}\binom{a-1}{i}\binom{i}{j}\binom{j+1}{m}.$$

The first four moments of X are

$$\mu_1' = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,1,\lambda)\,,$$

$$\mu_2' = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,2,\lambda)\,,$$

$$\mu_3' = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,3,\lambda)\,,$$

$$\mu_4' = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,4,\lambda)\,.$$

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

The μ_1', variance, skewness and kurtosis of the TLGLi distribution are computed numerically for selected values of λ, β, and θ using the Mathcad program Version 15.0. The numerical values displayed in Table 1 indicate that the skewness of the new model is always positive and can range in the interval (0.68, 3.18). The spread for its kurtosis is larger ranging from 3.77 to 16.92.

The moment generating function (mgf) $M_X(t) = \mathbf{E}(e^{tX})$ of X can be derived using Equation (3.1) as

$$M_X(t) = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\;\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,0,\lambda)|_{(\lambda>t)}\,.$$

The characteristic function (cf) of X, $\boldsymbol{\phi}_X(t) = \mathbf{E}(e^{itX})$, and the cumulative generating function (cgf) of X, $\mathbf{K}_X(t) = \log\boldsymbol{\phi}_X(t)$, are given by

$$\boldsymbol{\phi}_X(t) = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\;\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,0,i\lambda)\,,$$

and

$$\mathbf{K}_X(t) = \frac{(\theta\boldsymbol{\zeta}+\theta\beta)\,\lambda^2}{1+\lambda}\sum_{\boldsymbol{\zeta}=0}^{\infty}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\;\log\left[\mathbb{k}\,((\theta\boldsymbol{\zeta}+\theta\beta)\,,\lambda,0,i\lambda)\right],$$

Table 1: Mean, variance, skewness and kurtosis of the Topp Leone Generated Lindley distribution with different values of parameters

λ	β	θ	μ_1'	Variance	Skewness	Kurtosis
		0.5	0.5283	0.9050	**3.1750**	**16.9179**
		1.5	1.7810	2.6030	1.4835	5.9315
0.5	0.5	3.0	3.0484	3.5869	1.0063	4.3890
		5.0	4.1345	4.0290	0.8156	3.9783
		10	5.7107	4.2990	**0.6821**	**3.7671**
		0.5	0.5165	0.3515	2.1602	9.6757
		1.5	1.4047	0.6927	1.1384	4.9824
1.0	1.5	3.0	2.1364	0.8069	0.8925	4.3499
		5.0	2.7128	0.8435	0.8038	4.1739
		10	3.5140	0.8562	0.7464	4.0793
		0.5	0.3065	0.0882	1.9651	8.7477
		1.5	0.7665	0.1568	1.1332	5.0869
2.0	2.5	3.0	1.1286	0.1785	0.9343	4.5435
		5.0	1.4109	0.1858	0.8612	4.3793
		10	1.8025	0.1887	0.8127	4.2841
		0.5	0.2006	0.0229	1.6979	7.5324
		1.5	0.4420	0.0351	1.1204	5.1709
4.0	5.0	3.0	0.6206	0.0386	0.9823	4.7674
		5.0	0.7580	0.0399	0.9301	4.6324
		10	0.9478	0.0406	0.8939	4.5461
		0.5	0.1006	0.0036	1.5163	6.8066
		1.5	0.1982	0.0049	1.1258	5.2763
10	10	3.0	0.2674	0.0053	1.0308	4.9781
		5.0	0.3201	0.0055	0.9936	4.8694
		10	0.3927	0.0056	0.9663	4.7934

respectively, where $i = \sqrt{-1}$.

The s^{th} incomplete moment, say $c_s(t)$, of X can be expressed as

$$\begin{aligned} c_s(t) &= \sum_{\zeta=0}^{\infty} \boldsymbol{\upsilon}_{\boldsymbol{\zeta}} \int_{-\infty}^{t} x^s \, \mathbf{h}_{(\theta\boldsymbol{\zeta}+\theta\beta),\lambda}(x) dx \\ &= \frac{(\theta\boldsymbol{\zeta} + \theta\beta)\, \lambda^2}{1+\lambda} \sum_{\zeta=0}^{\infty} \boldsymbol{\upsilon}_{\boldsymbol{\zeta}} \tau\left((\theta\boldsymbol{\zeta} + \theta\beta), \lambda, s, \lambda, t\right), \end{aligned}$$

where

$$\begin{aligned} \tau(a, b, s, \delta, t) &= \int_{t}^{\infty} x^s (1+x) \left(1 - \mathrm{e}^{-bx} \frac{(1+x)\, b + 1}{1+b}\right)^{a-1} \mathrm{e}^{-\delta x} dx \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{m=0}^{j+1} \zeta_{i,j,m}^{(a,s)} \gamma\left(s + m + 1, (bi + \delta)\, t\right), \end{aligned}$$

and $\gamma(\zeta, q)$ is the incomplete gamma function

$$\begin{aligned} \gamma(\xi_1, \xi_2)\,|_{(\xi_1 \neq 0,-1,-2,\ldots)} &= \int_0^{\xi_2} t^{\xi_1 - 1} \mathrm{e}^{-t} dt \\ &= \frac{\xi_2^{\xi_1}}{\xi_1} \left\{ {}_1\mathbf{F}_1 \left[\xi_1; \xi_1 + 1; -\xi_2\right] \right\} = \sum_{\zeta=0}^{\infty} \frac{(-1)^{\zeta}}{\zeta!\, (\xi_1 + \zeta)} \xi_2^{\xi_1 + \zeta}, \end{aligned}$$

where ${}_1\mathbf{F}_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. If s is an integer, then (a,b,s,δ,t) can be simplified to

$$\tau(a,b,r,\delta,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{l=0}^{r+m}\sum_{m=0}^{j+1}\zeta_{i,j,m}^{(a,s)}\frac{(bi+\delta)^{l}(r+m)!}{l!\mathrm{e}^{(bi+\delta)t}}.$$

The n^{th} moment of the residual life, say

$$m_n(t)=\mathbf{E}\left[(X-t)^n\,\middle|\,\binom{(n=1,2,\ldots)}{(X>t)}\right],$$

The n^{th} moment of the residual life of X is given by

$$m_n(t)=\frac{\int_t^{\infty}(x-t)^n f_{\beta,\theta,\lambda}(x)dx}{1-F_{\beta,\theta,\lambda}(t)}.$$

Therefore

$$m_n(t)=\frac{(\theta\boldsymbol{\zeta}+\theta\beta)\lambda^2}{[1-F(t)](1+\lambda)}\sum_{\boldsymbol{\zeta}=0}^{\infty}\sum_{r=0}^{n}(1-t)^n\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\tau((\theta\boldsymbol{\zeta}+\theta\beta),\lambda,n,\lambda,t).$$

The n^{th} moment of the reversed residual life can be expressed as

$$M_n(t)=\mathbf{E}\left[(t-X)^n\middle|\binom{(n=1,2,\ldots)}{(X\leq t,\;t>0)}\right]$$

or

$$M_n(t)=\frac{\int_0^{t}(t-x)^n f_{\beta,\theta,\lambda}(x)dx}{F_{\beta,\theta,\lambda}(t)}.$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$M_n(t)=\frac{(\theta\boldsymbol{\zeta}+\theta\beta)\lambda^2}{F(t)(1+\lambda)}\sum_{\boldsymbol{\zeta}=0}^{\infty}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}\boldsymbol{\upsilon}_{\boldsymbol{\zeta}}\tau((\theta\boldsymbol{\zeta}+\theta\beta),\lambda,n,\lambda,t).$$

The reliability, say $\mathbf{R}$, of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent r.v.'s with TLGLi$(\beta_1,\theta_1,\lambda)$ and TLGLi$(\beta_2,\theta_2,\lambda)$ distributions. Thus $\mathbf{R}_{(X_1>X_2)}(X_1,X_2)$ can be expressed as

$$\mathbf{R}_{(X_1>X_2)}(X_1,X_2)=\Pr(X_1>X_2)=\sum_{\boldsymbol{\zeta},j=0}^{\infty}\Omega_{\boldsymbol{\zeta},j},$$

where

$$\Omega_{\boldsymbol{\zeta},j}=\sum_{\boldsymbol{\zeta},j=0}^{\infty}\frac{(-1)^{\boldsymbol{\zeta}+j}2^{\beta_1+\beta_2-\boldsymbol{\zeta}-j}}{[(\beta_2+j)\theta_2]\{[(\beta_1+\boldsymbol{\zeta})\theta_1]+[(\beta_2+j)\theta_2]\}}\binom{\beta_1}{\boldsymbol{\zeta}}\binom{\beta_2}{j}.$$

Let $X_1, \ldots, X_n$ be a random sample (r.s.) from the TLGLi distribution and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. The pdf of i^{th} order statistic, $X_{i:n}$, can be written as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{\mathbf{B}(i, n-i+1)} f(x) F^{j+i-1}(x), \tag{3.2}$$

where $B(\cdot, \cdot)$ is the beta function. Substituting (1.1) and (1.2) in Equation (3.2), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{w,d=0}^{\infty} t_{j,w,d} h_{w+d,\lambda}(x),$$

where

$$t_{j,w,d} = \frac{w(-1)^j t_w \delta_{j+i-1,d}}{\mathbf{B}(i, n-i+1)(w+d)},$$

and $\delta_{j+i-1,d}$ can be obtained recursively from

$$\delta_{j+i-1,d}|_{(d\geq 1)} = \frac{1}{dt_0} \sum_{m=0}^{d} t_m \left[m(j+i) - d\right] \delta_{j+i-1,d-m},$$

where $\delta_{j+i-1,0} = t_0^{j+i-1}$. Then, the q^{th} moment of $X_{i:n}$ can be expressed as

$$\mathbf{E}\left(X_{i:n}^q\right) = \alpha \frac{\lambda^2}{1+\lambda} \sum_{j=0}^{n-i} \sum_{w,d=0}^{\infty} \mathbb{k}(w+d, \lambda, q, \lambda).$$

4. Different methods of estimation

4.1. Maximum likelihood method

Let $X_1, X_2, \ldots, X_n$ be a r.s. from the TLGLi distribution with observed values $x_1, x_2, \ldots, x_n$, and $\mathbf{\Psi} = (\beta, \theta, \lambda)^T$ be the vector of the model parameters. The log likelihood function for $\mathbf{\Psi}$ may be expressed as

$$\begin{aligned}\ell = \ell(\mathbf{\Psi}) &= n\log 2 + n\log\beta + n\log\theta + 2n\log\lambda - n\log(1+\lambda) \\ &+ \sum_{i=1}^{n} \log(1+x_i) - \lambda\sum_{i=1}^{n} x_i + (\theta\beta - 1)\sum_{i=1}^{n}\log q_i \\ &+ \sum_{i=1}^{n}\log\left(1 - q_i^{\theta}\right) + (\beta - 1)\sum_{i=1}^{n}\log\left(2 - q_i^{\theta}\right),\end{aligned} \tag{4.1}$$

where

$$q_i = \left(1 - \mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda + 1}{1+\lambda}\right).$$

The maximum likelihood estimators (MLEs) of β, θ and λ are obtained from differentiating Equation (4.1) with respect to β, θ, and λ

$$U_\beta = \frac{n}{\beta} + \theta \sum_{i=1}^{n} \log q_i + \sum_{i=1}^{n} \log\left(2 - q_i^\theta\right),$$

$$U_\theta = \frac{n}{\theta} + \beta \sum_{i=1}^{n} \log q_i - \sum_{i=1}^{n} \frac{q_i^\theta \log q_i}{1 - q_i^\theta} - (\beta - 1) \sum_{i=1}^{n} \frac{q_i^\theta \log q_i}{2 - q_i^\theta},$$

$$U_\lambda = \frac{2n}{\lambda} - \frac{n}{1+\lambda} - \sum_{i=1}^{n} x_i + (\theta\beta - 1) \sum_{i=1}^{n} \frac{w_i}{q_i} - \sum_{i=1}^{n} \frac{\theta w_i q_i^{\theta-1}}{1 - q_i^\theta} - (\beta - 1) \sum_{i=1}^{n} \frac{\theta w_i q_i^{\theta-1}}{2 - q_i^\theta},$$

where

$$w_i = -\frac{1}{1+\lambda}\left[-x_i\left(\lambda x + \lambda + 1_i\right) e^{-\lambda x_i} + (1 + x_i)\, e^{-\lambda x_i}\right].$$

Setting them equal to zero and solving the system simultaneously yields the MLE $\hat{\mathbf{\Psi}} = (\hat{\beta}, \hat{\theta}, \hat{\lambda})^T$ of $\mathbf{\Psi} = (\beta, \theta, \lambda)^T$. The above equations cannot be solved analytically. So, statistical software can be used to solve them numerically using the iterative methods such as the Newton-Raphson type algorithms (for more details see Casella and Berger (2002)).

4.2. Method of ordinary least square and weighted least square estimation

The theory of ordinary least square (OLS) estimation and weighted least square (WLS) estimation was originally proposed by Swain *et al.* (1988) to estimate the parameters of the Beta distribution. This theory is based on minimizing the sum of the square of differences of theoretical cdf and empirical cdf.

Suppose that $F_{\beta,\theta,\lambda}(X_{i:n})$ denotes the distribution function of TLGLi distribution and if $x_1 < x_2 < \cdots < x_n$ be the n ordered random sample. The OLS estimators (OLSE) of the parameters (β, θ, λ) are obtained by minimizing

$$\mathbf{OLS}_{(\beta,\theta,\lambda)} = \sum_{i=1}^{n} \left\{F(x_i, \beta, \theta, \lambda) - \left(\frac{i}{1+n}\right)\right\}^2,$$

then

$$\mathbf{OLS}_{(\beta,\theta,\lambda)} = \sum_{i=1}^{n} \left[\left\{\left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta \left[2 - \left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta\right]\right\}^\beta - \left(\frac{i}{1+n}\right)\right]^2.$$

The OLSE of the parameters (β, θ, λ) are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} \left[\left\{\left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta \left[2 - \left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta\right]\right\}^\beta - \left(\frac{i}{1+n}\right)\right] \boldsymbol{\eta}_\beta(x_i, \beta, \theta, \lambda) = 0,$$

$$\sum_{i=1}^{n} \left[\left\{\left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta \left[2 - \left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta\right]\right\}^\beta - \left(\frac{i}{1+n}\right)\right] \boldsymbol{\eta}_\theta(x_i, \beta, \theta, \lambda) = 0,$$

$$\sum_{i=1}^{n} \left[\left\{\left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta \left[2 - \left(1 - e^{-\lambda x_i} \frac{(1+x_i)\lambda + 1}{1+\lambda}\right)^\theta\right]\right\}^\beta - \left(\frac{i}{1+n}\right)\right] \boldsymbol{\eta}_\lambda(x_i, \beta, \theta, \lambda) = 0,$$

where $\boldsymbol{\eta}_\beta(x_i,\beta,\theta,\lambda)\boldsymbol{\eta}_\theta(x_i,\beta,\theta,\lambda)$, and $\boldsymbol{\eta}_\lambda(x_i,\beta,\theta,\lambda)$ are the values of the first derivatives of the cdf with respect to (w.r.t.) parameters of TLGLi model.

The OLSE of the parameters (β,θ,λ) are obtained via solving the above simultaneous equations by using any numerical approximation techniques. The WLS estimates (WLSE) are obtained by minimizing the given form of equation w.r.t. the parameters

$$\mathbf{WLS}_{(\beta,\theta,\lambda)} = \sum_{i=1}^{n} \mathbf{w}_i \left[F(x_i;\beta,\theta,\lambda) - \left(\frac{i}{1+n}\right)\right]^2.$$

The WLSE of the parameters are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} \mathbf{w}_i \left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta} - \left(\frac{i}{1+n}\right)\right]\boldsymbol{\eta}_\beta(x_i,\beta,\theta,\lambda) = 0,$$
$$\sum_{i=1}^{n} \mathbf{w}_i \left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta} - \left(\frac{i}{1+n}\right)\right]\boldsymbol{\eta}_\theta(x_i,\beta,\theta,\lambda) = 0,$$
$$\sum_{i=1}^{n} \mathbf{w}_i \left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta} - \left(\frac{i}{1+n}\right)\right]\boldsymbol{\eta}_\lambda(x_i,\beta,\theta,\lambda) = 0,$$

where $\boldsymbol{\eta}_\beta(x_i,\beta,\theta,\lambda)$, $\boldsymbol{\eta}_\theta(x_i,\beta,\theta,\lambda)$, and $\boldsymbol{\eta}_\lambda(x_i,\beta,\theta,\lambda)$ are the values of first derivatives of the cdf of TLGLi distribution and

$$\mathbf{w}_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

4.3. Method of Cramer-Von-Mises estimation

The Cramer-Von- Mises estimation method of the parameters is based on the theory of minimum distance estimation (MacDonald, 1971). The Crammer-Von Mises estimates (CVME) of the parameter β, θ, and λ are obtained by minimizing the following expression w.r.t. the parameters β, θ, λ respectively.

$$\mathbf{CVM}_{(\beta,\theta,\lambda)} = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_{\beta,\theta,\lambda}(x_{i:n}) - \frac{-1+2i}{2n}\right]^2,$$

and

$$\mathbf{CVM}_{(\beta,\theta,\lambda)} = \sum_{i=1}^{n}\left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta} - \frac{-1+2i}{2n}\right]^2.$$

The CVME of the parameters are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta} - \frac{-1+2i}{2n}\right]\boldsymbol{\eta}_\beta(x_i,\beta,\theta,\lambda) = 0,$$
$$\sum_{i=1}^{n}\left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta} - \frac{-1+2i}{2n}\right]\boldsymbol{\eta}_\theta(x_i,\beta,\theta,\lambda) = 0,$$

and

$$\sum_{i=1}^{n}\left[\left\{\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\left[2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right]\right\}^{\beta}-\frac{-1+2i}{2n}\right]\boldsymbol{\eta}_{\lambda}(x_i,\beta,\theta,\lambda)=0,$$

where $\boldsymbol{\eta}_{\beta}(x_i,\beta,\theta,\lambda)$, $\eta_{\theta}(x_i,\beta,\theta,\lambda)$, and $\eta_{\lambda}(x_i,\beta,\theta,\lambda)$ are the values of the first derivatives of the cdf of TLGLi distribution w.r.t. β,θ,λ respectively.

4.4. Bayes estimation

Using the squared error loss function (SELF), the Bayes estimators are computed under informative gamma priors for all β,θ, and λ. The joint prior is given by

$$p(\beta,\theta,\lambda)\propto\beta^{a_1-1}\theta^{a_2-1}\lambda^{a_3-1}e^{-b_1\beta-b_2\theta-b_3\lambda};\quad\beta,\theta,\lambda>0,$$

where the hyperparameters a_1,b_1,a_2,b_2,a_3, and b_3 assume to be known and positive and

$$\mathbf{SELF}\left(\hat{\delta},\delta\right)=\mathbf{E}(\hat{\delta}-\delta)^2$$

and $\hat{\delta}$ is the estimated value of δ. The posterior distribution is needed to derive the Bayes estimators. The posterior distribution of the TLGLi distribution can be written as

$$\begin{aligned}\pi(\beta,\theta,\lambda|\underline{x})\propto\beta^{n+a_1-1}\theta^{n+a_2-1}\frac{\lambda^{2n+a_3-1}}{(1+\lambda)^n}e^{-b_1\beta-b_2\theta-b_3\lambda-\lambda\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta\beta-1}\\ \times\prod_{i=1}^{n}\left[\left\{1-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right\}\left\{2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right\}^{\beta-1}\right].\end{aligned}$$

Under this loss function, posterior mean is the Bayes estimate of the respective parameter. Thus, the Bayes estimators under SELF are obtained as

$$\hat{\beta}_B=\mathbf{E}(\beta|\theta,\lambda,x)=\int_{\theta,\lambda}\beta^{n+a_1}\theta^{n+a_2-1}\frac{\lambda^{2n+a_3-1}}{(1+\lambda)^n}e^{-b_1\beta-b_2\theta-b_3\lambda-\lambda\sum_{i=1}^{n}x_i}\Phi(x_i)d\theta d\lambda,$$

where

$$\begin{aligned}\Phi(x_i)&=\prod_{i=1}^{n}\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta\beta-1}\\&\times\left[\left\{1-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right\}\left\{2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\lambda+1}{1+\lambda}\right)^{\theta}\right\}^{\beta-1}\right],\end{aligned}$$

$$\hat{\theta}_B=\mathbf{E}(\theta|\beta,\lambda,x)=\int_{\beta,\lambda}\beta^{n+a_1-1}\theta^{n+a_2}\frac{\lambda^{2n+a_3-1}}{(1+\lambda)^n}e^{-b_1\beta-b_2\theta-b_3\lambda-\lambda\sum_{i=1}^{n}x_i}\Phi(x_i)\,d\beta d\lambda,$$

$$\hat{\lambda}_B=\mathbf{E}(\lambda|\beta,\theta,x)=\int_{\beta,\theta}\beta^{n+a_1-1}\theta^{n+a_2-1}\frac{\lambda^{2n+a_3}}{(1+\lambda)^n}e^{-b_1\beta-b_2\theta-b_3\lambda-\lambda\sum_{i=1}^{n}x_i}\Phi(x_i)\,d\beta d\theta,$$

respectively. The above equations cannot be solved analytically, thus we use Markov chain Monte Carlo (MCMC) technique to generate the posterior sample from the full conditional posterior distribution. The full conditional posterior distributions are given by

$$\pi_1(\beta|\theta,\lambda,\underline{x}) \propto \beta^{n+a_1-1}e^{-b\beta}\prod_{i=1}^{n}\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta\beta-1}\left\{2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta}\right\}^{\beta-1},$$

$$\begin{aligned}\pi_2(\theta|\beta,\lambda,\underline{x}) \propto{}& \theta^{n+a_2-1}e^{-b_2\theta}\prod_{i=1}^{n}\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta\beta-1}\\ &\times\prod_{i=1}^{n}\left[\left\{1-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta}\right\}\left\{2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta}\right\}^{\beta-1}\right],\\ \pi_3(\lambda|\beta,\theta,\underline{x}) \propto{}& \frac{\lambda^{2n+a_3-1}}{(1+\lambda)^n}e^{-b_3\lambda-\lambda\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta\beta-1}\\ &\times\prod_{i=1}^{n}\left[\left\{1-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta}\right\}\left\{2-\left(1-\mathrm{e}^{-\lambda x_i}\frac{(1+x_i)\,\lambda+1}{1+\lambda}\right)^{\theta}\right\}^{\beta-1}\right].\end{aligned}$$

The following steps are used to extract the posterior samples from full-conditional posterior density

- Starts with $j = 1$ and set initial values of β, θ, λ say $\beta_0, \theta_0, \lambda_0$;
- Generate posterior samples from full conditional distribution using normal distribution as a proposal density;
- Repeat the above step for $j = 1, 2, \ldots, \mathbf{M}$ and simulate $(\beta_1, \theta_1, \lambda_1), (\beta_1, \theta_1, \lambda_1), \ldots, (\beta_{\mathbf{M}}, \theta_{\mathbf{M}}, \lambda_{\mathbf{M}})$;
- Under SELF, the Bayes estimates of β, θ, λ are given by

$$\begin{aligned}\hat{\beta}_S &= \frac{\sum_{j=1}^{\mathbf{M}-\mathbf{M}_0}\beta_j}{\mathbf{M}-\mathbf{M}_0},\\ \hat{\theta}_S &= \frac{\sum_{j=1}^{\mathbf{M}-\mathbf{M}_0}\theta_j}{\mathbf{M}-\mathbf{M}_0},\\ \hat{\lambda}_S &= \frac{\sum_{j=1}^{\mathbf{M}-\mathbf{M}_0}\lambda_j}{\mathbf{M}-\mathbf{M}_0},\end{aligned}$$

  where, $\mathbf{M}_0$ is the burn in period.

5. Data analysis

In this section, real data sets are used to demonstrate the real life applicability of the proposed model using MLE, LS, WLS, and CVM. We consider the Cramér-Von Mises (W^*), the Anderson-Darling (A^*), and the Kolmogorov-Smirnov (KS) statistic. The W^* and A^* statistics are given by

$$W^* = \left(1+\frac{1}{2n}\right)\left(\frac{1}{12n}+\sum_{h=1}^{n}\tau_h^{(n)}\right),$$

Table 2: The values of estimators, KS, p-values, W^*, and A^*

Method	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	KS	p-value	W^*	A^*
MLE	19.04243	3.126965	1.249401	0.08472	0.7564	0.0658	0.3444
LS	18.06670	2.310553	1.128924	**0.06427**	**0.9571**	**0.0628**	**0.3331**
WLS	21.02794	2.369555	1.168441	0.07204	0.8993	0.0640	0.3372
CVM	21.01143	2.227783	1.145041	0.06687	0.9408	0.0635	0.3353

KS = Kolmogorov-Smirnov; W^* = Cramér-Von Mises; A^* = Anderson-Darling; MLE = maximum likelihood estimators; LS = least squares; WLS = weighted LS; CVM = Crammer-Von Mises.

and

$$A^* = a_{(n)}\left(n + n^{-1}\sum\nolimits_{h=1}^{n} v_h^{(n)}\right),$$

where

$$\tau_h^{(n)} = \left(z_h - \frac{2h-1}{2n}\right)^2,$$
$$a_{(n)} = \frac{3n^{-1}}{4} + \frac{9n^{-2}}{4} + 1,$$
$$v_h^{(n)} = \log\left[\left(1 - z_{n-h+1}\right) z_i\right](2h-1),$$

where $z_h = F(y_h)$ and the y_h's values are the ordered observations.

5.1. Application 1

The first data with size 63 shows the strength measured in GPa for single carbon fibers and impregnated at gaugelengths of 20mm. The data are:

$1.9010, 2.132, 2.203, 2.2280, 2.257, 2.350, 2.361, 2.3960, 2.397, 2.445, 2.4540, 2.474, 2.518, 2.5220, 2.525, 2.5320, 2.575, 2.614, 2.616, 2.6180, 2.624, 2.659, 2.6750, 2.738, 2.740, 2.8560, 2.917, 2.928, 2.9370, 2.937, 2.9770, 2.9960, 3.030, 3.125, 3.139, 3.1450, 3.220, 3.223, 3.2350, 3.243, 3.264, 3.272, 3.2940, 3.332, 3.346, 3.377, 3.408, 3.4350, 3.493, 3.501, 3.537, 3.554, 3.5620, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.$

In general, the smaller value of KS, the better fit to the data. Table 2 gives the values of estimators of α, β, and λ, the KS test statistics and its p-value for TLGLi for the four different estimation methods.

From Table 2 we conclude that the LS method is the best method for modelling the carbon fibers with KS = 0.06427, p-value = 0.9571, $W^* = 0.0628$, and $A^* = 0.3331$. However all other methods performed well. Finally, we can say that the CVM is better that the WLS method in modelling the carbon fibers. The fitted density, the relative histogram with the fitted density of the proposed model for various methods and fitted survival function for data-I, are also piloted and given in Figure 3.

5.2. Application 2

The second data set represents on the relief times of twenty patients receiving an analgesic. The data are:

$1.10, 1.4, 1.30, 1.7, 1.90, 1.8, 1.6, 2.20, 1.7, 2.70, 4.1, 1.80, 1.5, 1.20, 1.4, 30, 1.7, 2.30, 1.6, 2$

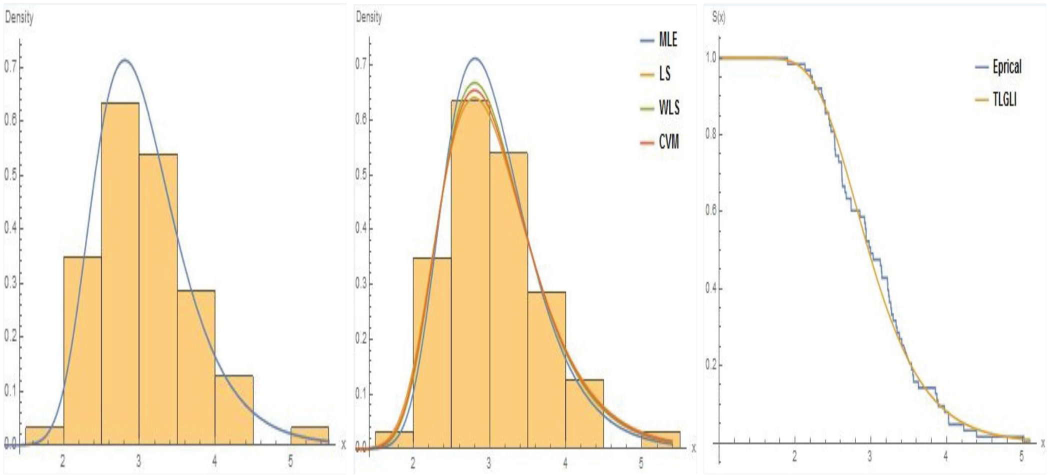

Figure 3: *Fitted density, the histograms with the fitted density of the TLGLi distribution for various methods and fitted survival function for data I. TLGLi = Topp Leone Generated Lindley; MLE = maximum likelihood estimator; LS = Least square; WLS = weighted least square; CVM = Crammer-Von Mises.*

Table 3: The values of estimators, KS, p-values, W^*, and A^*

Method	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	KS	p-value	W^*	A^*
MLE	83.98383	0.450189	1.322298	0.13861	0.8369	0.0578	0.3399
LS	87.57996	0.45128	1.389133	0.11138	0.9651	0.0577	0.3396
WLS	89.28296	0.41184	1.316787	0.10808	0.9736	0.0591	0.3477
CVM	101.4514	0.52226	1.499851	**0.09269**	**0.9954**	**0.0525**	**0.3078**

KS = Kolmogorov-Smirnov; W^* = Cramér-Von Mises; A^* = Anderson-Darling; MLE = maximum likelihood estimator; LS = Least square; WLS = weighted LS; CVM = Crammer-Von Mises.

(Gross and Clark, 1975).

From Table 3 we conclude that the CVM method is the best method for modelling the relief times data with KS = 0.09269, p-value = 0.9954, $W^* = 0.0525$, and $A^* = 0.3078$. However all other methods performed well.

The fitted density plot, the relative histogram plot with the fitted density of the proposed model and fitted survival function for data-II are given in Figure 3. Afterall, from Figures 3 and 4, it has been noticed that the proposed model fitted well to the considered data set.

6. Goodness-of-fit test for TLGLi model in complete data case

6.1. Nikulin-Rao-Robson statistic test for complete data

The NRR statistic ($Y_n^2(\hat{\mathbf{\Psi}}_n)$) given by Nikulin (1973) and Rao-Robson (1974) is a famous modified chi-squared goodness-of-fit test (Pearson, 1900). This statistical test is based on the differences between theoretical and empirical probabilities that fall into grouping cells. It uses the maximum likelihood estimation on the initial data (for more details see Voinov *et al.* (2013)). Let $X = (X_1, X_2, \ldots, X_n)^T$ be a random sample of n independent and identically distributed r.v.s. We want to test the following null composite hypothesis H_0

$$H_0 : P_{\mathbf{\Psi}}(X_i \leq x) = F(x, \mathbf{\Psi}), \quad x \in \mathcal{R}, \;\; \mathbf{\Psi} = (\theta_1, \theta_2, \ldots, \theta_s)^T,$$

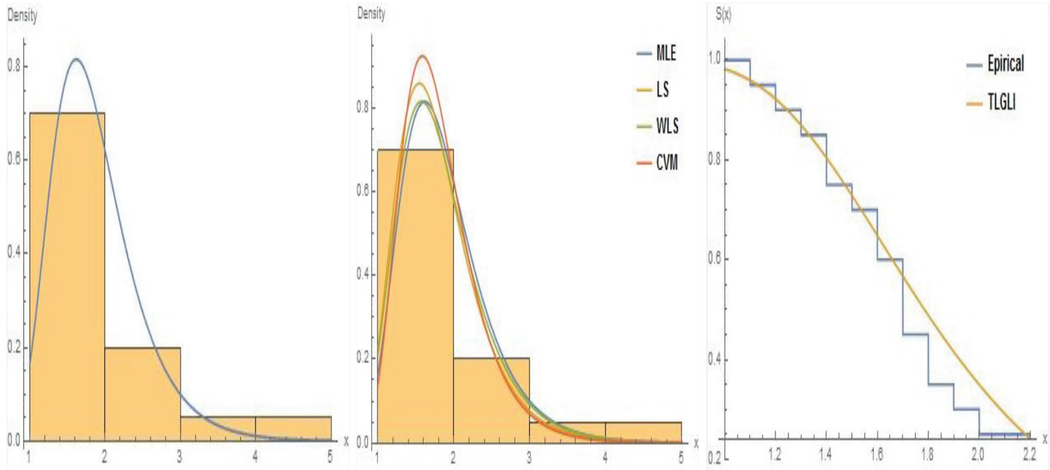

Figure 4: *Fitted density, the histograms with the fitted density of the TLGLi distribution for various methods and fitted survival function for data II. TLGLi = Topp Leone Generated Lindley. TLGLi = Topp Leone Generated Lindley; MLE = maximum likelihood estimator; LS = least square; WLS = weighted least square; CVM = Crammer-Von Mises.*

and $\nu_j = (\nu_1, \nu_2, \ldots, \nu_r)^T$ is the vector of frequencies (where $\sum_{j=1}^{\zeta} \nu_j = n$) obtained by grouping X_i into r intervals I_j :

$$I_j = \left]a_{j-1}, a_j\right]; \quad -\infty < a_1 < \cdots < a_{r-1} < a_r = +\infty.$$

The boundaries of the intervals a_j are:

$$a_j|_{(j=1,\ldots,r-1)} = F^{-1}\left(\frac{j}{r}\right).$$

The NRR statistic, with $\hat{\boldsymbol{\Psi}}_n$ as a maximum likelihood estimator of the parameter vector θ, is defined by

$$Y_n^2\left(\hat{\boldsymbol{\Psi}}_n\right) = X_n^2\left(\hat{\boldsymbol{\Psi}}_n\right) + \frac{1}{n}L^T\left(\hat{\boldsymbol{\Psi}}_n\right)\left(I\left(\hat{\boldsymbol{\Psi}}_n\right) - J\left(\hat{\boldsymbol{\Psi}}_n\right)\right)^{-1} L\left(\hat{\boldsymbol{\Psi}}_n\right).$$

The Pearson's statistic $X_n^2(\boldsymbol{\Psi}) = X_n^T(\boldsymbol{\Psi})X_n(\boldsymbol{\Psi})$, where

$$X_n(\theta) = \left(\frac{\nu_1 - n\,\mathrm{Pr}_1(\boldsymbol{\Psi})}{[n\,\mathrm{Pr}_1(\boldsymbol{\Psi})]^{\frac{1}{2}}}, \frac{\nu_2 - n\,\mathrm{Pr}_2(\boldsymbol{\Psi})}{[n\,\mathrm{Pr}_2(\boldsymbol{\Psi})]^{\frac{1}{2}}}, \ldots, \frac{\nu_r - n\,\mathrm{Pr}_r(\boldsymbol{\Psi})}{[n\,\mathrm{Pr}_r(\boldsymbol{\Psi})]^{\frac{1}{2}}}\right)^T,$$

where vector of probabilities $\mathrm{Pr}(\boldsymbol{\Psi}) = (\mathrm{Pr}_1(\boldsymbol{\Psi}), \mathrm{Pr}_2(\boldsymbol{\Psi}), \ldots, \mathrm{Pr}_r(\boldsymbol{\Psi}))^T$, with

$$\mathrm{Pr}_j(\boldsymbol{\Psi})\,|_{(j=1,2,\ldots,r)} = \int_{a_{j-1}}^{a_j} f(x, \boldsymbol{\Psi})\,dx.$$

The $I(\boldsymbol{\Psi}_n)$ is the Fisher information matrix, and

$$l(\boldsymbol{\Psi}) = (l_1(\boldsymbol{\Psi}), \ldots, l_s(\boldsymbol{\Psi}))^T; \quad \text{with } l_\zeta(\boldsymbol{\Psi}) = \sum_{i=1}^{r} \frac{\nu_i}{\mathrm{Pr}_i} \frac{\partial}{\partial \boldsymbol{\Psi}_\zeta} [\mathrm{Pr}_i(\boldsymbol{\Psi})].$$

$J(\boldsymbol{\Psi}) = B(\boldsymbol{\Psi})^T B(\boldsymbol{\Psi})$ represents the Fisher information matrix of any distribution with parameters $\Pr(\boldsymbol{\Psi})$, where

$$B(\boldsymbol{\Psi}) = \left[\frac{1}{\sqrt{\Pr_i}} \frac{\partial \Pr_i(\boldsymbol{\Psi})}{\partial \mu} \right]_{r\times s} \Bigg|_{(i=1,2,\ldots,r \text{ and } \zeta=1,\ldots,s)}.$$

Under the null hypothesis H_0, the NRR ($Y_n^2(\hat{\boldsymbol{\Psi}}_n)$) statistic follow a chi-square distribution with $r-1$ degrees of freedom. For any fixed $x > 0$, we have:

$$\lim_{n\to\infty} \Pr\left(Y_n^2\left(\hat{\boldsymbol{\Psi}}_n\right) \geq x\right) = \Pr\left(\chi^2_{r-1} \geq x\right).$$

6.2. Validity of TLGLi model

We test the following null hypothesis H_0. The distribution of the sample $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$ is given by

$$H_0 : P(X_i \leq x) = F_{\beta,\theta,\lambda}(x), \quad \boldsymbol{\psi} = (\beta, \theta, \lambda)^T, \ x \geq 0,$$

where $F_{\beta,\theta,\lambda}(x)$ is given by (1.1). To reveal the usefulness of the modified chi-square NRR ($Y_n^2(\hat{\boldsymbol{\psi}}_n)$) statistic test and the pertinence of the TLGLi distribution, we consider two real demonstrative data sets.

Example 1. Data was reported by Badar and Priest (1982) with size 63 shows the strength measured in GPa for single carbon fibers and impregnated at gaugelengths of 20mm. We choose $r = 5$ intervals, Using the BB package of R language, the MLE's parameters of the TLGLi(β, θ, λ) distribution are the following:

$$\hat{\beta} = 1.6115, \qquad \hat{\theta} = 1.6404, \qquad \hat{\lambda} = 0.5442.$$

We can aver that

$$Y_n^2\left(\hat{\boldsymbol{\psi}}_n\right) \leq \chi^2_{r-1,\alpha=0.05} = 9.4877,$$

since

$$Y_n^2\left(\hat{\boldsymbol{\psi}}_n\right) = 7.9904$$

thus, the strength measured in GPa for single carbon fibers follows a TLGLi distribution with $\hat{\boldsymbol{\psi}}_n$ parameters.

Example 2. The data constitutes twenty observations of the lifetime data relating to relief times (in minutes) of patients receiving an analgesic. This data was reported by Gross and Clark (1975). We take $r = 4$ intervals; then the values of the estimated parameters of our TLGLi (β, θ, λ) are:

$$\hat{\beta} = 2.0579, \qquad \hat{\theta} = 2.1668, \qquad \hat{\lambda} = 0.9837.$$

The NRR statistic tests $Y_n^2(\hat{\boldsymbol{\psi}}_n) = 7.4017$. It is clear that

$$Y_n^2\left(\hat{\boldsymbol{\psi}}_n\right) \leq \chi^2_{r-1,\alpha=0.05} = 7.8147.$$

We conclude that we have a concordance of the analgesic failure time data and our TLGLi model.

Table 4: Average values of estimates and mean squared errors (in parentheses) for $n = 20$

Parameters	MLE	LS	WLS	CVM	Bayesian
$\beta = 2.0$	2.3532 (0.38887)	2.1393 (2.08396)	1.8615 (1.17179)	1.8403 (0.78797)	1.9825 (0.36291)
$\theta = 0.9$	1.0345 (0.05332)	1.0082 (0.28217)	1.0695 (0.37182)	0.9998 (0.27709)	0.9725 (0.05102)
$\lambda = 1.5$	1.4189 (0.04491)	1.6744 (0.49578)	1.3668 (0.56632)	1.8278 (0.75370)	1.4062 (0.02231)
$\beta = 1.2$	1.3733 (0.14894)	1.7486 (1.73004)	1.6431 (1.47016)	1.6111 (0.89955)	1.0090 (0.11238)
$\theta = 1.25$	1.3593 (0.07589)	1.5477 (0.49841)	1.4652 (0.53282)	1.5500 (0.44179)	1.1924 (0.06207)
$\lambda = 0.5$	0.4919 (0.00380)	0.5382 (0.14299)	0.6003 (0.31093)	0.5259 (0.14304)	0.4726 (0.00175)
$\beta = 2.5$	2.6570 (0.56914)	2.4663 (1.45153)	2.2088 (1.86197)	2.4313 (1.59231)	2.5121 (0.53294)
$\theta = 0.8$	0.8241 (0.02121)	0.9339 (0.10929)	0.9314 (0.21013)	0.9520 (0.14371)	0.8321 (0.02019)
$\lambda = 0.3$	0.3012 (0.00128)	0.3745 (1.19557)	0.3345 (0.06211)	0.5054 (3.00787)	0.2752 (0.00089)

MLE = maximum likelihood estimator; LS = least square; WLS = weighted least square; CVM = Crammer-Von Mises.

Table 5: Average values of estimates and mean squared errors for $n = 50$

Parameters	MLE	LS	WLS	CVM	Bayesian
$\beta = 2.0$	2.2743 (0.15772)	1.8969 (0.39803)	1.8156 (0.93618)	1.8654 (0.38163)	1.9625 (0.13194)
$\theta = 0.9$	1.0101 (0.02351)	0.8766 (0.06023)	1.0299 (0.32983)	0.9116 (0.18209)	0.8562 (0.02127)
$\lambda = 1.5$	1.4205 (0.02041)	1.6965 (0.32712)	1.4223 (0.52627)	1.8211 (0.59441)	1.2213 (0.01749)
$\beta = 1.2$	1.3246 (0.05230)	1.6139 (0.54912)	1.6080 (1.11703)	1.6155 (0.65038)	1.1230 (0.05067)
$\theta = 1.25$	1.3331 (0.02870)	1.4953 (0.18328)	1.4126 (0.51303)	1.5109 (0.29059)	1.1960 (0.02513)
$\lambda = 0.5$	0.4908 (0.00147)	0.4684 (0.01874)	0.5942 (0.29852)	0.4873 (0.09689)	0.4238 (0.00071)
$\beta = 2.5$	2.5587 (0.17512)	3.1008 (1.41519)	2.3333 (1.73189)	2.7694 (0.97663)	2.5670 (0.16327)
$\theta = 0.8$	0.8084 (0.00710)	0.9115 (0.04340)	0.9317 (0.14571)	0.9154 (0.05181)	0.7865 (0.00591)
$\lambda = 0.3$	0.3009 (0.00048)	0.3104 (0.36087)	0.3183 (0.06067)	0.3284 (1.04234)	0.3421 (0.00029)

MLE = maximum likelihood estimator; LS = least square; WLS = weighted least square; CVM = Crammer-Von Mises.

7. Simulation study

A MCMC simulation study is conducted in this section, to compare the performance of the different estimators of the unknown parameters of the TLGLi distribution. This performance is evaluated regarding their mean squared errors (MSEs). Computations in this section are done by 'Mathcad program Version 15.0'. We generate 1,000 samples of the TLGLi distribution, where $n = (20, 50, 100, 200)$ and choosing $(\beta, \theta, \lambda) = (2, 1.5, 0.9)$, $(0.9, 1.25, 2)$, and $(2.5, 0.8, 0.3)$. The average values of estimates and MSEs of MLEs, LSEs, WLSEs, CVMEs, and Bayesian estimators are obtained and reported in Tables 4–7. The Bayesian estimators of the parameters are evaluated with flexible gamma prior under the SELF by using the MCMC technique. The values of the hyperparameters are assumed known and chosen in such a way that the prior mean is equal to the true value, and prior variance is unity. From Tables 4–7, we observe that all the estimates show the property of consistency, i.e., the MSEs decrease as sample size increase. The MSEs of the Bayesian estimators are also less when compared to the other estimators; in addition, sometimes the MSEs of the Bayes and MLEs are very close to each other.

8. Conclusion

This paper introduces a new extension of the Lindley model. The estimation of the parameters is carried out via different methods. Bayes estimation is computed under gamma informative prior under the squared error loss function. The performances of the proposed estimation methods are studied through Monte Carlo simulations. The potentiality of the proposed model is analyzed through two data sets. A modified goodness-of- fit test using the NRR statistic test is investigated via two examples. Certain characterizations of the proposed distribution are presented. A modified goodness-

Table 6: Average values of estimates and mean squared errors for $n = 100$

Parameters	MLE	LS	WLS	CVM	Bayesian
$\beta = 2.0$	2.2659 (0.09170)	1.7649 (0.17525)	1.8554 (0.79055)	1.8139 (0.17631)	1.9920 (0.08923)
$\theta = 0.9$	1.0084 (0.01832)	0.8492 (0.05150)	0.9972 (0.27431)	0.8493 (0.05137)	0.9321 (0.01652)
$\lambda = 1.5$	1.4113 (0.01541)	1.6787 (0.17699)	1.4944 (0.47237)	1.6295 (0.19440)	1.4621 (0.01469)
$\beta = 1.2$	1.3222 (0.02832)	1.5396 (0.27328)	1.6031 (0.93027)	1.5404 (0.27639)	1.0951 (0.02343)
$\theta = 1.25$	1.3334 (0.01804)	1.4637 (0.09777)	1.5482 (0.48893)	1.4641 (0.09843)	1.1970 (0.01601)
$\lambda = 0.5$	0.4885 (0.00082)	0.4587 (0.00348)	0.5638 (0.27453)	0.4631 (0.01789)	0.4235 (0.00063)
$\beta = 2.5$	2.5538 (0.09009)	2.9853 (0.67365)	2.4582 (1.57146)	2.9866 (0.68129)	2.6120 (0.07321)
$\theta = 0.8$	0.8083 (0.00367)	0.8962 (0.02395)	0.9102 (0.10320)	0.8963 (0.02412)	0.7126 (0.00296)
$\lambda = 0.3$	0.2997 (0.00024)	0.2826 (0.00084)	0.3148 (0.05267)	0.2826 (0.00084)	0.3712 (0.00018)

MLE = maximum likelihood estimator; LS = least square; WLS = weighted least square; CVM = Crammer-Von Mises.

Table 7: Average values of estimates and mean squared errors for $n = 200$

Parameters	MLE	LS	WLS	CVM	Bayesian
$\beta = 2.0$	2.0022 (0.02182)	1.7879 (0.10016)	2.1824 (0.61904)	1.7845 (0.09545)	1.8921 (0.00861)
$\theta = 0.9$	1.0113 (0.01584)	0.8332 (0.01154)	0.9739 (0.07393)	0.8316 (0.01086)	0.7829 (0.01329)
$\lambda = 1.5$	1.4067 (0.01229)	1.6824 (0.09175)	1.6873 (0.18729)	1.6813 (0.08527)	1.5700 (0.01017)
$\beta = 1.2$	1.3189 (0.02400)	1.4896 (0.13249)	1.4998 (0.61342)	1.4914 (0.13387)	1.1921 (0.02091)
$\theta = 1.25$	1.3331 (0.01279)	1.4420 (0.05639)	1.5302 (0.37676)	1.4431 (0.05649)	1.3624 (0.00972)
$\lambda = 0.5$	0.4875 (0.00051)	0.4576 (0.00268)	0.4958 (0.07906)	0.4573 (0.00264)	0.3905 (0.00047)
$\beta = 2.5$	2.5496 (0.04815)	2.9140 (0.34917)	2.4723 (1.41766)	2.9138 (0.32176)	2.6412 (0.02938)
$\theta = 0.8$	0.8080 (0.00195)	0.8860 (0.08602)	0.9093 (0.08668)	0.8863 (0.01300)	0.9214 (0.00072)
$\lambda = 0.3$	0.2991 (0.00012)	0.2819 (0.00059)	0.2911 (0.01006)	0.2819 (0.00057)	0.2764 (0.00011)

MLE = maximum likelihood estimator; LS = least square; WLS = weighted least square; CVM = Crammer-Von Mises.

of-fit test for the new model in complete data case is investigated via two examples. We propose the construction of a modified chi-squared goodness of fit statistic test for the new TLGLi model in complete data case. The new test is based on the NRR statistic separately proposed by Nikulin (1973) and Rao and Robson (1974). As a second step, an application to real data has been proposed to show the applicability of the proposed test and the new TLGLi model for modeling different data sets.

Appendix: Theorem 1.

Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$, $e = \infty$ be allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X)|X \geq x\right] = \mathbf{E}\left[q_1(X)|X \geq x\right]\eta(X), \quad x \in H,$$

is defined with some real function η. Assume that $q_1,\ q_2 \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2, and η, particularly

$$F(X) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_1(u) - q_2(u)}\right| \mathrm{e}^{-s(u)}\,du,$$

where the function s is a solution of the differential equation $s' = (\eta' q_1)/(\eta q_1 - q_2)$ and C is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (Glänzel, 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $q_{1,n}$, $q_{2,n}$, and $\eta_n (n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $q_{1,n} \to q_1$, $q_{2,n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1,n}(X)$ and $q_{2,n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution only if η_n converges to η, where

$$\eta(X) = \frac{E\left[q_2(X)|X \geq x\right]}{E\left[q_1(X)|X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2, and η, respectively. It guarantees the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in Glänzel and Hamedani (2001).

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $q_1(X) \equiv 1$, as we did in Proposition 3, which reduces the condition of Theorem 1 to

$$\mathbf{E}\left[q_2(X)|X \geq x\right] = \eta(X), \quad x \in H.$$

We, however, believe that employing three functions q_1 , q_2 and η will enhance the domain of applicability of Theorem 1.

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