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A FIXED POINT APPROACH TO THE STABILITY OF A QUADRATIC-CUBIC-QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the stability problems for a functional equation

$$f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x)-2f(2y)$$
$$+12f(y)-4f(-y)=0$$

by using the fixed point theory in the sense of L. Cădariu and V. Radu.

1. Introduction

In 1940, Ulam [14] posed the problem concerning the stability of group homomorphisms. In the following year, Hyers [7] gave an affirmative answer to this problem for additive mappings between Banach spaces. Thereafter, many mathematicians have dealt with this problem (cf. [4, 9, 10, 11, 12]).

Throughout this paper, let V and W be real vector spaces and Y be a real Banach space.

For a given mapping $f : V \to W$, we use the following abbreviations

$$\begin{aligned}
f_o(x) &:= \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2},\\
Qf(x,y) &:= f(x+y)+f(x-y)-2f(x)-2f(y),\\
Cf(x,y) &:= f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y),\\
Q'f(x,y) &:= f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y)-24f(y),\\
Ef(x,y) &:= f(x+2y)-4f(x+y)+6f(x)-4f(x-y)+f(x-2y),\\
Df(x,y) &:= f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x)-2f(2y)\\
&\quad +12f(y)-4f(-y)
\end{aligned}$$

for all $x, y \in V$. Each solution of $Qf(x,y)=0$, $Cf(x,y)=0$, and $Q'f(x,y)=0$ are called a quadratic mapping, a cubic mapping, and a quartic mapping,

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respectively. Every mapping $f : V \to W$ is called a QCQ (quadratic-cubic-quartic) mapping if f can be expressed by the sum of a quadratic mapping, a cubic mapping, and a quartic mapping. Gordji et al. [5, 6] proved the stability of the two kinds of function equations whose solutions are QCQ mappings.

Now, we consider the mapping $f : V \to W$ satisfying the following functional equation

$$Df(x, y) = 0 \tag{1}$$

for all $x, y \in V$. The mapping $f(x) = ax^4 + bx^3 + cx^2$ is a solution of this functional equation, where $f : \mathbb{R} \to \mathbb{R}$ and a, b, c are real constants.

In this paper, we will show that every solution of functional equation (1) is a QCQ mapping and we introduce a strictly contractive mapping which allows us to use the fixed point theory in the sense of L. Cădariu and V. Radu [1, 2]. And then we can adopt the fixed point method for proving the stability of the functional equation (1). Namely, starting from the given mapping f that approximately satisfies the functional equation (1), a solution F of the functional equation (1) is explicitly constructed by using the formula

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left[\frac{(-1)^{n-i}16^i}{64^n} f_o\big(2^{2n-i}x\big) + \frac{(-1)^{n-i}20^i}{64^n} f_e\big(2^{2n-i}x\big) \right]$$

or

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left[20^i(-96)^{n-i} f_o\Big(\frac{x}{2^{2n-i}}\Big) + 20^i(-64)^{n-i} f_e\Big(\frac{x}{2^{2n-i}}\Big) \right],$$

which approximates the mapping f.

2. Main theorems

The following fixed point theorem of Margolis and Diaz [3] is necessary to prove the main theorems.

Theorem 2.1. ([3], [13]) *Assume that (X, d) is a complete generalized metric space, which means that the metric d may assume infinite values, and $J : X \to X$ is a strictly contractive mapping with the Lipschitz constant $0 < L < 1$. It then holds that, for each given $x \in X$, either*

$$d\big(J^n x, J^{n+1}x\big) = \infty \text{ for all } n \in \mathbb{N}_0,$$

or there exists a nonnegative integer k such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \ge k$;*
(2) *The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $Y = \big\{y \in X : d(J^k x, y) < \infty\big\}$;*
(4) *$d\big(y, y^*\big) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Lemma 2.2. ([8]) *Suppose $f : V \to W$ is an odd mapping satisfying $Ef(x, y) = 0$ for all $x, y \in V$ and $f(2x) = 8f(x)$ for all $x \in V$. Then f is a cubic mapping.*

Lemma 2.3. ([8]) *Suppose* $f : V \to W$ *is an even mapping satisfying the equalities* $Ef(x, y) = 0$ *for all* $x, y \in V$ *and* $f(2x) = 4f(x)$ *for all* $x \in V$. *Then* f *is a quadratic mapping.*

In the following theorem, we investigate the general solution of the functional equation $Df(x, y) = 0$.

Theorem 2.4. *A mapping* $f : V \to W$ *satisfies* $Df(x, y) = 0$ *for all* $x, y \in V$ *if and only if* f *is a QCQ mapping.*

Proof. First, we assume that a mapping $f : V \to W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$. Then, the odd mapping f_o satisfies the equalities $Ef_o(x, y) = Df_o(x, y) = 0$ for all $x, y \in V$ and $f_o(2x) - 8f_o(x) = \frac{-Df_o(0,x)}{2} = 0$. It follows from Lemma 2.2 that f_o is a cubic mapping. Let $f_1(x) := \frac{-f_e(2x)+16f_e(x)}{12}$ and $f_2(x) := \frac{f_e(2x)-4f_e(x)}{12}$ for all $x \in V$.

By a direct calculation and using the assumption that $Df(x, y) = 0$ for all $x, y \in V$, we obtain the equalities $f_1(2x) = 4f_1(x)$ and $f_2(2x) = 16f_2(x)$ from the equalities

$$\begin{aligned} 4f_1(x) - f_1(2x) &= f_2(2x) - 16f_2(x) \\ &= \frac{f_e(4x) - 20f_e(2x) + 64f_e(x)}{12} \\ &= \frac{Df_e(2x, x) + 4Df_e(x, x)}{12} \end{aligned}$$

for all $x \in V$. Since $f_1(2x) = 4f_1(x)$ and $f_2(2x) = 16f_2(x)$ hold for all $x \in V$, the equalities $Ef_1(x, y) = 0$ and $Q'f_2(x, y) = 0$ can be derived from the equalities $Ef_1(x, y) = Df_1(x, y)$ and $Q'f_2(x, y) = Df_2(x, y)$ for all $x, y \in V$. According to Lemma 2.3, f_1 is a quadratic mapping. Moreover, f_2 is a quartic mapping because $Q'f_2(x, y) = 0$ holds for all $x, y \in V$.

Conversely, assume that f_1, f_2, f_3 are mappings such that the equalities $f(x) := f_1(x) + f_2(x) + f_3(x)$, $Qf_1(x, y) = 0$, $Cf_2(x, y) = 0$, and $Q'f_3(x, y) = 0$ hold for all $x, y \in V$. Then the equalities $f_1(x) = f_1(-x)$, $f_2(x) = -f_2(-x)$, $f_3(x) = f_3(-x)$ $f_1(2x) = 4f_1(x)$, $f_2(2x) = 8f_2(x)$, and $f_3(2x) = 16f_3(x)$ hold for all $x \in V$. From the above equalities, we obtain the equalities

$$\begin{aligned} Df_1(x, y) &= Qf_1(x, 2y) - 4Qf_1(x, y), \\ Df_2(x, y) &= Cf_2(x, y) - Cf_2(x - y, y), \\ Df_3(x, y) &= Q'f_3(x, y) \end{aligned}$$

for all $x, y \in V$, which mean that

$$Df(x, y) = Df_1(x, y) + Df_2(x, y) + Df_3(x, y) = 0$$

as we desired. □

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (1) by using the fixed point theory.

Theorem 2.5. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0,\infty)$ such that the inequality*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2}$$

holds for all $x, y \in V$ and $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ satisfies the condition

$$\varphi(2x,2y) \le 2(\sqrt{41}-5)L\varphi(x,y) \tag{3}$$

for all $x, y \in V$, then there exists a unique solution $F : V \to Y$ of (1) satisfying the inequality

$$\|f(x) - F(x)\| \le \frac{1}{64(1-L)}\Phi(x) \tag{4}$$

for all $x \in V$, where $\Phi(x) := \varphi(2x,x) + 4\varphi(x,x) + 2\varphi(0,x) + \varphi(-2x,-x) + 4\varphi(-x,-x) + 2\varphi(0,-x)$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left[\frac{(-1)^{n-i}16^i}{64^n} f_o\big(2^{2n-i}x\big) + \frac{(-1)^{n-i}20^i}{64^n} f_e\big(2^{2n-i}x\big)\right] \tag{5}$$

for all $x \in V$.

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g,h) = \inf\big\{K \in \mathbb{R}_+ \,:\, \|g(x) - h(x)\| \le K\Phi(x) \text{ for all } x \in V\big\}.$$

It is not difficult to see that (S, d) is a generalized complete metric space.

We now consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := -\frac{g(4x)}{64} + \frac{18g(2x)}{64} + \frac{2g(-2x)}{64}$$

for all $x \in V$. We will show that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^i}{64^n} g_o\big(2^{2n-i}x\big) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^i}{64^n} g_e\big(2^{2n-i}x\big) \tag{6}$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Clearly, the equality (6) holds for $n = 1$. We assume that the equality (6) holds for some $n \in \mathbb{N}$. Then we have

$$
\begin{aligned}
&J^{n+1}g(x)\\
&= J^n Jg(x)\\
&= \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^i}{64^n} Jg_o(2^{2n-i}x) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^i}{64^n} Jg_e(2^{2n-i}x)\\
&= \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^i}{16^n} \frac{-1}{64} g_o(2^{2n+2-i}x) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^i}{16^n} \frac{18}{64} g_o(2^{2n+1-i}x)\\
&\quad + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^i}{16^n} \frac{2}{64} g_o(-2^{2n+1-i}x) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^i}{64^n} \frac{-1}{64} g_e(2^{2n+2-i}x)\\
&\quad + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^i}{64^n} \frac{18}{64} g_e(2^{2n+1-i}x) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^i}{64^n} \frac{2}{64} g_e(-2^{2n+1-i}x)\\
&= \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n+1-i}16^i}{64^{n+1}} g_o\ (2^{2n+2-i}x) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^{i+1}}{64^{n+1}} g_o(2^{2n+1-i}x)\\
&\quad + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n+1-i}20^i}{64^{n+1}} g_e\big(2^{2n+2-i}x\big) + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^{i+1}}{64^{n+1}} g_e\big(2^{2n+1-i}x\big)\\
&= \sum_{i=0}^{n+1} {}_{n+1}C_i \frac{(-1)^{n+1-i}16^i}{64^{n+1}} g_o(2^{2n+2-i}x) + \sum_{i=0}^{n+1} {}_{n+1}C_i \frac{(-1)^{n+1-i}20^i}{64^{n+1}} g_e(2^{2n+2-i}x)
\end{aligned}
$$

holds for $n+1$. By mathematical induction, the equality (6) holds for all natural numbers.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$
\begin{aligned}
\|Jg(x) - Jh(x)\| &\leq \frac{1}{64}\|g(4x) - h(4x)\|\\
&\quad + \frac{18}{64}\|g(2x) - h(2x)\| + \frac{2}{64}\|g(-2x) - h(-2x)\|\\
&\leq K\left(\frac{1}{64}\Phi(4x) + \frac{20}{64}\Phi(2x)\right)\\
&\leq K\left(\frac{4\big(\sqrt{41}-5\big)^2}{64} L^2\Phi(x) + \frac{40\big(\sqrt{41}-5\big)}{64} L\Phi(x)\right)\\
&\leq LK\Phi(x)
\end{aligned}
$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L.

Moreover, by (2), we see that

$$\begin{aligned}\|f(x) - Jf(x)\| &= \frac{\|Df(2x,x) + 4Df(x,x) - 2Df(0,x)\|}{64} \\ &\le \frac{\varphi(2x,x) + 4\varphi(x,x) + 2\varphi(0,x)}{64} \\ &\le \frac{1}{64}\Phi(x)\end{aligned}$$

for all $x \in V$. It means that $d(f, Jf) \le \frac{1}{64}$ by the definition of d. Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S : d(f,g) < \infty\}$, which is represented by (5) for all $x \in V$.

Notice that

$$d(f,F) \le \frac{1}{1-L} d(f,Jf) \le \frac{1}{64(1-L)},$$

which implies (4).

By the definition of F, together with (2) and (3), we have

$$\begin{aligned}&\|DF(x,y)\| \\ &= \lim_{n\to\infty} \|DJ^n f(x,y)\| \\ &= \lim_{n\to\infty} \Bigg\| \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}16^i}{64^n} Df_o\big(2^{2n-i}x, 2^{2n-i}y\big) \\ &\qquad\qquad + \sum_{i=0}^{n} {}_nC_i \frac{(-1)^{n-i}20^i}{64^n} Df_e\big(2^{2n-i}x, 2^{2n-i}y\big) \Bigg\| \\ &\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{20^i}{64^n} \Big(\varphi\big(2^{2n-i}x, 2^{2n-i}y\big) + \varphi\big(-2^{2n-i}x, -2^{2n-i}y\big)\Big) \\ &\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{20^i[2(\sqrt{41}-5)]^{n-i}}{64^n} L^{n-i} \Big(\varphi\big(2^n x, 2^n y\big) + \varphi\big(-2^n x, -2^n y\big)\Big) \\ &\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \frac{\big(\sqrt{41}-5\big)^{n-i}10^i}{32^n} \Big(\varphi\big(2^n x, 2^n y\big) + \varphi\big(-2^n x, -2^n y\big)\Big) \\ &\le \lim_{n\to\infty} \frac{\big(\sqrt{41}+5\big)^n}{32^n} \Big(\varphi\big(2^n x, 2^n y\big) + \varphi\big(-2^n x, -2^n y\big)\Big) \\ &\le \lim_{n\to\infty} \frac{\big(\sqrt{41}+5\big)^n\big(\sqrt{41}-5\big)^n}{16^n} L^n \big(\varphi(x,y) + \varphi(-x,-y)\big) \\ &\le \lim_{n\to\infty} L^n \big(\varphi(x,y) + \varphi(-x,-y)\big) \\ &= 0\end{aligned}$$

for all $x, y \in V$, *i.e.*, F is a solution of the functional equation (1). Notice that if F is a solution of the functional equation (1), then the equality

$$F(x) - JF(x) = \frac{DF(2x, x) + 4DF(x, x) - 2DF(0, x)}{64}$$

implies that F is a fixed point of J. □

Theorem 2.6. *Let $f : V \to Y$ be a mapping for which there exists a mapping $\varphi : V^2 \to [0, \infty)$ such that the inequality (2) holds for all $x, y \in V$ and $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ satisfies the condition*

$$L\varphi(2x, 2y) \geq 24\varphi(x, y) \tag{7}$$

for all $x, y \in V$, then there exists a unique solution $F : V \to Y$ of (1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \frac{L^2\Phi(x)}{24^2(1 - L)} \tag{8}$$

for all $x \in V$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i \left[20^i(-96)^{n-i} f_o\left(\frac{x}{2^{2n-i}}\right) + 20^i(-64)^{n-i} f_e\left(\frac{x}{2^{2n-i}}\right) \right] \tag{9}$$

for all $x \in V$.

Proof. Let us define the set (S, d) as in the proof of Theorem 2.5. We now consider the mapping $J : S \to S$ defined by

$$Jg(x) := 20g\left(\frac{x}{2}\right) - 80g\left(\frac{x}{4}\right) + 16g\left(\frac{-x}{4}\right)$$

for all $x \in V$. Notice that the equality

$$J^n g(x) = \sum_{i=0}^{n} {}_nC_i 20^i(-96)^{n-i} g_o\left(\frac{x}{2^{2n-i}}\right) + \sum_{i=0}^{n} {}_nC_i 20^i(-64)^{n-i} g_e\left(\frac{x}{2^{2n-i}}\right)$$

holds for all $n \in \mathbb{N}$ and $x \in V$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &\leq 20\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| \\
&\quad + 80\left\|g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right)\right\| + 16\left\|g\left(\frac{-x}{4}\right) - h\left(\frac{-x}{4}\right)\right\| \\
&\leq 96K\Phi\left(\frac{x}{4}\right) + 20K\Phi\left(\frac{x}{2}\right) \\
&\leq L^2\frac{96}{24^2}K\Phi(x) + \frac{20}{24}LK\Phi(x) \\
&\leq LK\Phi(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L.

Moreover, by (2), we see that

$$\begin{aligned}
\|f(x) - Jf(x)\| &= \left\| Df\left(\frac{x}{2}, \frac{x}{4}\right) + 4Df\left(\frac{x}{4}, \frac{x}{4}\right) \right\| \\
&\leq \varphi\left(\frac{x}{2}, \frac{x}{4}\right) + 4\varphi\left(\frac{x}{4}, \frac{x}{4}\right) \\
&\leq \frac{L^2}{24^2}\big(\varphi(2x, x) + 4\varphi(x, x)\big) \\
&\leq \frac{L^2}{24^2}\Phi(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(f, Jf) \leq \frac{L^2}{24^2} < \infty$$

by the definition of d.

Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (9) for all $x \in V$. Notice that

$$d(f, F) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{L^2}{24^2(1 - L)},$$

which implies the validity of (8).

By the definition of F, together with (2) and (7), we have

$$\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty} \|DJ^n f(x,y)\| \\
&= \lim_{n\to\infty} \Bigg\| \sum_{i=0}^{n} {}_nC_i(-96)^{n-i}20^i f_o\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \\
&\qquad + \sum_{i=0}^{n} {}_nC_i(-64)^{n-i}20^i f_e\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right)\Bigg\| \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} \frac{{}_nC_i}{2}\big(96^{n-i}20^i + 64^{n-i}20^i\big)\times \\
&\qquad \times \left(\varphi\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) + \varphi\left(\frac{-x}{2^{2n-i}}, \frac{-y}{2^{2n-i}}\right)\right) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 96^{n-i}20^i \left(\varphi\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) + \varphi\left(\frac{-x}{2^{2n-i}}, \frac{-y}{2^{2n-i}}\right)\right) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 96^{n-i}20^i \frac{L^{n-i}}{24^{n-i}} \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right) \\
&\le \lim_{n\to\infty} \sum_{i=0}^{n} {}_nC_i 20^i 4^{n-i} \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right) \\
&\le \lim_{n\to\infty} 24^n \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right)\right) \\
&\le \lim_{n\to\infty} 24^n \frac{L^n}{24^n}\big(\varphi(x,y) + \varphi(-x,-y)\big) \\
&\le \lim_{n\to\infty} L^n\big(\varphi(x,y) + \varphi(-x,-y)\big) \\
&= 0
\end{aligned}$$

for all $x, y \in V$, *i.e.*, F is a solution of the functional equation (1). Notice that if F is a solution of the functional equation (1), then the equality

$$F(x) - JF(x) = DF\left(\frac{x}{2}, \frac{x}{4}\right) + 4DF\left(\frac{x}{4}, \frac{x}{4}\right)$$

implies that F is a fixed point of J. □

References

[1] L. Cădariu and V. Radu, *Fixed points and the stability of quadratic functional equations*, An. Univ. Timisoara Ser. Mat.-Inform. **41** (2003), 25–48.

[2] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach in Iteration Theory*, Grazer Mathematische Berichte, Karl-Franzens-Universitäet, Graz, Graz, Austria **346** (2004), 43–52.

[3] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[5] M. E. Gordji, S. Kaboli, and S. Zolfaghari, *Stability of a mixed type quadratic, cubic and quartic functional equation*, arXiv preprint arXiv:0812.2939 (2008).
[6] M. E. Gordji, H. Khodaei, and R. Khodabakhsh, *General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces*, U.P.B. Sci. Bull., Series A, **72** Iss. 3, (2010) 69–84.
[7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[8] Y.-H. Lee, *Hyers-Ulam-Rassias stability of a quadratic-additive type functional equation on a restricted domain*, Int. J. Math. Anal. (Ruse) **7** (2013), 2745–2752.
[9] Y.-H. Lee, *Stability of a monomial functional equation on a restricted domain*, Mathematics **5(4)** (2017), 53.
[10] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quartic functional equation*, East Asian Math. J. **35** (2019), 351–356.
[11] Y.-H. Lee, *On the stability of a general quadratic-cubic functional equation in non-archimedean normed spaces*, East Asian Math. J. **35** (2019), 331–340.
[12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[13] I. A. Rus, *Principles and Applications of Fixed Point Theory*, Ed. Dacia, Cluj-Napoca 1979 (in Romanian).
[14] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

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