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DEFINING FIELDS OF SPECIAL SUPERSINGULAR K3 SURFACES

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Abstract. In this paper, we prove that a special supersingular K3 surface of Artin invariant σ over a field of odd characteristic p has a model over a finite field of $p^{2\sigma}$ elements.

1. Introduction

Let k be an algebraically closed field of odd characteristic p.

A K3 surface X over k is supersingular if the rank of the Neron-Severi lattice $NS(X)$ is 22. A K3 surface X over k is supersingular if and only if the height of the formal Brauer group $\widehat{Br}_X$ is infinite. Let X be a supersingular K3 surface over k. The signature of $NS(X)$ is (1,21). The discriminant group of the Neron-Severi group of X

$$l(NS(X)) = NS(X)^*/NS(X)$$

is isomorphic to $(\mathbb{Z}/p)^{2\sigma}$. Here σ is an integer between 1 and 10. We call σ the Artin-invariant of X. The discriminant of the induced quadratic form on $l(NS(X))$ is $(-1)^\sigma\Delta$. Here Δ is a non quadratic residue modulo p. Hence there is no σ-dimensional isotropic $(\mathbb{Z}/p)-$subspace of $l(NS(X))$. The integral lattice satisfying all these conditions is unique up to isomorphism. Therefore the Neron-Severi lattice $NS(X)$ is uniquely determined by the base characteristic p and the Artin-invariant σ up to isomorphism. All the supersingular K3 surfaces of Artin-invariant σ form a family of $\sigma - 1$ dimension. A supersingular K3 surface of Artin-invariant 1 is unique up to isomorphism. For detail explanation and references, see [2], section 2.

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For a K3 surface X over k, we say the order of the image of a natural representation

$$\rho_X : \mathrm{Aut}(X) \to GL(H^0(X, \Omega^2_{X/k}))$$

is the non-symplectic index of X and we denote this by N_X. In a previous work ([3]), we showed that if X is a supersingular K3 surface of Artin-invariant σ, the non-symplectic index of X, $N_X = p^m + 1$ for $m = 0$ or m is a positive integer such that σ/m is an odd integer. Also we proved that there exists a unique supersingular K3 surface of Artin-invariant σ over k such that the non-symplectic index is equal to $p^\sigma + 1$. We call this unique supersingular K3 surface is a special supersingular K3 surface of Artin-invariant σ and we denote this by $X_{p,\sigma}$. Because, over any algebraically closed field of characteristic p, there exists a unique special supersingular K3 surface of Artin-invariant σ, $X_{p,\sigma}$ has a model over an algebraic closure of the prime field $\mathbb{F}_p$, in particular, $X_{p,\sigma}$ has a model over a finite field. In [3], we see that many special supersingular K3 surfaces are defined over prime fields. Also we raised a question whether every special supersingular K3 surface has a model over a prime field. In this article, we will give a partial answer to that question. Precisely we prove the following.

Theorem 1.1. *A special supersingular K3 surface $X_{p,\sigma}$ has a model over a finite field $\mathbb{F}_{p^{2\sigma}}$ of $p^{2\sigma}$ elements.*

2. Preliminary : Period space of supersingular K3 surface

In this section, we review the classification of supersingular K3 surfaces in terms of the period space. For the detail, we refer to [3], section 3.

Let k be an algebraically closed field of odd characteristic p. Let W and K be the ring of Witt vectors of k and the fraction field of W respectively. Assume X is a supersingular K3 surface of Artin invariant σ over k. Let us fix an abstract lattice $N_{p,\sigma}$ which is isomorphic to $NS(X)$. Let $l_{p,\sigma}$ is the discriminant group of $N_{p,\sigma}$. Hence $l_{p,\sigma}$ is a 2σ-dimensional $\mathbb{F}_p$-quadratic space. We set a Frobenius semi-linear endomorphism of $l_{p,\sigma} \otimes k$, $f = id \otimes F_k$. Here F_k is the Frobenius morphism of k. A σ-dimensional isotropic k-subspace $\mathcal{K}$ of $l_{p,\sigma} \otimes k$ is a strictly characteristic subspace of $l_{p,\sigma} \otimes k$ if

(1) $\mathcal{K} + f(\mathcal{K})$ is of dimension $\sigma + 1$
(2) $\mathcal{K}^{f=id} = 0$

Let $\mathcal{K}$ be a strictly characteristic space of $l_{p,\sigma} \otimes k$. Then

$$l(\mathcal{K}) = \mathcal{K} \cap f^{-1}(\mathcal{K}) \cap \cdots \cap f^{1-\sigma}(\mathcal{K})$$

is a line in $l_{p,\sigma} \otimes k$. We choose a nonzero vector $v_1 \in l(\mathcal{K})$. Let $v_i = f^{i-1}(v_1)$ for $i = 2, \cdots, 2\sigma$. If the pairing $v_1 \cdot v_{\sigma+1} = 0$, then $\mathcal{K} + f(\mathcal{K})$ is an isotropic subspace of dimension $\sigma + 1$. It is impossible, so $v_1 \cdot v_{\sigma+1} \neq 0$. If we replace v_1 by ξv_1, then v_i is changed into $\xi^{p^{i-1}} v_i$. After a suitable scalar multiplication, we may assume $v_1 \cdot v_{\sigma+1} = 1$. In this case, v_1 is uniquely determined up to multiplication by a $p^\sigma+1$-th root of unity. Let $a_i = v_1 \cdot v_{\sigma+i+1} \in k$ for $i = 1, \cdots, \sigma-1$. Since the pairing of $l_{p,\sigma} \otimes k$ is defined over $\mathbb{F}_p$, $v_{1+j} \cdot v_{\sigma+1+i+j} = f^k(v_1) \cdot f^k(v_{\sigma+i+1}) = F_k^j(a_i)$. The intersection matrix of $l_{p,\sigma}$ in terms of the base $v_1, \cdots, v_{2\sigma}$ is $\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 1 & a_1 & a_2 & a_3 & \cdots & a_{\sigma-1} \\ 0 & 1 & F_k(a_1) & F_k(a_2) & \cdots & F_k(a_{\sigma-2}) \\ 0 & 0 & 1 & F_k^2(a_1) & \cdots & F_k^2(a_{\sigma-3}) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{2.1}$$

For special supersingular K3 surface $X_{p,\sigma}$, the matrix A in (2.1) is equal to the identity matrix of rank σ, I_σ. (See [3], page 8) If we replace v_1 by ξv_1 for a $p^\sigma+1$-th root of unity ξ, then a_i is changed into $\xi^{p^{\sigma+i}+1}a_i$. We give an action of $\mu_{p^\sigma+1}$ on $\mathbb{A}^{\sigma-1}$ by

$$\xi \cdot (x_1, \cdots, x_{\sigma-1}) = (\xi^{p^{\sigma+1}+1}x_1, \xi^{p^{\sigma+2}+1}x_2, \cdots, \xi^{p^{2\sigma-1}+1}x_{\sigma-1}).$$

Then $(a_1, \cdots, a_{\sigma-1}) \in \mathbb{A}^{\sigma-1}/\mu^{p^\sigma+1}$ is determined by $\mathcal{K}$. If $g \in O(l_{p,\sigma})$, $g(\mathcal{K})$ is also a strictly characteristic subspace and $f(\mathcal{K})$ gives the same element $(a_1, \cdots, a_{\sigma-1}) \in \mathbb{A}^{\sigma-1}/\mu_{p^\sigma+1}$ with $\mathcal{K}$ in the above construction. Let $\mathcal{M}$ be the set of $O(l_{p,\sigma})$-conjugacy classes of strictly characteristic subspaces of $l_{p,\sigma}$. By all the above, we have a map

$$\Phi : \mathcal{M} \to \mathbb{A}/\mu_{p^\sigma+1}, \ \mathcal{K} \mapsto (a_1, \cdots, a_{\sigma-1}).$$

It is known that Φ is bijective. ([5], 3.21) In other words, all the $O(l_{p,\sigma})$-conjugacy classes of strictly characteristic subspaces are classified by $\mathbb{A}^{\sigma-1}/\mu_{p^\sigma+1}$.

Let X be a supersingular K3 surface of Artin-invariant σ over k. The second crystalline cohomology of X, $H^2_{cris}(X/W)$ is a free W module of rank 22 equipped with a unimodular lattice structure. The cycle map gives the following chain of W-lattices of same rank

$$NS(X) \otimes W \subset H^2_{cris}(X/W) \subset NS(X)^* \otimes W (\subset NS(X) \otimes K).$$

The cokernel of the cycle map $\mathcal{K}_X = H^2_{cris}(X/W)/(NS(X) \otimes W)$ is a σ-dimensional isotropic k-subspace of $l(NS(X)) \otimes k = (NS(X)^* \otimes W)/(NS(X) \otimes W)$. We say $\mathcal{K}_X$ is the period space of X. It is known that $\mathcal{K}_X$ is a strictly characteristic subspace of $l(NS(X)) \otimes k \simeq l_{p,\sigma} \otimes k$. ([5], 3.20) Hence we have a map

$$\Psi : \{\text{isomorphic classes of supersingular K3 surfaces of Artin invariant } \sigma\} \to \mathcal{M}.$$

Moreover the following is known.

Theorem 2.1 ([6], Theorem III)**.** *Ψ is bijective.*

Therefore the composition $\Phi \circ \Psi$ is bijective, so the isomorphic classes of supersingular K3 surfaces of Artin-invariant σ are classified by $\mathbb{A}^{\sigma-1}/\mu_{p^\sigma+1}$.

3. Galois descent

Let E be a field and F be a finite Galois extension of E. Let $G = \mathrm{Gal}(F/E)$. Set $F' = F \otimes_E F$, $F'' = F \otimes_E F \otimes_E F$. There are two projections $p_1, p_2 : \operatorname{Spec} F' \to \operatorname{Spec} F$ and three projections $q_{12}, q_{13}, q_{23} : \operatorname{Spec} F'' \to \operatorname{Spec} F'$. We have the canonical identities

$$p_1 \circ q_{12} = p_1 \circ q_{13},\ p_2 \circ q_{13} = p_2 \circ q_{23} \text{ and } p_2 \circ q_{12} = p_1 \circ q_{23}.$$

Let us set $\operatorname{Spec} F' = \coprod_{\tau \in G} \operatorname{Spec} F_\tau$ and $\operatorname{Spec} F'' = \coprod_{\tau,\eta \in G} \operatorname{Spec} F_{\tau,\eta}$. Here each of F_τ and $F_{\tau,\eta}$ is isomorphic to F over E, so we may set $F_\tau = F$. Also, we may regard the restriction $(p_1|_{\operatorname{Spec} F_\tau})^* : F \to F_\tau$ as the identity for each $\tau \in G$ and $(p_2|_{\operatorname{Spec} F_\tau})^* = \tau : F \to F_\tau$. In a similar way, we may set $F_{\tau,\eta} = F$. Then $q_{12}|_{\operatorname{Spec} F_{\tau,\eta}} : \operatorname{Spec} F_{\tau,\eta} \to \operatorname{Spec} F_\tau$ and we may regard $(q_{12}|\operatorname{Spec} F_{\tau,\eta})^* : F_\tau \to F_{\tau,\eta}$ as the identity. Also we may regard $(q_{13}|_{\operatorname{Spec} F_{\tau,\eta}})^* : F_\eta \to F_{\tau,\eta}$ as the identity and $(q_{23}|_{\operatorname{Spec} F_{\tau,\eta}})^* = \tau : F_{\tau^{-1}\eta} \to F_{\tau,\eta}$.

Assume X is a scheme over $\operatorname{Spec} F$. Let τX $(\tau \in G)$ be the base change of $X/\operatorname{Spec} F$ by $\tau : F \to F$.

$$\begin{array}{ccc} \tau X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} F & \xrightarrow{\operatorname{Spec} \tau} & \operatorname{Spec} F \end{array}$$

A descent datum of $X/F/E$ is an isomorphism $\gamma : p_1^* X \to p_2^* X$ over $\operatorname{Spec} F'$ such that

$$q_{23}^*(\gamma) \circ q_{12}^*(\gamma) = q_{13}^*(\gamma) : q_{12}^* p_1^* X \to q_{23}^* p_2^* X.$$

To give a descent datum of $X/F/E$ is equivalent to the following :

(1) There is an F-isomorphism $g_\tau : X \to \tau X$ for all $\tau \in G$.
(2) $(\operatorname{Spec} \tau)^*(g_\eta) \circ g_\tau : X \to \tau X \to \tau(\eta X) = \tau\eta X$ is equal to $g_{\tau\eta} : X \to \tau\eta X$.

Here, note that for $\tau, \eta \in G$, $\tau\eta = \tau \circ \eta \in G$ and $\operatorname{Spec} \eta \circ \operatorname{Spec} \tau = \operatorname{Spec} \tau\eta$. Hence $\tau(\eta X)$, the base change of ηX by τ is equal to $\tau\eta X$, the base change of X by $\tau\eta$.

If there is an F-isomorphism $\beta : X \to Y \otimes_E F$ for an E-scheme Y, it gives the canonical descent datum

$$p_1^* X \xrightarrow{p_1^*(\beta)} p_1^*(Y \otimes_E F) = p_2^*(Y \otimes_E F) \xrightarrow{p_2^*(\beta^{-1})} p_2^*(X).$$

This kind of descent datum is called an effective descent datum. It is known that if X/F is a quasi-projective variety, every descent datum of $X/F/E$ is an effective descent datum. ([4], 16.25)

Now assume E is a finite field $\mathbb{F}_{p^r}$ and F is a finite Galois extension of E, $\mathbb{F}_{p^{rm}}$. Let α be the p^r-th power Frobenius morphism of F. Hence $G = \mathrm{Gal}(F/E) =< \alpha >$ and $G \simeq \mathbb{Z}/m$.

Lemma 3.1. *Assume X is a quasi-projective variety over F. If there exists an F-isomorphism $g_\alpha : X \to \alpha X$ such that the composition*

$$\alpha^{(m-1)*}(g_\alpha)\circ\alpha^{(m-2)*}(g_\alpha)\circ\cdots\circ\alpha^*(g_\alpha)\circ g_\alpha : X \to \alpha X \to \cdots \to \alpha^{m-1}X \to \alpha^m X = X$$

is the identity, then $X = Y \otimes_E F$ for a variety Y over E.

Proof. It is enough to give a descent datum for $X/F/E$. We set $g_{id} : X \to X = id$ and

$$g_{\alpha^i} = \alpha^{(i-1)*}(g_\alpha) \circ \cdots \circ \alpha^*(g_\alpha) \circ g_\alpha : X \to \alpha^i X$$

for $1 \leq i \leq m-1$. And we set

$$g = \textstyle\coprod_{0\leq i \leq m-1} g_{\alpha^i} : p_1^* X = \coprod X \to \coprod \alpha^i X = p_2^* X.$$

If $i + j \leq m - 1$, then, by definition,

$$\alpha^{i*}(g_{\alpha^j}) \circ g_{\alpha^i} = g_{\alpha^{i+j}} = g_{\alpha^j\alpha^i} : X \to \alpha^i X \to \alpha^j\alpha^i X.$$

If $i + j \geq m$, then

$$\alpha^{i*}(g_{\alpha^j}) \circ g_{\alpha_i} = g_{\alpha^{i+j-m}} \circ id = g_{\alpha^j\alpha^i} : X \to X \to \alpha^{i+j-m}X = \alpha^j\alpha^i X.$$

Therefore g is a descent datum for $X/F/E$. □

4. Proof of Theorem 1.1

Let k be an algebraic closure of the prime field $\mathbb{F}_p$ for an odd prime number p. Let X be a supersingular K3 surface of Artin-invariant σ over k. The Frobenius morphism F_k is a topological generator of $\mathrm{Gal}(k/\mathbb{F}_p)$. Let $F_k^r X$ be the base change of X by $F_k^r : k \to k$.

$$\begin{array}{ccc} F_k^r X & \xrightarrow{F_k^r} & X \\ \downarrow & & \downarrow \\ k & \xrightarrow{F_k^r} & k \end{array}$$

The induced map $F_k^{r*} : NS(X) \to NS(F_k^r X)$ is an isomorphism. Also $NS(X)$ and $NS(F_k^r X)$ are isomorphic to $N_{p,\sigma}$. We may set $NS(X) = N_{p,\sigma} = NS(F_k^r X)$ and we may regard $F_k^{r*} : NS(X) \to NS(F_k^r X)$ as the identity map of $N_{p,\sigma}$. We let $F_W : W \to W$ and $F_K : K \to K$ be the Frobenius morphisms of W and K respectively. Let W_r be the free W-module of rank 1, W via $F_W^r : W \to W$.

The crystalline cohomology $H^2_{cris}(F^r_k X/W)$ is identified with $H^2_{cris}(X/W)\otimes W_r$ and the following diagram commutes.

$$\begin{array}{ccc} NS(F^r_k X) & \xleftarrow{\quad F^{r*}_k \quad} & NS(X) \\ \downarrow & & \downarrow \\ H^2_{cris}(X/W)\otimes W_r & \xleftarrow{id\otimes 1 = F^{r*}_k} & H^2_{cris}(X/W). \end{array}$$

Here the vertical arrows are the cycle maps. Set $\mathbf{f} = id\otimes F_K : N_{p,\sigma}\otimes K \to N_{p,\sigma}\otimes K$. If we regard $H^2_{cris}(X/W)$ and $H^2_{cris}(F^r_k X/W)$ as W-lattices inside $N_{p,\sigma}\otimes K$, then $H^2_{cris}(F^r_k X/W) = \mathbf{f}^r(H^2_{cris}(X/W))$. Therefore $\mathcal{K}(F^r_k X) = f^r(\mathcal{K}(X))$ in $l_{p,\sigma}\otimes k$.

Now assume X is a special supersingular K3 surface of Artin-invariant σ over k. Then the matrix A in (2.1) is the identity matrix I_σ and it is not difficult to see that $f(v_{2\sigma}) = v_1$ in the notation of section 1. It follows that $\mathcal{K}(F^{2\sigma}_k X) = \mathcal{K}(X)$. Now it is obvious that $F^{2\sigma *}_k : NS(X) \to NS(F^{2\sigma}_k X)$ sends the ample cone to the ample cone and the period space to the period space. Hence, by the crystalline Torelli theorem ([6] Theorem II, [1] Theorem 5.1.9), there exists a unique isomorphism $g_{2\sigma} : F^{2\sigma}_k X \to X$ over k such that

$$g^*_{2\sigma} = F^{2\sigma *}_k : NS(X) \to NS(F^{2\sigma}_k X).$$

We set $E = \mathbb{F}_{p^{2\sigma}}$ and $F = \mathbb{F}_{p^{2\sigma m}}$ such that X, $g_{2\sigma}$ and all the classes of $NS(X)$ are defined over F. Let α be the $p^{2\sigma}$-th power morphism of F. Then

$$\alpha^{(m-1)^*}(g_{2\sigma}) \circ \cdots \alpha^*(g_{2\sigma}) \circ g_{2\sigma} : X \to X$$

induces the identity morphism on $NS(X)$, so it is the identity map of X by the crystalline Torelli theorem. Therefore $g_{2\sigma}$ satisfies the condition of Lemma 2.1 and it gives a descent datum for $X/F/E$ and the proof is complete. □

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