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THE CARATHÉODORY-CARTAN-KAUP-WU THEOREM ON KOBAYASHI HYPERBOLIC ALMOST COMPLEX MANIFOLDS

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Abstract. The purpose of this article is to present The Carathéodory-Cartan-Kaup-Wu theorem for connected Kobayashi hyperbolic almost complex manifolds.

1. Introduction

For a holomorphic mapping $f : \Omega \to \Omega$ of a domain Ω in the complex Euclidean space $\mathbb{C}^n$, denote by $Jf(p)$ the Jacobian determinant of the holomorphic differential of f at $p \in \Omega$. The classic Carathéodory-Cartan-Kaup-Wu Theorem is stated as follows:

Theorem 1.1. (Carathéodory-Cartan-Kaup-Wu) *Let Ω be a bounded domain in $\mathbb{C}^n$ and $f : \Omega \to \Omega$ a holomorphic mapping such that $f(p) = p$, then $|J_{\mathbb{C}}f(p)| = 1$ for some $p \in \Omega$ implies f is an automorphism.*

This has turned out to be one of the most influential theorems in the study of automorphism groups of bounded domains. Thus it was only natural that it has been generalized to various important cases. Generalization to the Kobayashi hyperbolic manifolds was by S. Kobayashi himself (see [3]). It was also generalized by Cima, Graham, Kim and Krantz to the bounded domains in the infinite dimensional separable Hilbert spaces ([1]). As the study of pseudo-holomorphic objects progresses rather fast in these days, further generalization of this theorem, especially to the collection of almost complex manifolds seems in order.

Indeed, the primary goal of this article is to present the following theorem

Theorem 1.2. *Let (M,J) be a connected Kobayashi hyperbolic almost complex manifold equipped with a smooth almost complex structure J. If $f : M \to M$ is a pseudo-holomorphic mapping such that*

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(a) *$f(p) = p$*
(b) $\det(df_p) = 1$

for some point $p \in M$, then f is an automorphism of (M, J).

The definition of $\det(df_p)$ will be introduced in Section 2. Notice that the condition (b) is not at all restrictive. If one interprets the condition $|J_{\mathbb{C}} f(p)| = 1$ for the holomorphic f with the terminology and notation for the almost complex case, it is exactly the same as (b).

2. Almost complex manifolds and pseudo-holomorphic maps

Let M be a smooth manifold of dimension $2n$. An *almost complex structure* is a bundle endomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{Id}$. A pair (M, J) is called an *almost complex manifold.* Given almost complex manifolds (M, J) and $(\tilde{M}, \tilde{J})$, a C^1 mapping $f : M \to \tilde{M}$ is called $(J, \tilde{J})$-*holomrphic* (or simply, *pseudo-holomorphic*) if its differential preserves almost complex structures, in the sense that

$$\tilde{J} \circ df = df \circ J.$$

For example, on $\mathbb{C}^n$

$$J_{st} = \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & \cdot\cdot & & & \\ & & & \cdot\cdot & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}$$

is a standard almost complex structure. Denote by

$$\mathcal{O}_{(J,\tilde{J})}(M, \tilde{M}) = \{f : M \to \tilde{M} |\ f \text{ is } (J, \tilde{J})\text{-holomorphic}\}.$$

These are general version of the complex manifold and the holomorphic mapping. A smooth mapping $f : (M, J) \to (\tilde{M}, \tilde{J})$ is called a $(J, \tilde{J})$-*biholomorphism* (or simply, *biholomorphism*) if f and f^{-1} are pseudo-holomorphic diffeomorphisms. We denote by

$$\mathrm{Aut}(M, J) = \{f : (M, J) \to (M, J) :\ f \text{ is a biholomorphism}\}.$$

If there is no confusion, we simply denote $\mathrm{Aut}(M)$ by $\mathrm{Aut}(M, J)$.

For any (J, J)-mapping $f : M \to M$ such that $f(p) = p$, df_p is represented by a $2n \times 2n$ matrix

$$[df_p] := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \text{for some } A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R}),$$

with respect to a basis $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ of T_pM. The determinant of the real differential of f at p is defined by $\det(df_p) = \det([df_p])$. Note that this definition does not depends on the choice of the basis.

3. Kobayashi metric and hyperbolicity

In 1967, S. Kobayashi introduced a pseudo-distance on every complex manifold by chains of holomorphic discs ([3]). By the existance theorem of pseudo-holomorphic discs due to Nijenhuis and Woolf ([6]), we can define the Kobayashi pseudo-distance and the Kobayashi-Royden pseudo-metric for the almost complex manifolds ([3]).

Let (M, J) be an almost complex manifold. Given two points p and q in M, a finite sequence of pseudo-holomorphic discs $\Sigma = \{\phi_j\}_{j=1,\ldots,k} \subset \mathcal{O}_{(J_{st},J)}(\mathbf{D}, M)$ is called a chain of pseudo-holomorphic discs from p to q if there are points $p = p_0, p_1, \ldots, p_k = q$ in M and $a_1, a_2, \ldots, a_k$ in $\mathbf{D}$ such that

$$\phi_j(0) = p_{j-1} \quad \text{and} \quad \phi_j(a_j) = p_j$$

for $j = 1, \ldots, k$. For this chain, we define its length $l(\Sigma)$ by

$$l(\sigma) = \log \frac{1 + |a_1|}{1 - |a_1|} + \cdots + \log \frac{1 + |a_k|}{1 - |a_k|}.$$

Note that $\log \frac{1+|z|}{1-|z|}$ is the Poincaré distance from 0 to z in $\mathbf{D}$. The Kobayashi pseudo-distance $d_{(M,J)}$ on (M, J) is then defined by

$$d_{(M,J)}(p, q) = \inf l(\Sigma),$$

where the infimum is taken over all chains of pseudo-holomorphic discs from p to q.

The Kobayashi-Royden pseudo-metric $F_{(M,J)}$ is the infinitesimal version of the Kobayashi pseudo-distance defined by

$$F_{(M,J)}(p, v) = \inf \left\{ \frac{1}{|a|} : \phi \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, M) \quad \text{with} \quad \phi(0) = p, \quad d\phi(\mathbf{e}) = av \right\}$$

where $\mathbf{e}$ is the unit vector in $T_0\mathbf{D}$ and $p \in M$ and $v \in T_pM$. For the case J is of Hölder class $C^{1,\alpha}$, $F_{(M,J)}(p, v)$ is upper semi-continuous and

$$d_{(M,J)}(p, q) = \inf \int_0^1 F_{(M,J)}(\gamma(t), \gamma'(t))dt$$

where the infimum is taken over all piecewise smooth paths $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. For pseudo-holomorphic map $f : (M, J) \to (\tilde{M}, \tilde{J})$ and $p, q \in M$ and $v \in T_pM$,

$$d_{(\tilde{M},\tilde{J})}(f(p), f(q)) \leq d_{(M,J)}(p, q)$$

and

$$F_{(\tilde{M},\tilde{J})}(f(p), df_p(v)) \leq F_{(M,J)}(p, v).$$

Definition 1. (M, J) is said to be (Kobayashi) hyperbolic if the Kobayashi pseudo-distance $d_{(M,J)}$ is a proper distance. For a hyperbolic manifold (M, J), if the Kobayashi ball

$$B^K_{(M,J)}(p, r) = \{q \in M \mid d_{(M,J)}(p, q) < r\}$$

is relatively compact in M for any $p \in M$ and any $r > 0$, (M, J) is said to be complete hyperbolic.

4. Proof of Theorem 1.2

We start with some algebraic facts. Let

$$\mathbb{S} = \{x \in \mathrm{GL}(2n, \mathbb{R}) \mid x = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \text{for some} \quad A, B \in \mathrm{M}_{n\times n}(\mathbb{R})\}.$$

This is a subgroup of $\mathrm{GL}(2n, \mathbb{R})$, group of general linear maps from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$. We can identify with $\mathrm{GL}(n, \mathbb{C})$ and $\mathbb{S}$ by

$$g : \mathbb{S} \to \mathrm{GL}(n, \mathbb{C}^n) \ \text{ with } \ g\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = A + iB.$$

Obviously, this map is a (Lie group) isomorphism.

Proof of Theorem 1.2

Let $r > 0$ be a sufficiently small positive number such that

$$B_r = \{x \in M \mid d_{(M,J)}(x, p) \leq r\}$$

is compact. Let

$$F_p = \{h : B_r \to B_r \mid h \text{ is pseudo-holomorphic and } h(p) = p\}.$$

Since pseudo-holomorphic mappings have distance decreasing property, F_p is an equicontinuous family. On the other hand, since $f(B_r) \subset B_r$ and B_r is compact, F_p is pointwise bounded. So by the Arzela-Ascoli theorem and Lemma 2.11 in [2], F_p is compact.

From now on, let f be a restriction of f on B_r. Denote f^k by the k-fold composition $f \circ \cdots \circ f$. Obviously $f^k \in F_p$ by the distance decreasing property of the Kobayashi distance. Then there exists a subsequence $\{f^{k_j}\}_{j=1,2,...}$ of $\{f^k\}_{k=1,2,...}$ and pseudo-holomorphic mapping $\widehat{f} : B_r \to B_r$ such that $\{f^{k_j}\}_{j=1,2,...}$ converges to $\hat{f}$ in compact-open topology, and $\{d(f^{k_j})_p\}_{j=1,2,...}$ goes to $d\widehat{f}_p$ in operator norm. Fix a basis $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ of T_pM. By abusing a notation, we identify df_p with $[df_p]$ the matrix representation of df_p with respect to $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$. Let

$$d(f^{k_j})_p = \begin{pmatrix} A_{k_j} & B_{k_j} \\ -B_{k_j} & A_{k_j} \end{pmatrix}$$

and

$$d\widehat{f}_p = \begin{pmatrix} \widehat{A} & \widehat{B} \\ -\widehat{B} & \widehat{A} \end{pmatrix}.$$

Then $g(d(f^{k_j})_p) = A_{k_j} + iB_{k_j}$, $g(d\widehat{f}_p) = \widehat{A} + i\widehat{B}$ and $A_{k_j} + iB_{k_j} \to \widehat{A} + i\widehat{B}$. By the above Lemma, we can see that $|\det(g(df_p))| = 1$.

Let $F_{(M,J)}$ be the Kobayashi-Royden metric on M and λ be a eigenvalue of $g(df_p) = A_{k_1} + iB_{k_1}$. Then

$$\lambda F_{(M,J)}(p, v) = F_{(M,J)}(p, \lambda v) = F_{(M,J)}(p, df_p v) \le F_{(M,J)}(p, v).$$

So every eigenvalue of $A_{k_1} + iB_{k_1}$ has modulus less than or equal to 1. But if there exists an eigenvalue λ with $\mid \lambda \mid < 1$, then

$$|\det(A_{k_1} + iB_{k_1})| < 1.$$

This contradiction implies that every eigenvalue has modulus 1.

Suppose $A_{k_1} + iB_{k_1}$ is not diagonalizable. Then $A_{k_1} + iB_{k_1}$ has a Jordan canonical form, which has diagonal blocks of the form

$$\begin{pmatrix} e^{i\theta} & 1 & 0 \\ & \ddots & 1 \\ 0 & & e^{i\theta} \end{pmatrix}$$

Then $(A_{k_1} + iB_{k_1})^{k_j}$ has the diagonal block of the form

$$\begin{pmatrix} e^{ik_j\theta} & (k_j-1)e^{i\theta} & * \\ & \ddots & (k_j-1)e^{i\theta} \\ 0 & & e^{k_j\theta} \end{pmatrix}.$$

This matrix diverges as j tends to ∞. This contradiction implies that $A_{k_1}+iB_{k_1}$ has the diagonal form

$$\begin{pmatrix} e^{i\theta_1} & & & & & & 0 \\ & \ddots & & & & & \\ & & e^{i\theta_1} & & & & \\ & & & \ddots & & & \\ & & & & e^{i\theta_k} & & \\ & & & & & \ddots & \\ 0 & & & & & & e^{i\theta_k} \end{pmatrix}.$$

Then there exists a subsequence $\{A_{k_{j_l}} + iB_{k_{j_l}}\}_{l=1,2,\ldots}$ such that $A_{k_{j_l}} + iB_{k_{j_l}}$ converges to $\mathrm{Id}_{n\times n}$ by Kronecker's theorem. So $d\widehat{f_p} = \mathrm{Id}_{2n\times 2n}$. By Cartan's Uniqueness theorem ([2]), $\widehat{f} = \mathrm{Id}$ on B_r. On the other hand, there exists a subsequence of $\{f^{k_{j_l}-1}\}$ that converges to pseudo-holomorphic mapping $h : B_r \to B_r$. Then $f \circ h = h \circ f = \mathrm{Id}$ on B_r. So $f \in \mathrm{Aut}(B_r, J)$.

To prove that f is an automorphism of M, we follow the Kobayashi's argument in [3]. Let W be the largest neighborhood of p such that for some subsequence $\{f^{k_j}\}_{j=1,2,\ldots}$ converges to Id on W in the compact-open topology. We may assume that $\{f^{k_j}\}_{j=1,2,\ldots}$ converges to Id. Since M is connected, it is enough to show that W is closed.

For $q \in \overline{W}$, there exists a sequence $\{q_l\}_{l=1,2,\ldots}$ in W such that $q_l \to q$.

$$d_{(M,J)}(f^{k_j}(q),q) \le d_{(M,J)}(f^{k_j}(q),f^{k_j}(q_l)) + d_{(M,J)}(f^{k_j}(q_l),q_l) + d_{(M,J)}(q_l,q)$$
$$\le 2d_{(M,J)}(q,q_l) + d_{(M,J)}(f^{k_j}(q_l),q_l).$$

Given $\epsilon > 0$, we choose $l > 0$ such that $2d_{(M,J)}(q,q_l) < \epsilon/2$. We choose also positive integer n_0 such that $d_{(M,J)}(f^{k_j}(q_l),f(q_l)) < \epsilon/2$ for $j > n_0$. This implies that

$$d_{(M,J)}(f^{k_j}(q),q) < \epsilon$$

for $j > n_0$. So $\lim_{j\to\infty} f^{k_j}(q) = q$.

Now we choose a sufficiently small $\delta > 0$ such that Kobayashi ball centered at q of radius 2δ is relatively compact. Again, Arzela-Ascoli theorem implies that

$$\Psi_q = \{\phi : B_\delta(q) \to B_{2\delta}(q) \mid \phi \text{ is pseudo-holomorphic}\}$$

is compact. Since $d_{(M,J)}(f^{k_j}(q),q) < \delta$ for sufficiently large j, it follows that

$$d_{(M,J)}(f^{k_j}(z),q) < d_{(M,J)}(f^{k_j}(z),f^{k_j}(q)) + d_{(M,J)}(f^{k_j}(q),q) < 2\delta,$$

for every $z \in B_\delta$ by the distance decreasing property of $d_{(M,J)}$. This means that f^{k_j} belongs to Ψ_q for sufficiently large j. So there exists a pseudo-holomorphic mapping $\hat{f}$, such that the subsequence of $\{f^{k_j}\}_{j=1,2,\dots}$ converging to $\hat{f}$. While on $B_\epsilon(q) \cap W$, $\hat{f} = \mathrm{Id}$. The unique continuation principle for pseudo-holomorphic mappings implies that $\hat{f} = \mathrm{Id}$ on $B_\epsilon(q)$. It follows that $q \in W$. This completes the proof. □

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