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# NECESSARY CONDITIONS FOR OPTIMAL CONTROL GOVERNED BY SOME ODE-PDE SYSTEMS

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Abstract. This paper is concerned with the necessary conditions for optimal control problem governed by some ODE-PDE systems. That is, we obtain the necessary conditions for optimal control by showing the differentiability of the solution with respect to the control.

## 1. Introduction

In this paper, we consider the necessary conditions for optimal control problem governed by some controlled ODE-PDE systems

$$\begin{aligned}
&\frac{\partial y}{\partial t} = d\frac{\partial^2 y}{\partial x^2} - \gamma(\rho)y - fy + g\rho \qquad \text{in } I \times (0,T],\\
&\frac{\partial \rho}{\partial t} = fy - h\rho - u(t)\rho \qquad \text{in } I \times (0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = 0 \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } I.
\end{aligned} \tag{1.1}$$

Here, $I = (0, L)$ is a bounded interval in $\mathbb{R}$. $y(x,t)$ and $\rho(x,t)$ are the variable representing tree densities of young age and old age class in $I$ at time $t$. $d > 0$ is the diffusion rate of the young trees. $g > 0$ is fertility of the species. $h > 0$ and $f > 0$ denote death and aging rates. $\gamma(\rho)$ is a mortality rate function of the young trees with $\gamma(\rho) = a(\rho - b)^2 + c$ $(a, b, c > 0)$. $u(t)$ denotes the control term.

More precisely, we consider the necessary conditions for optimal control minimizing the cost functional $\mathcal{J}(u)$ of the form

$$\mathcal{J}(u) = \int_0^T \|y(u) - y_d\|^2_{L^2(I)} dt + \int_0^T \|\rho(u) - \rho_d\|^2_{L^2(I)} dt + \gamma\|u\|^2_{L^2(0,T)},$$

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where $y(u)$ and $\rho(u)$ are the solution of (1.1) corresponding to $u$. Here, $y_d$ and $\rho_d$ are the given ideal (or desired) state and $\gamma > 0$ is the control parameter.

A simple model (1.1) for mono species distribution with mixed ages in forest was introduced by Antonovsky et al.([1]). In [2], authors analyzed time behavior of small perturbations of the standing front as solutions of (1.1). In [6], the author showed the global existence of strong solution and the stability of the solution with respect to the control.

The optimal control problem for the ODE-PDE systems was studied in a few papers. The optimal control problem for FitzHugh-Nagumo equation was considered in [4]. In [3], the optimal control problem for prey-predator as the ODE-PDE systems was studied. Ryu([5]) considered the local existence of strong solution for (1.1) and the existence of optimal control. In this paper, we obtain the necessary conditions for optimal control by showing the differentiability of the solution with respect to the control.

The paper is organized as follows. Section 2 is a preliminary section reviewing the global existence of strong solution for (1.1) and the existence of optimal control. The necessary conditions for optimal control are derived in Section 3.

**Notations.** Let $J$ be an interval in the real line $\mathbb{R}$. Let $\mathcal{H}$ be a Hilbert space. $L^p(J;\mathcal{H})$, $1 \le p \le \infty$, denotes the $L^p$ space of $\mathcal{H}$-valued measurable functions in $J$. $\mathcal{C}(J;\mathcal{H})$ denotes the space of $\mathcal{H}$-valued continuous functions in $J$. We denote $W^{1,2}(J;\mathcal{H}) = \{y : y^{(j)} \in L^2(J;\mathcal{H}),\ j = 0,1\}$, where $y^{(j)}$ is the derivative of order $j$ of $y$ in the sense of distributions. For simplicity, we shall use a universal constant $C$ to denote various constants which are determined in each occurrence in a specific way by $a, b, c, d, f, g, h$, $m$, $l$ and $I$.

## 2. Preliminaries

In this section, we recall the global existence of strong solution for (1.1) and the existence of optimal control(see [5] and [6]). We rewrite (1.1) as an abstract semilinear equation in a Hilbert space $\mathcal{H} = L^2(I) \times L^2(I)$. To this end, let us define the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ as follows:

$$AY = \begin{pmatrix} d\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \rho \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in D(A).$$

Here, $D(A) = \left\{ Y = \binom{y}{\rho} \in H^2(I) \times L^\infty(I), \frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(L) = 0 \right\}$. Then $A$ is a self adjoint dissipative operator in $\mathcal{H}$.

Let $F(t, Y(t)) : [0,T] \times \mathcal{H} \to \mathcal{H}$ be the nonlinear operator defined by

$$F(t, Y(t)) = \begin{pmatrix} -\gamma(\rho)y - fy + g\rho \\ fy - h\rho - u(t)\rho \end{pmatrix}.$$

Then, (1.1) is expressed as an abstract semilinear equation

$$\begin{aligned}&\frac{dY}{dt} + AY = F(t, Y(t)), \quad 0 < t \leq T,\\ &Y(0) = Y_0\end{aligned}$$

in the space $\mathcal{H}$. Here, $Y_0 = \binom{y_0}{\rho_0}$ and $U_{ad} = \{u \in H^1(0,T); \|u\|_{H^1(0,T)} \leq m,\ 0 \leq u(t) \leq l\}$. Using a truncation procedure for $F(\cdot, Y(\cdot))$ and some estimates, we have the followig result for the global strong solution to (1.1)(see [5] and [6]).

**Proposition 2.1.** *For any $0 \leq y_0 \in H^2(I)$ and $0 \leq \rho_0 \in H^1(I)$ and $u \in U_{ad}$, (1.1) has a unique global strong solution $Y = \binom{y}{\rho} \in W^{1,2}(0,T;\mathcal{H})$ such that*

$$\begin{aligned}&0 \leq y \in L^\infty((0,T)\times I) \cap L^\infty(0,T;H^1(I)) \cap L^2(0,T;H^2(I)),\\ &0 \leq \rho \in L^\infty((0,T)\times I) \cap L^\infty(0,T;H^1(I)).\end{aligned}$$

*Moreover, the estimates*

$$\left\|\frac{\partial y}{\partial t}\right\|_{L^2(0,T;L^2(I))} + \|y\|_{L^2(0,T;H^2(I))} + \|y\|_{H^1(I)} + \|y\|_{L^\infty((0,T)\times I)} \leq C \tag{2.1}$$

*and*

$$\left\|\frac{\partial \rho}{\partial t}\right\|_{L^2(0,T;L^2(I))} + \|\rho\|_{L^\infty((0,T)\times I)} + \|\rho\|_{H^1(I)} \leq C \tag{2.2}$$

*hold, where $C$ is also determined by $\|y_0\|_{L^\infty(I)}$ and $\|\rho_0\|_{L^\infty(I)}$.*

Moreover, we obtain the continuous dependence of solution on the control(see [6]).

**Proposition 2.2.** *For any $u_1, u_2 \in U_{ad}$, we have*

$$\begin{aligned}&\|y_1(t) - y_2(t)\|^2_{L^2(I)} + \|\rho_1(t) - \rho_2(t)\|^2_{L^2(I)}\\ &\qquad \leq C\|u_1(t) - u_2(t)\|^2_{H^1(0,T)}, \quad 0 \leq t \leq T, \end{aligned}\tag{2.3}$$

*where $(y_1, \rho_1)$ and $(y_2, \rho_2)$ are the solutions of (1.1) with respect to $u_1$ and $u_2$, respectively.*

Let $T > 0$ be such that for each $u \in U_{ad}$, (1.1) has a unique strong solution $Y(u) = \binom{y(u)}{\rho(u)} \in W^{1,2}(0,T;\mathcal{H})$ satisfying (2.1) and (2.2). Thus, we consider the following optimal control problem

$$(\mathbf{P}) \qquad \text{minimize } \ \mathcal{J}(u)$$

with the cost functional $\mathcal{J}(u)$ of the form

$$\begin{aligned}\mathcal{J}(u) = \int_0^T \|y(u) - y_d\|^2_{L^2(I)}dt + \int_0^T \|\rho(u) - \rho_d\|^2_{L^2(I)}dt&\\ + \gamma\|u\|^2_{L^2(0,T)}, \quad u \in U_{ad}.&\end{aligned}$$

Here, $y_d, \rho_d \in L^2(0,T;H^1(I))$ are fixed elements. $\gamma$ is a positive constant. $U_{ad} = \{u \in H^1(0,T); \|u\|_{H^1(0,T)} \leq m,\ 0 \leq u(t) \leq l\}$. Then, we obtain the existence result(see [5]).

**Proposition 2.3.** *There exists an optimal control* $\bar{u} \in U_{ad}$ *for* $(\mathbf{P})$ *such that* $\mathcal{J}(\bar{u}) = \min_{u \in U_{ad}} \mathcal{J}(u)$.

## 3. Necessary conditions of optimal control

In this section, we derive the ncessary conditions for optimal control by showing the differentiability of the solution with respect to the control. Let $Y = \binom{y}{\rho}$ be a state solution of (1.1) corresponding to the control $u$. At first, we consider the following linear systems

$$\begin{aligned} &\frac{dZ}{dt} + AZ - F_Y(t, Y)Z = B_v(Y), \quad 0 < t \leq T, \\ &Z(0) = 0 \end{aligned}$$

in the space $\mathcal{H}$. Here,

$$F_Y(t, Y)Z = \begin{pmatrix} -\gamma(\rho)z - 2a(\rho - b)yw - fz + gw \\ fz - hw - u(t)w \end{pmatrix}, \; B_v(Y) = \begin{pmatrix} 0 \\ v(t)\rho \end{pmatrix}, \; Z = \begin{pmatrix} z \\ w \end{pmatrix}.$$

In detail, this can be written as

$$\begin{aligned} &\frac{\partial z}{\partial t} = d\frac{\partial^2 z}{\partial x^2} - \gamma(\rho)z - 2a(\rho - b)yw - fz + gw \quad \text{in } I \times (0, T], \\ &\frac{\partial w}{\partial t} = fz - hw - u(t)w + v(t)\rho \quad \text{in } I \times (0, T], \\ &\frac{\partial z}{\partial x}(0, t) = \frac{\partial z}{\partial x}(L, t) = 0 \quad \text{on } (0, T], \\ &z(x, 0) = 0, \quad w(x, 0) = 0 \quad \text{in } I. \end{aligned} \tag{3.1}$$

**Proposition 3.1.** *Under the hypotheses of Proposition 2.1, if* $u, v \in U_{ad}$ *and* $Y = \binom{y}{\rho}$ *be a strong solution of (1.1) corresponding to* $u$*, then (3.1) has a unique strong solution* $Z = \binom{z}{w} \in W^{1,2}(0, T; \mathcal{H})$ *such that*

$$\begin{aligned} &z \in L^\infty((0, T) \times I) \cap L^\infty(0, T; H^1(I)) \cap L^2(0, T; H^2(I)), \\ &w \in L^\infty((0, T) \times I) \cap L^\infty(0, T; L^2(I)). \end{aligned}$$

*Proof.* The result can be shown completely analogously to Proposition 2.1. Since (3.1) is linear, a truncation procedure is not needed. Then, we obtain that there exists a unique strong sloution $Z = \binom{z}{w} \in W^{1,2}(0, T; \mathcal{H})$ such that $z \in L^\infty(0, T; H^1(I)) \cap L^2(0, T; H^2(I))$.

Now, we will show $z, w \in L^\infty((0, T) \times I)$ by three steps.

*Step 1.* Multiply the first equation of (3.1) by $z$ and integrate the product in $I$. Then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L z^2dx+\delta\|z\|^2_{H^1(I)}\\
&\quad\le\int_0^L z^2dx-\int_0^L\big(\gamma(\rho)+f\big)z^2dx-2a\int_0^L(\rho-b)ywzdx+g\int_0^L wzdx\\
&\qquad\le C\Big(\|y\|^2_{L^\infty(I)}+\|\rho\|^2_{L^\infty(I)}+1\Big)\Big(\|z\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\Big), \qquad (3.2)
\end{aligned}$$

where $\delta=\min\{d,1\}$. Multiply the second equation of (3.1) by $w$ and integrate the product in $I$. Then, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^L w^2dx&=f\int_0^L zwdx-\big(h+u(t)\big)\int_0^L w^2dx+\int_0^L v(t)\rho wdx \qquad (3.3)\\
&\le C\Big(\|z\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\Big)+\frac{1}{2}v^2(t)\|\rho\|^2_{L^2(I)}.
\end{aligned}$$

Thus, we obtain from (3.2) and (3.3) that

$$\begin{aligned}
&\frac{d}{dt}\int_0^L(z^2+w^2)dx+2\delta\|z\|^2_{H^1(I)}\\
&\le C\Big(\|y\|^2_{L^\infty(I)}+\|\rho\|^2_{L^\infty(I)}+1\Big)\Big(\|z\|^2_{L^2(I)}+\|w\|^2_{L^2(I)}\Big)+v^2(t)\|\rho\|^2_{L^2(I)}. \qquad (3.4)
\end{aligned}$$

By using Gronwall's Lemma, we have

$$\|z(t)\|^2_{L^2(I)}+\|w(t)\|^2_{L^2(I)}\le C,\quad 0\le t\le T. \qquad (3.5)$$

If we use (3.2), we obtain

$$\int_0^t\|z(s)\|^2_{H^1(I)}ds\le C,\quad 0\le t\le T. \qquad (3.6)$$

*Step 2. We will estimate the norm $\|w\|_{L^\infty(I)}$.* Since $u(t),v(t)$ and $\rho(x,t)$ are non-negative, we have

$$\begin{aligned}
w(t)=&f\int_0^t e^{-\int_s^t(h+u(\tau))d\tau}z(x,s)ds+\int_0^t e^{-\int_s^t(h+u(\tau))d\tau}v(s)\rho(x,s)ds \qquad (3.7)\\
\le&f\int_0^t e^{-h(t-s)}|z(x,s)|ds+\int_0^t e^{-h(t-s)}v(s)\rho(x,s)ds.
\end{aligned}$$

Since

$$\begin{aligned}
\Big\|\int_0^t e^{-h(t-s)}|z(x,s)|ds\Big\|_{H^1(I)}&\le\int_0^t e^{-h(t-s)}\|z(x,s)\|_{H^1(I)}ds \qquad (3.8)\\
&\le\frac{1}{\sqrt{2h}}\|z\|_{L^2((0,S;H^1(I))}
\end{aligned}$$

and

$$\Big\| \int_0^t e^{-h(t-s)} v(s)\rho(x,s)ds \Big\|_{H^1(I)} \le \int_0^t e^{-h(t-s)} \|v(s)\rho(x,s)\|_{H^1(I)} ds \qquad (3.9)$$
$$\le \frac{1}{\sqrt{2h}} \|v(t)\|_{L^2(0,T)} \|\rho\|_{L^\infty((0,T;H^1(I))},$$

we obtain from (3.6), (3.7), (3,8), (3.9) and $\rho \in L^\infty(0,T;H^1(I))$ that

$$\|w\|_{H^1(I)} \le C.$$

Since $H^1(I) \subset L^\infty(I)$, we have

$$\|w\|_{L^\infty(I)} \le C. \qquad (3.10)$$

*Step 3. We will estimate the norm* $\|z\|_{L^\infty(I)}$. Multiply the first equation of (3.1) by $-\frac{\partial^2 z}{\partial x^2}$ and integrate the product in $I$. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Big|\frac{\partial z}{\partial x}\Big|^2 dx + d\int_0^L \Big|\frac{\partial^2 z}{\partial x^2}\Big|^2 dx$$
$$= \int_0^L \gamma(\rho) z \frac{\partial^2 z}{\partial x^2} dx + 2a\int_0^L (\rho - b) y w \frac{\partial^2 z}{\partial x^2} dx + f\int_0^L z\frac{\partial^2 z}{\partial x^2} dx - g\int_0^L w \frac{\partial^2 z}{\partial x^2} dx.$$

Therefore, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Big|\frac{\partial z}{\partial x}\Big|^2 dx + f\int_0^L \Big|\frac{\partial z}{\partial x}\Big|^2 dx \le C\int_0^L \Big(w^2 + \gamma(\rho)^2 z^2 + (\rho - b)^2 y^2 w^2\Big) dx.$$

Since

$$\int_0^L \Big(w^2 + \gamma(\rho)^2 z^2 + (\rho - b)^2 y^2 w^2\Big) dx$$
$$\le C\Big(\|\rho\|^4_{L^\infty(I)} + \|y\|^4_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)} + 1\Big)\Big(\|z\|^2_{L^2(I)} + \|w\|^2_{L^2(I)}\Big),$$

it follows from (2.1), (2.2) and (3.5) that

$$\frac{d}{dt}\int_0^L \Big|\frac{\partial z}{\partial x}\Big|^2 dx + 2f\int_0^L \Big|\frac{\partial z}{\partial x}\Big|^2 dx \le C, \quad 0 \le t \le T.$$

If we solve the differential inequality, we have

$$\Big\|\frac{\partial z}{\partial x}\Big\|^2_{L^2(I)} \le C, \quad 0 \le t \le T. \qquad (3.11)$$

From (3.5) and (3.11). we obtain

$$\|z\|_{L^\infty(I)} \le C\|z\|_{H^1(I)} \le C.$$

$\square$

**Proposition 3.2.** *The mapping $u \to Y(u)$ from $U_{ad}$ into $W^{1,2}(0,T;\mathcal{H})$ is differentiable in the sense*

$$\frac{Y(u_\epsilon)-Y(u)}{\epsilon} \to Z \text{ strongly in } L^2(0,T;\mathcal{H})$$

*as $\epsilon \to 0$, where $u, v \in U_{ad}$ and $u_\epsilon = u + \epsilon v \in U_{ad}$. Here, $Y(u_\epsilon) = \binom{y(u_\epsilon)}{\rho(u_\epsilon)}$ and $Y(u) = \binom{y(u)}{\rho(u)}$ are the solution of (1.1) corresponding to $u_\epsilon$ and $u$, respectively. Moreover, $Z = \binom{z}{w}$ satisfies the linear equations (3.1).*

*Proof.* Let $u, v \in U_{ad}$ and $0 < \epsilon \le 1$. Let $Y_\epsilon = \binom{y_\epsilon}{\rho_\epsilon}$ and $Y = \binom{y}{\rho}$ be the strong solutions of (1.1) corresponding to $u_\epsilon$ and $u$, respectively.

*Step 1. $Y_\epsilon \to Y$ strongly in $L^2(0,T;\mathcal{H})$ as $\epsilon \to 0$.* By using (2.3), we obtain

$$\begin{aligned}&\|y_\epsilon(t)-y(t)\|^2_{L^2(I)} + \|\rho_\epsilon(t)-\rho(t)\|^2_{L^2(I)}\\ &\qquad \le C\|u_\epsilon(t)-u(t)\|^2_{H^1(0,T)} \le \epsilon C\|v(t\|^2_{H^1(0,T)}, \quad 0 \le t \le T.\end{aligned}$$

As $\epsilon \to 0$, we infer that

$$y_\epsilon \to y,\ \rho_\epsilon \to \rho \text{ strongly in } L^2(0,T;L^2(I)). \tag{3.12}$$

*Step 2.* If we let $z_\epsilon = \frac{y_\epsilon - y}{\epsilon}$ and $w_\epsilon = \frac{\rho_\epsilon-\rho}{\epsilon}$, then $Z_\epsilon = \binom{z_\epsilon}{w_\epsilon}$ satisfy the following equations

$$\begin{aligned}&\frac{\partial z_\epsilon}{\partial t} = d\frac{\partial^2 z_\epsilon}{\partial x^2} - \gamma(\rho_\epsilon)z_\epsilon - a(\rho_\epsilon+\rho-2b)yw_\epsilon - fz_\epsilon + gw_\epsilon \quad \text{in } I\times(0,T],\\ &\frac{\partial w_\epsilon}{\partial t} = fz_\epsilon - hw_\epsilon - u_\epsilon(t)w_\epsilon + v(t)\rho \quad \text{in } I\times(0,T],\\ &\frac{\partial z_\epsilon}{\partial x}(0,t) = \frac{\partial z_\epsilon}{\partial x}(L,t) = 0 \quad \text{on } (0,T],\\ &z_\epsilon(x,0)=0, \quad w_\epsilon(x,0)=0 \quad \text{in } I.\end{aligned} \tag{3.13}$$

Indeed, by using the similar methods used in Proposition 3.1, we see that there exists a unique strong solution $Z_\epsilon = \binom{z_\epsilon}{w_\epsilon} \in W^{1,2}(0,T;\mathcal{H})$ such that

$$\begin{aligned}&z_\epsilon \in L^\infty((0,T)\times I)\cap L^\infty(0,T;H^1(I))\cap L^2(0,T;H^2(I)),\\ &w_\epsilon \in L^\infty((0,T)\times I)\cap L^\infty(0,T;L^2(I)).\end{aligned}$$

*Step 3. $Z_\epsilon \to Z$ strongly in $L^2(0,T;\mathcal{H})$ as $\epsilon \to 0$.* From (3.1) and (3.13), we see that $\tilde{z} = z_\epsilon - z$ and $\tilde{w} = w_\epsilon - w$ satisfy the following equations

$$\begin{aligned}
\frac{\partial \tilde{z}}{\partial t} =& d\frac{\partial^2 \tilde{z}}{\partial x^2} - \big[\gamma(\rho_\epsilon)\tilde{z} + \big(\gamma(\rho_\epsilon) - \gamma(\rho)\big)z\big] \\
&- a\big[(\rho_\epsilon + \rho - 2b)\tilde{w} + (\rho_\epsilon - \rho)w\big]y - f\tilde{z} + g\tilde{w} \quad \text{in } I \times (0,T], \\
\frac{\partial \tilde{w}}{\partial t} =& f\tilde{z} - h\tilde{w} - u_\epsilon(t)\tilde{w} + \epsilon v(t)w \quad \text{in } I \times (0,T], \\
\frac{\partial \tilde{z}}{\partial x}&(0,t) = \frac{\partial \tilde{z}}{\partial x}(L,t) = 0 \quad \text{on } (0,T], \\
\tilde{z}(x,0) &= 0, \quad \tilde{w}(x,0) = 0 \quad \text{in } I.
\end{aligned} \tag{3.14}$$

Multiply the first equation of (3.14) by $\tilde{z}$ and integrate the product in $I$. Then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \tilde{z}^2 dx + d\int_0^L \Big|\frac{\partial \tilde{z}}{\partial x}\Big|^2 dx \\
&= -\int_0^L \gamma(\rho_\epsilon)\tilde{z}^2 dx - \int_0^L \big(\gamma(\rho_\epsilon) - \gamma(\rho)\big)z\tilde{z}dx - a\int_0^L (\rho_\epsilon + \rho - 2b)y\tilde{w}\tilde{z}dx \\
&\quad - a\int_0^L (\rho_\epsilon - \rho)yw\tilde{z}dx - f\int_0^L \tilde{z}^2 dx + g\int_0^L \tilde{w}\tilde{z}dx.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^L (\rho_\epsilon^2 - \rho^2)z\tilde{z}dx &\le C\Big(\|\rho_\epsilon\|^2_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)}\Big)\int_0^L (\rho_\epsilon - \rho)^2 z^2 dx + C\|\tilde{z}\|^2_{L^2(I)} \\
&\le C\Big(\|\rho_\epsilon\|^2_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)}\Big)\|z\|^2_{L^\infty(I)}\|\rho_\epsilon - \rho\|^2_{L^2(I)} + C\|\tilde{z}\|^2_{L^2(I)},
\end{aligned}$$

$$\int_0^L \rho y\tilde{w}\tilde{z}dx \le C\Big(\|y\|^2_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)}\Big)\Big(\|\tilde{z}\|^2_{L^2(I)} + \|\tilde{w}\|^2_{L^2(I)}\Big)$$

and

$$\int_0^L (\rho_\epsilon - \rho)yw\tilde{z}dx \le C\|y\|^2_{L^\infty(I)}\|w\|^2_{L^\infty(I)}\|\rho_\epsilon - \rho\|^2_{L^2(I)} + C\|\tilde{z}\|^2_{L^2(I)},$$

we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^L \tilde{z}^2 dx &\le C\Big(\|y\|^2_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)} + \|\rho_\epsilon\|^2_{L^\infty(I)} + 1\Big)\Big(\|\tilde{z}\|^2_{L^2(I)} + \|\tilde{w}\|^2_{L^2(I)}\Big) \\
+ C\Big(\|\rho_\epsilon\|^2_{L^\infty(I)} &+ \|\rho\|^2_{L^\infty(I)} + \|y\|^2_{L^\infty(I)}\Big)\Big(\|z\|^2_{L^\infty(I)} + \|w\|^2_{L^\infty(I)}\Big)\|\rho_\epsilon - \rho\|^2_{L^2(I)}.
\end{aligned} \tag{3.15}$$

Multiply the second equation of (3.14) by $\tilde{w}$ and integrate the product in $I$. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L \tilde{w}^2 dx &= f\int_0^L \tilde{z}\tilde{w}dx - \int_0^L u_\epsilon(t)\tilde{w}^2 dx - \epsilon\int_0^L v(t)w\tilde{w}dx \\ &\le C\Big(\|\tilde{z}\|^2_{L^2(I)} + \|\tilde{w}\|^2_{L^2(I)}\Big) + \frac{\epsilon^2}{2}v^2\|w\|^2_{L^2(I)}. \quad (3.16)\end{aligned}$$

Then, we obtain from (3.15) and (3.16) that

$$\begin{aligned}&\frac{d}{dt}\int_0^L (\tilde{z}^2+\tilde{w}^2)dx \le C\Big(\|y\|^2_{L^\infty(I)}+\|\rho\|^2_{L^\infty(I)}+\|\rho_\epsilon\|^2_{L^\infty(I)}+1\Big)\Big(\|\tilde{z}\|^2_{L^2(I)}+\|\tilde{w}\|^2_{L^2(I)}\Big) \\ &+C\Big(\|\rho_\epsilon\|^2_{L^\infty(I)}+\|\rho\|^2_{L^\infty(I)}+\|y\|^2_{L^\infty(I)}\Big)\Big(\|z\|^2_{L^\infty(I)}+\|w\|^2_{L^\infty(I)}\Big)\|\rho_\epsilon-\rho\|^2_{L^2(I)}+\epsilon^2v^2\|w\|^2_{L^2(I)}.\end{aligned}$$

By using Gronwall's Lemma and (3.12), we have

$$\begin{aligned}\int_0^L (\tilde{z}^2 + \tilde{w}^2)dx \le\, &C\Big(\|\rho_\epsilon\|^2_{L^\infty((0,T))\times I)} + \|\rho\|^2_{L^\infty((0,T))\times I)} + \|y\|^2_{L^\infty((0,T))\times I)}\Big) \\ &\times\Big(\|z\|^2_{L^\infty((0,T))\times I)} + \|w\|^2_{L^\infty((0,T))\times I)}\Big)\|\rho_\epsilon - \rho\|^2_{L^2(0,T;L^2(I))} \\ &+ \epsilon^2\|v\|^2_{L^2(I)}\|w\|^2_{L^\infty(0,T;L^2(I))} \\ &\to 0\end{aligned}$$

as $\epsilon \to 0$. Hence, we infer that

$$z_\epsilon \to z,\ w_\epsilon \to w \text{ strongly in } L^2(0,T;L^2(I)).$$

$\square$

**Theorem 3.3.** *Let $\bar{u}$ be an optimal control of $(\mathbf{P})$ and let $\overline{Y} = \binom{\bar{y}}{\bar{\rho}}$ be the optimal state, that is $\overline{Y}$ is the solution to (1.1) with the control $\bar{u}$. Then, there exists a unique solution $P = \binom{p_1}{p_2} \in W^{1,2}(0,T;\mathcal{H})$ such that*

$$\begin{aligned}&p_1 \in L^\infty((0,T)\times I)\cap L^\infty(0,T;H^1(I))\cap L^2(0,T;H^2(I)),\\ &p_2 \in L^\infty((0,T)\times I)\cap L^\infty(0,T;L^2(I))\end{aligned}$$

*to the linear problem*

$$\begin{aligned}&-\frac{\partial p_1}{\partial t} - d\frac{\partial^2 p_1}{\partial x^2} + \gamma(\bar{\rho})p_1 + fp_1 - fp_2 = \bar{y} - y_d \quad \text{in } I\times(0,T],\\ &-\frac{\partial p_2}{\partial t} + 2a(\bar{\rho}-b)\bar{y}p_1 - gp_1 + hp_2 + \bar{u}(t)p_2 = \bar{\rho} - \rho_d \quad \text{in } I\times(0,T], \quad (3.17)\\ &\frac{\partial p_1}{\partial x}(0,t) = \frac{\partial p_1}{\partial x}(L,t) = 0 \quad \text{on } (0,T],\\ &p_1(x,T) = 0, \quad p_2(x,T) = 0 \quad \text{in } I.\end{aligned}$$

*Moreover, $\bar{u}$ satisfies*

$$\int_0^T \langle p_2, (v-\bar{u})\bar{\rho}\rangle_{L^2(I)} dt + \gamma\langle \bar{u}, v-\bar{u}\rangle_{L^2(0,T)} \geq 0 \quad \textit{for all } v \in U_{ad}.$$

*Proof.* The existence and uniqueness of the strong solution for (3.17) can be obtained by employing the same methods used in Proposition 2.1. Let $\bar{u}$ be an optimal control of (**P**). Let $\bar{u}$ and $u_\epsilon = \bar{u} + \epsilon(v - \bar{u}) \in U_{ad}$, $0 < \epsilon \leq 1$. Since $\bar{u}$ is optimal, we have $\mathcal{J}(u_\epsilon) - \mathcal{J}(\bar{u}) \geq 0$, i.e.,

$$\int_0^T \langle y_\epsilon + \bar{y} - 2y_d, y_\epsilon - \bar{y}\rangle_{L^2(I)} dt + \int_0^T \langle \rho_\epsilon + \bar{\rho} - 2\rho_d, \rho_\epsilon - \bar{\rho}\rangle_{L^2(I)} dt \\ + \gamma\langle u_\epsilon + \bar{u}, \epsilon(v-\bar{u})\rangle_{L^2(0,T)} \geq 0,$$

where $y_\epsilon$ and $\rho_\epsilon$ are solution of (1.1) with respect to $u_\epsilon$. By dividing $\epsilon > 0$, it can be written as

$$\int_0^T \langle y_\epsilon + \bar{y} - 2y_d, z_\epsilon\rangle_{L^2(I)} dt + \int_0^T \langle \rho_\epsilon + \bar{\rho} - 2\rho_d, w_\epsilon\rangle_{L^2(I)} dt \\ + \gamma\langle u_\epsilon + \bar{u}, v-\bar{u}\rangle_{L^2(0,T)} \geq 0,$$

where $z_\epsilon = \frac{y_\epsilon - \bar{y}}{\epsilon}$ and $w_\epsilon = \frac{\rho_\epsilon - \bar{\rho}}{\epsilon}$. Since $y_\epsilon \to \bar{y}$, $\rho_\epsilon \to \bar{\rho}$ strongly in $L^2(0,T;L^2(I))$ and $z_\epsilon \to z$, $w_\epsilon \to w$ strongly in $L^2(0,T;L^2(I))$, we obtain

$$\int_0^T \langle \bar{y} - y_d, z\rangle_{L^2(I)} dt + \int_0^T \langle \bar{\rho} - \rho_d, w\rangle_{L^2(I)} dt + \gamma\langle \bar{u}, v-\bar{u}\rangle_{L^2(0,T)} \geq 0$$

as $\epsilon \to 0$. Here, $Z = \binom{z}{w}$ is the solution of (3.1) with respect to $\bar{u}$ and $v - \bar{u}$ instead of $u$ and $v$. Then, we have

$$\begin{aligned}
&\int_0^T \langle \bar{y} - y_d, z\rangle_{L^2(I)} dt \\
&= \int_0^T \Big\langle -\frac{\partial p_1}{\partial t} - d\frac{\partial^2 p_1}{\partial x^2} + \gamma(\bar{\rho})p_1 + fp_1 - fp_2, z\Big\rangle_{L^2(I)} dt \\
&= \int_0^T \Big\langle p_1, \frac{\partial z}{\partial t} - d\frac{\partial^2 z}{\partial x^2} + \gamma(\bar{\rho})z + fz\Big\rangle_{L^2(I)} dt + \int_0^T \langle p_2, -fz\rangle_{L^2(I)} dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \langle \bar{\rho} - \rho_d, w\rangle_{L^2(I)} dt \\
&= \int_0^T \Big\langle -\frac{\partial p_2}{\partial t} + 2a(\bar{\rho} - b)\bar{y}p_1 - gp_1 + hp_2 + \bar{u}(t)p_2, w\Big\rangle_{L^2(I)} dt \\
&= \int_0^T \Big\langle p_2, \frac{\partial w}{\partial t} + hw + \bar{u}(t)w\rangle_{L^2(I)} dt + \int_0^T \Big\langle p_1, 2a(\bar{\rho} - b)\bar{y}w - gw\rangle_{L^2(I)} dt.
\end{aligned}$$

By using (3.1) having $\bar{u}$ and $v-\bar{u}$ instead of $u$ and $v$, we have

$$\begin{aligned}
&\int_0^T \langle \bar{y}-y_d, z\rangle_{L^2(I)} dt + \int_0^T \langle \bar{\rho}-\rho_d, w\rangle_{L^2(I)} dt \\
&= \int_0^T \Big\langle p_1, \frac{\partial z}{\partial t} - d\frac{\partial^2 z}{\partial x^2} + \gamma(\bar{\rho})z + fz + 2a(\bar{\rho}-b)\bar{y}w - gw \Big\rangle_{L^2(I)} dt \\
&\quad + \int_0^T \Big\langle p_2, \frac{\partial w}{\partial t} + hw - fz + \bar{u}(t)w \Big\rangle_{L^2(I)} dt \\
&= \int_0^T \langle p_2, (v-\bar{u})\bar{\rho}\rangle_{L^2(I)} dt.
\end{aligned}$$

Hence, we obtain

$$\int_0^T \langle p_2, (v-\bar{u})\bar{\rho}\rangle_{L^2(I)} dt + \gamma\langle \bar{u}, v-\bar{u}\rangle_{L^2(0,T)} \geq 0 \quad \text{for all } v \in U_{ad}.$$

□

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