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FIRST PASSAGE TIME UNDER A REGIME-SWITCHING JUMP-DIFFUSION MODEL AND ITS APPLICATION IN THE VALUATION OF PARTICIPATING CONTRACTS

Yinghui Dong, Wenxin Lv, and Sang Wu

Abstract. We investigate the valuation of participating life insurance policies with default risk under a geometric regime-switching jump-diffusion process. We derive explicit formula for the Laplace transform of the price of participating contracts by solving integro-differential system and then price them by inverting Laplace transforms.

1. Introduction

Participating contracts are very popular since they provide a basic benefit to the policyholders. There is a lot of literature investigating the valuation of participating policies. See for example, Briys and de Varenne [2, 3] price the participating policies under the assumption that default can occur only at maturity within the framework of Black-Scholes model. Grosen and Jøgrgensen [13] focus on the modeling of early default of the participating contracts under a diffusion model. Dong [10] and Dong and Wang [9] derive the Laplace transform for the price of participating contracts under a two-sided jump-diffusion model.

As life insurance contracts are long term products, these instruments should be subject to the changes of economic regimes. Yet, the aforementioned literature on the valuations of participating life insurance contracts does not take into account changes of market regimes. Regime-switching models have been widely used in financial economics and insurance, see Buffington and Elliott [4], Siu et al. [6], Xu et al. [20], Dong et al. [11], Jin et al. [15] and Fan et al. [12]. Regime switches are triggered by structural changes in different stages of business cycles. Siu [18] investigates the fair valuation of a participating life insurance policy under a regime-switching geometric Brownian motion.

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Regime-switching jump-diffusion processes have attracted a lot of attention since they can capture both the long-run and short-run behaviors of the underlying funds simultaneously. See for example, Siu et al. [19] consider the valuation of participating life insurance products for default at maturity under a generalized jump-diffusion model which includes regime-switching and jump-diffusion models. Hieber [14] investigates the pricing of equity-linked life insurance contracts which offer cliquet-style return guarantees in a regime-switching Lévy model. Since they mainly price maturity return guarantees, the pricing formulas they derive are not associated with the joint distribution of the first passage time of the overshoot. To better protect the policyholders' benefits, regulatory authorities set early default mechanisms to monitor financial status of insurance companies, which forces the insurance company to be liquidated once a preset default threshold is achieved. In this paper we will derive the valuation of participating contracts under a geometric regime-switching jump-diffusion process for early default mechanisms, in which the valuation of the embedded options are linked to the joint Laplace transform of the first passage time and the overshoot.

Recently, Siu et al. [6] investigate the valuation of equity-linked life insurance contracts with various embedded options under a regime-switching, double exponential jump-diffusion process by using the Laplace transform of the first passage time. By using the conditional independence and conditional memoryless properties of the exponential distribution, Kijima and Siu [16] also present the analytical solution for the Laplace transform of the joint distribution of the first passage time and the overshoot under a regime-switching double-exponential jump-diffusion model. Note that, the results presented in Siu et al. [6] and Kijima and Siu [16] both rely on the conditional independence of the exponential distribution. Xu et al. [20] consider a structural form credit risk model in which the value of a firm and the default threshold are described by two dependent double exponential regime-switching jump-diffusion processes. They derive the integro-differential equations satisfied by the Laplace transform of the first passage time and the expected discounted ratio of the firm value to the default threshold at default and explicitly solve them.

Under the process with jumps, the value of the firm jumps due to the arrival of unexpected market information or special events. Intuitively, there may be a variety of information since a firm may invest in multiple risky assets. If different kinds of information are assumed to arrive as regime-switching Poisson processes and the jump sizes caused by the same kind of information are assumed to follow a regime-switching double-exponential distribution, then the jumps of the firm can be described by a regime-switching hyper-exponential jump-diffusion process, which is much more flexible than a regime-switching double-exponential jump-diffusion process. So, we generalize the model in Xu et al. [20] to a regime-switching hyper-exponential jump-diffusion process and extend the application of regime-switching jump-diffusion processes in the credit risk modeling to the valuation of participating contracts. Furthermore, the pricing of the defaultable bond in Xu et al. [20] is only associated with the joint Laplace transform of the first passage time and the overshoot, while the valuation of participating contracts depends on not only the joint Laplace transform but the distribution of the Markov chain at the first passage time. Therefore, we derive the explicit formulas for the joint Laplace transform and the distribution of the Markov chain at the first passage time when the size of jumps has a regime-switching hyper-exponential distribution, which is much more complex than that in Xu et al. [20]. Based on the results, we present the Laplace transform of the price of participating contracts and then numerically price participating contracts via inverting the Laplace transform. In fact, the results we present could be very useful for the valuation of some complex path-dependent options under a general regime-switching jump-diffusion model, since the hyper-exponential distribution is very rich enough to approximate many other distributions, including the normal distribution and various heavy-tailed distributions such as Gamma and Pareto distributions.

This article is organized as follows. In Section 2, we present the participating contracts. Section 3 introduces the underlying dynamics and provides the derivation of the joint distribution of the first passage time, the overshoot and the Markov chain at the first passage time under a generalized regime-switching jump-diffusion model. In Section 4, we derive the closed-form expression for the joint Laplace transform of the first passage time and the overshoot and the distribution of the Markov chain at the first passage time when the size of jumps has a regime-switching hyper-exponential distribution. Based on the result, we give the closed-form formula for the Laplace transform of the price of participating life insurance contracts. Section 5 carries out some numerical analysis by inverting the Laplace transforms via the Euler inversion algorithm. Section 6 concludes. The appendix presents the derivations of the characteristic equation for solving the integro-differential system satisfied by the joint Laplace transform under the regime-switching hyper-exponential jump-diffusion process.

2. The contract

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space where the filtration $\mathbb{F} := \{\mathcal{F}_t \,|\, t \in \mathcal{T}\}$ satisfies the usual conditions with $\mathcal{T} = [0, T]$ and $T < \infty$. As in Dong (2011), we assume that an equivalent martingale measure Q, under which the discounted prices at the riskfree interest rate r are Q-martingales, has been chosen.

Consider that an insurance company has only two types of agents: policyholders and shareholders. At the initiation of the contract, the policyholder provides a lump sum L_0 in a single premium contract; the insurer invests initial equity $E_0 > 0$. Consequently, the initial asset value of the insurance company is $A_0 = L_0 + E_0$. Denote the liability-to-asset ratio by $\alpha = \frac{L_0}{A_0} \in (0, 1)$. The policyholders are guaranteed a fixed amount $L_T^g = L_0 e^{r_g T}$ at the maturity T, where $r_g < r$ is the guaranteed interest rate. However, the terminal payoff to the policyholders is associated with financial risks and default risk of the insurance company. Now we specify these payments to the policyholders and the insurance company and the early default mechanism.

Following Le Courtois and Quittard-Pinon [7], we use the so-called structural approach of bankruptcy to define the early default. Let $B_t = \gamma L_0 e^{r_g t}$, $0 < \gamma < 1$, be the default barrier. The default time is defined as the first hitting time:

$$\tau = \inf\{t \in [0,T] \mid A_t \le B_t\}. \tag{2.1}$$

Note that the asset value at default A_τ is completely distributed to the policyholders since $\gamma < 1$.

In the participating contract, if there is no default before maturity, then the policyholder receives the amount $\Theta_L(T)$ as follows (see Briys and De Varenne [2]):

$$\Theta_L(T) = \begin{cases} A_T, & A_T < L_T^g, \\ L_T^g, & L_T^g \le A_T \le \frac{L_T^g}{\alpha}, \\ L_T^g + \beta(\alpha A_T - L_T^g), & A_T > \frac{L_T^g}{\alpha}. \end{cases}$$

The payoff $\Theta_L(T)$ can be rewritten in a more compact form:

$$\Theta_L(T) = L_T^g + \beta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+, \tag{2.2}$$

where we have used $x^+ = \max\{x, 0\}$. It is obvious that $\Theta_L(T)$ is a combination of a fixed payment L_T^g, a bonus call and a shorted put option on the insurance company's assets.

If the default occurs, then the policyholders receives the asset value at default, A_τ. To sum up, the policyholder's payoff at T can also be expressed as

$$V_L(T) = 1_{\{\tau > T\}} \Theta_L(T) + 1_{\{\tau \le T\}} A_\tau e^{r(T-\tau)}.$$

The policyholder will retain the residual assets at maturity: $A_T - V_L(T)$.

3. The model

Assume that there exists a process $\mathbf{X} := \{X_t\}_{t \ge 0}$ which represents the switches among the different macro economic states. Let $\mathbf{X}$ be a continuous-time Markov chain taking values in $\mathcal{E} = \{e_1, \dots, e_N\}$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)^* \in R^N$, a^* denotes the transpose of a vector or a matrix a. The generator of the Markov chain is denoted by $A = (a_{ij})_{i,j=1,2,\dots,N}$. Denote by $\mathcal{F}^{\mathbf{X}} := \{\mathcal{F}_t^{\mathbf{X}} \mid t \in \mathcal{T}\}$ the filtration generated by $\mathbf{X}$.

Assume that the initial asset value A_0 of the insurance company may be invested in some risky and risk-free assets. Therefore, under the process with jumps, the asset value A_t may have different classes of jumps related to the invested assets. We assume the insurance company invests in m different kinds of stocks and we use the regime-switching pure jump process to model the jump component in the asset value process A_t.

Consider a regime-switching pure jump process, $\mathbf{S} := \{S(t)\,|\,t \in \mathcal{T}\}$, such that

$$S(t) = \sum_{l=1}^{m} \sum_{i=1}^{N_l(t)} \xi_i^{(l)}, \tag{3.1}$$

where $\mathbf{N}_l := \{N_l(t)\,|\,t \in \mathcal{T}\}$ is a regime-switching Poisson process with the stochastic intensity $\lambda_t^l = \langle \boldsymbol{\lambda}^l, X_t \rangle$, for a vector $\boldsymbol{\lambda}^l = (\lambda^{l1}, \ldots, \lambda^{lN})^* \in R^N$ with $\lambda^{lj} > 0$ for $l = 1, \ldots, m$, $j = 1, \ldots, N$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product in R^N. The jump amounts $\xi_j^{(l)}$, $j = 1, 2, \ldots$ are assumed to be independent and identically distributed with the common conditional density $f_t^l(y)$ for each $l = 1, \ldots, m$, conditional on the Markov chain $\mathbf{X}$. Assume $f_t^l(\cdot) = \langle \mathbf{f}^l(\cdot), X_t \rangle$, where $\mathbf{f}^l(\cdot) = (f^{l1}(\cdot), \ldots, f^{lN}(\cdot))^* \in R^N$ and for each $l = 1, \ldots, N$, f^{lj} satisfies $\int_{-\infty}^{\infty} e^{yx} f^{lj}(x)dx < \infty$ for some $y > 0$. Furthermore, we assume $\mathbf{N}_1, \ldots, \mathbf{N}_m$, $\{\xi_j^{(1)}, j = 1, 2, \ldots\}, \ldots \{\xi_j^{(m)}, j = 1, 2, \ldots\}$ are mutually independent, given the path of the Markov chain $\mathbf{X}$.

Lemma 3.1. *Let $\{S(t)\}_{t \geq 0}$ be defined in (3.1). Then $\{S(t)\}_{t \geq 0}$ is a regime-switching Compound Poisson process, which can be expressed as*

$$S(t) = \sum_{i=1}^{N(t)} Z_i, \tag{3.2}$$

where the intensity of the regime-switching Poisson $N(t)$ is $\lambda(t) = \langle \boldsymbol{\lambda}, X_t \rangle$, for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)^* \in R^N$ with $\lambda_j = \sum_{l=1}^m \lambda^{lj}$ for $j = 1, 2, \ldots, N$, the common density function is given by $f_t(\cdot) = \langle \mathbf{f}(\cdot), X_t \rangle$, for $\mathbf{f}(\cdot) = (f_1(\cdot), \ldots, f_N(\cdot))^* \in R^N$ with

$$f_j(x) = \sum_{l=1}^{m} \frac{\lambda^{lj}}{\lambda_j} f^{lj}(x). \tag{3.3}$$

Proof. From Lemma 3.2 of Dong et al. [11], we have

$$E\left[e^{yS(t)} | \mathcal{F}_t^{\mathbf{X}}\right] = \prod_{l=1}^{m} E\left[e^{y\sum\limits_{i=1}^{N_l(t)} X_i^{(l)}} | \mathcal{F}_t^{\mathbf{X}}\right] = e^{\int_0^t \langle \mathbf{G}(y), X_s \rangle ds},$$

where $\mathbf{G}(y) = (G^1(y), \ldots, G^N(y))^*$, with

$$G^j(y) = \lambda_j \Big(\sum_{l=1}^{m} \frac{\lambda^{lj}}{\lambda_j} \int_0^{\infty} e^{yz} f^{lj}(z)dz - 1\Big) < \infty.$$

Then Lemma A.1 of Buffington and Elliott [4] gives

$$E\left[e^{yS(t)}\right] = \langle \exp\{(\mathbf{diag}(\mathbf{G}(y)) + A)t\}\mathbf{1}, X_0 \rangle.$$

Similarly, we can also prove that

$$E\left[e^{y\sum_{i=1}^{N(t)} Z_i}\right] = \langle \exp\{(\mathbf{diag}(\mathbf{G}(y)) + A)t\}\mathbf{1}, X_0\rangle,$$

which concludes the proof. □

Remark 3.2. When f^{lj} is the density function of a double exponential distribution, then f_j is the density function of a hyper-exponential distribution. Therefore, the regime-switching hyper-exponential jump-diffusion process outperforms the regime-switching double exponential jump-diffusion process in modeling the dynamics of a company's asset value since the asset value of each company may be affected by different kinds of shock events.

Assume that the asset value process under the risk-neutral measure Q is given by

$$A_t = A_0 e^{\int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) + S(t)}, \tag{3.4}$$

for $t \geq 0$, where $A_0 > 0$ is the initial assets value;

$$\mu(t) = \langle \boldsymbol{\mu}, X_t\rangle, \sigma(t) = \langle \boldsymbol{\sigma}, X_t\rangle, \tag{3.5}$$

for constant vectors $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)^*, \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)^*$, with $\mu_i = r - \frac{1}{2}\sigma_i^2 - \lambda_i(\int_{-\infty}^{\infty} e^z f_i(z)dz - 1)$ and $\sigma_i > 0$ for each $i = 1, 2, \ldots, N$; $\mathbf{W} := \{W(t) \,|\, t \in \mathcal{T}\}$ is a standard Q-Brownian motion.

Let

$$\begin{aligned} Y(t) &= \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s) + S(t) - r_g t \\ &\doteq \int_0^t \overline{\mu}(s)ds + \int_0^t \sigma(s)dW(s) + S(t), \end{aligned}$$

where $\overline{\mu}(t) = \langle \overline{\boldsymbol{\mu}}, X_t\rangle$, for a constant vector $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_N)^*$, with $\overline{\mu}_i = \mu_i - r_g$.

We now specify the information structure of our model. Let $\mathcal{F}_t^{\mathbf{Y}} := \{\mathcal{F}_t^{\mathbf{Y}} \,|\, t \in \mathcal{T}\}$ be the right-continuous, P-completed, natural filtration generated by the process $\mathbf{Y} := \{Y(t) \,|\, t \in \mathcal{T}\}$. Denote the enlarged filtration by $\mathbb{F} := \{\mathcal{F}_t \,|\, t \in \mathcal{T}\}$, where for each $t \in \mathcal{T}$, $\mathcal{F}_t = \mathcal{F}_t^{\mathbf{Y}} \vee \mathcal{F}_t^{\mathbf{X}}$, be the minimal σ-field containing $\mathcal{F}_t^{\mathbf{Y}}$ and $\mathcal{F}_t^{\mathbf{X}}$.

Lemma 3.3. *Let* $\tilde{\mathbf{G}}(y) = (\tilde{G}^1(y), \ldots, \tilde{G}^N(y))^*$, *where* y *satisfies* $\tilde{G}^j(y) < \infty$ *and*

$$\tilde{G}^j(y) = \overline{\mu}_j + \frac{\sigma_j^2}{2} + \lambda_j(\sum_{l=1}^m \frac{\lambda^{lj}}{\lambda_j} \int_0^{\infty} e^{yz} f^{lj}(z)dz - 1).$$

Then for $t > s \geq 0$,

$$E\left[e^{yY(t)}|\mathcal{F}_s\right] = e^{yY(s)}\langle \exp\{(\mathbf{diag}(\tilde{\mathbf{G}}(y)) + A)(t-s)\}\mathbf{1}, X_s\rangle,$$

where $\mathbf{1} = (1, \ldots, 1)^*$.

Proof. Conditional on the path of the Markov chain,

$$E\left[e^{yY(t)}|\mathcal{F}_s\right] = e^{yY(s)}E\left[E\left[e^{y(Y(t)-Y(s))}|\mathcal{F}_t^{\mathbf{X}} \vee \mathcal{F}_s\right]|\mathcal{F}_s\right]$$
$$= e^{yY(s)}E\left[e^{\int_s^t \langle \mathbf{G}(y), X(u)\rangle du}|\mathcal{F}_s\right].$$

Then from Lemma A.1 of Buffington and Elliott [4] we can obtain the result. □

From (2.1), the default time can be rewritten as

$$\tau = \inf\{t \geq 0 : u + Y(t) \leq 0\},$$

where $u = \ln\left(\frac{A_0}{\gamma L_0}\right) > 0$. For $\delta > 0$ and each $i = 1, \dots, N$, define

$$\Xi_i(u, \delta, \eta, \boldsymbol{\theta}) = E\left[e^{-\delta\tau + \eta(u+Y(\tau))}\theta(X_\tau)1_{\{\tau<\infty\}}|X_0 = e_i, u + Y(0) = u\right],$$

where η satisfies $E\left[e^{\eta Y(\tau)}|X_0 = e_i, Y(0) = 0\right] < \infty$, and $\theta(X_t) = \langle \boldsymbol{\theta}, X_t\rangle$ with $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)^*$. To simplify the notation, we drop $\delta, \eta, \boldsymbol{\theta}$ in the parameters. Note that, if $\boldsymbol{\theta} = (1, \dots, 1)^*$, then

$$\Xi_i(u, \delta, \eta, \boldsymbol{\theta}) = E\left[e^{-\delta\tau + \eta(u+Y(\tau))}1_{\{\tau<\infty\}}|X_0 = e_i, u + Y(0) = u\right]$$

is the joint Laplace transform of τ and $Y(\tau)$. Xu et al. [20] derive the integro-differential equations satisfied by the joint Laplace transform. In this paper, we can use the same arguments as in Xu et al. [20] to derive the integro-differential equations for $\Xi_i(u)$'s.

Theorem 3.4. *Let $u > 0$. Then, $\Xi_i(u)$'s satisfy the integro-differential system*

$$(\delta - a_{ii} + \lambda_i)\Xi_i(u) - \overline{\mu}_i\Xi_i'(u) - \frac{\sigma_i^2}{2}\Xi_i''(u) - \lambda_i\int_{-\infty}^{-u}\theta_i e^{\eta(u+x)}f_i(x)dx$$
$$- \lambda_i\int_{-u}^{\infty}\Xi_i(u+x)f_i(x)dx = \sum_{j=1, j\neq i}^{N} a_{ij}\Xi_j(u), \tag{3.6}$$

with boundary conditions:

$$\Xi_i(u) = \theta_i e^{\eta u},\ i = 1, 2, \dots, N,\ u \leq 0. \tag{3.7}$$

Proof. The proof is similar to that of Theorem 3.1 in Xu et al. [20], so we omit it. □

Remark 3.5. Extending Siu et al. [6] and Kijima and Siu [16], we can obtain the joint Laplace transform under a regime-switching jump-diffusion process with a general jump size distribution by solving the boundary value problems (3.6) and (3.7) analytically or numerically. In particular, when the jumps follow a regime-switching exponential type distribution, we can derive the explicit formula.

4. Solution approach

In this section, we suppose that the Markov chain $\mathbf{X}$ only have two states, that is, $N = 2$. Suppose that state e_1 (state e_2) represents a "bad" ("good") economic state. The intensity matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $a_{11} = -a_{12} < 0, a_{22} = -a_{21} < 0$. Furthermore, for each $l = 1, \ldots, m, j = 1, 2$, we assume that

$$f^{lj}(y) = \frac{1}{2}\alpha_{l,j}e^{-\alpha_{l,j}y}1_{\{y\geq 0\}} + \frac{1}{2}\beta_{l,j}e^{\beta_{l,j}y}1_{\{y<0\}}, \tag{4.1}$$

where $1 < \alpha_{l,1} < \alpha_{l,2}, 0 < \beta_{l,1} < \beta_{l,2}$, for $l = 1, \ldots, m$. The condition $\alpha_{l,1} < \alpha_{l,2}, \beta_{l,1} < \beta_{l,2}$ holds due to the fact that the expectation of the jump size corresponding to the "bad" economic state should be greater than that corresponding to the "good" economic state. $\alpha_{l,1} > 1$ guarantees that the vector $\boldsymbol{\mu}$ in (3.5) is well-defined. For simplicity, we only consider the case when all $\alpha_{i,j}$'s are distinct and all $\beta_{i,j}$'s are also distinct since the analysis of the other case is more tedious, which will be further illustrated in the derivation of explicit formulas for $\Xi_i(u)$ below. Without loss of generality, we assume $\alpha_{i,1} < \alpha_{i,2} < \alpha_{i+1,1} < \alpha_{i+1,2}, \beta_{i,1} < \beta_{i,2} < \beta_{i+1,1} < \beta_{i+1,2}, i = 1, 2, \ldots, m-1$.

From Lemma 3.1, we have, for each $j = 1, 2$,

$$f_j(y) = \sum_{l=1}^{m} p_{lj}\alpha_{l,j}e^{-\alpha_{l,j}y}1_{\{y>0\}} + \sum_{l=1}^{m} q_{lj}\beta_{l,j}e^{\beta_{l,j}y}1_{\{y<0\}}, \tag{4.2}$$

where $p_{lj} = q_{lj} = \frac{\lambda^{lj}}{2\lambda_j} > 0$.

Then from Theorem 3.4, we can directly obtain the following result.

Corollary 4.1. *Let $u > 0$. Then, $\Xi_i(u)$'s satisfy the integro-differential system*

$$\begin{aligned}&(\delta - a_{11} + \lambda_1)\Xi_1(u) - \overline{\mu}_1\Xi_1^{'}(u) - \frac{\sigma_1^2}{2}\Xi_1^{''}(u) - \lambda_1\int_{-\infty}^{-u}\theta_1 e^{\eta(u+x)}f_1(x)dx \\ &- \lambda_1\int_{-u}^{\infty}\Xi_1(u+x)f_1(x)dx = -a_{11}\Xi_2(u)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&(\delta - a_{22} + \lambda_2)\Xi_2(u) - \overline{\mu}_2\Xi_2^{'}(u) - \frac{\sigma_2^2}{2}\Xi_2^{''}(u) - \lambda_2\int_{-\infty}^{-u}\theta_2 e^{\eta(u+x)}f_2(x)dx \\ &- \lambda_2\int_{-u}^{\infty}\Xi_2(u+x)f_2(x)dx = -a_{22}\Xi_1(u).\end{aligned} \tag{4.4}$$

Let $\mathbf{I}$ and $\mathbf{D}$ denote the identity operator and the differential operator, respectively. Define the differential operator polynomials

$$h_i(\mathbf{D}) = \frac{1}{2}\sigma_i^2\mathbf{D}^2 + \overline{\mu}_i\mathbf{D} - (\delta - a_{ii} + \lambda_i)\mathbf{I}.$$

Inserting (4.1) into (4.3) and (4.4) gives

$$\begin{aligned} h_1(\mathbf{D})\Xi_1(u) = & -\lambda_1\Big(\sum_{i=1}^m \frac{q_{i1}\theta_1\beta_{i,1}}{\beta_{i,1}+\eta}e^{-\beta_{i,1}u} + \int_0^u \Xi_1(s)\sum_{i=1}^m q_{i1}\beta_{i,1}e^{\beta_{i,1}(s-u)}ds \\ & + \int_u^\infty \Xi_1(s)\sum_{i=1}^m p_{i1}\alpha_{i,1}e^{-\alpha_{i,1}(s-u)}ds\Big) + a_{11}\Xi_2(u), \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} h_2(\mathbf{D})\Xi_2(u) = & -\lambda_2\Big(\sum_{i=1}^m \frac{q_{i2}\theta_2\beta_{i,2}}{\beta_{i,2}+\eta}e^{-\beta_{i,2}u} + \int_0^u \Xi_2(s)\sum_{i=1}^m q_{i2}\beta_{i,2}e^{\beta_{i,2}(s-u)}ds \\ & + \int_u^\infty \Xi_2(s)\sum_{i=1}^m p_{i2}\alpha_{i,2}e^{-\alpha_{i,2}(s-u)}ds\Big) + a_{22}\Xi_1(u). \end{aligned} \tag{4.6}$$

When $m = 2, \theta_i = 1$ and $\beta_{i,j} = \alpha_{i,j}$ for $i, j = 1, 2$, Xu et al. [20] have explicitly solved (4.5)-(4.6). Although the hyper-exponential distribution we consider in this paper may be asymmetric, for any $m \geq 1$ and any constants θ_i, we can use a similar procedure as in Xu et al. [20] (see the appendix) to obtain that the characteristic equation for solving (4.5)-(4.6) can be written as

$$\tilde{h}_1(x)\tilde{h}_2(x) = a_{11}a_{22}\prod_{j=1}^2\prod_{i=1}^m (x+\beta_{i,j})(x-\alpha_{i,j}). \tag{4.7}$$

Define

$$\hat{h}_j(x) = \frac{\sigma_j^2}{2}x^2 + \overline{\mu}_j x + \lambda_j\Big(\sum_{i=1}^m \frac{p_{ij}\alpha_{i,j}}{\alpha_{i,j}-x} + \sum_{i=1}^m \frac{q_{ij}\beta_{i,j}}{\beta_{i,j}+x}\Big) - (\delta - a_{jj} + \lambda_j).$$

Then (4.7) can be written as

$$\hat{h}_1(x)\hat{h}_2(x) = a_{11}a_{22}. \tag{4.8}$$

In order to derive the solution for $\Xi_i(u)$, it remains to investigate the roots of the equation (4.8). Dong et al. [8] have proved that the equation

$$\hat{h}_j(x) = 0$$

has $2m+2$ roots, $x_{1,j}, \ldots, x_{m+1,j}, y_{1,j}, \ldots, y_{m+1,j}$, satisfying

$$\begin{aligned} & -\infty < y_{m+1,j} < -\beta_{m,j} < y_{m+1,j} < -\beta_{m-1,j} < y_{m,j} < \cdots \\ & < -\beta_{1,j} < y_{1,j} < 0 < x_{1,j} < \alpha_{1,j} < \cdots < \alpha_{m,j} < x_{m+1,j} < \infty. \end{aligned}$$

Lemma 4.2. *For $\delta > 0$, the equation (4.8) has $2m+2$ distinct positive real roots, and $2m+2$ distinct negative real roots.*

Proof. Let

$$h(x) = \hat{h}_1(x)\hat{h}_2(x) - a_{11}a_{22}.$$

Note that

$$h(0) = (\delta - a_{11} + \lambda_1)(\delta - a_{22} + \lambda_2) - a_{11}a_{22} > 0,$$

$$h(x_{i,j}) = h(y_{i,j}) = -a_{11}a_{22} < 0,\ h(-\infty) = +\infty,\ h(+\infty) = +\infty,$$
$$h(\alpha_{i,j}-)h(\alpha_{i,j}+) = -\infty,\ h(-\beta_{i,j}-)h(\beta_{i,j}+) = -\infty.$$

Furthermore, we have (see Figure 1)

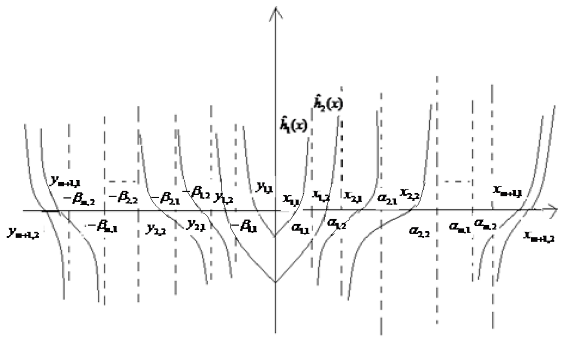

FIGURE 1. the roots of Eq. (4.8)

$$-\infty < y_{m+1,2} < y_{m+1,1} < -\beta_{m,2} < \cdots < -\beta_{1,2} < y_{1,2} < -\beta_{1,1} < y_{1,1}$$
$$< 0 < x_{1,1} < \alpha_{1,1} < x_{1,2} < \alpha_{1,2} < \cdots < \alpha_{m,2} < x_{m+1,1} < x_{m+1,2} < +\infty.$$

Under the given assumption, we observe that

$$h(\alpha_{i,j}-) = -\infty,\ h(\alpha_{i,j}+) = +\infty,\ h(-\beta_{i,j}-) = +\infty,\ h(-\beta_{i,j}+) = -\infty.$$

Hence, there exists at least one root at each of the $4m+4$ intervals,

$$(-\infty, y_{m+1,2}), \ldots, (y_{i+1,1}, -\beta_{i,2}), (y_{i,2}, -\beta_{i,1}), \ldots, (-y_{1,1}, 0),$$
$$(0, x_{1,1}), \ldots, (\alpha_{i,1}, x_{i,2}), (\alpha_{i,2}, x_{i+1,1}), \ldots, (x_{m+1,2}, +\infty),$$

for $i = 1, \ldots, m$. Since $h(x)$ is a polynomial of degree $4m+4$, there exists exactly one root at each of the above intervals, which concludes the result. $\square$

Remark 4.3. If all $\alpha_{i,j}$'s are not distinct or all $\beta_{i,j}$'s are not distinct, then the equation (4.8) may have complex roots or multiple roots, which makes the following analysis much more tedious but does not bring any essential extension of mathematics.

Theorem 4.4. *For any $\delta > 0$, $0 < \eta < \alpha_{11}$, we have*

$$\Xi_1(u) = \begin{cases} \sum_{j=1}^{2m+2} c_j e^{\rho_j u}, & u > 0, \\ \theta_1 e^{\eta u}, & u \le 0, \end{cases} \tag{4.9}$$

and

$$\Xi_2(u) = \begin{cases} \sum_{j=1}^{2m+2} d_j e^{\rho_j u}, & u > 0, \\ \theta_2 e^{\eta u}, & u \le 0, \end{cases} \tag{4.10}$$

where $\rho_1, \ldots, \rho_{2m+2}$ *are* $2m+2$ *distinct negative real roots of the Eq.* (4.8), $c_1, \ldots, c_{2m+2}$ *are determined by the following linear system*

$$\begin{aligned} &(c_1, \ldots, c_{2m+2})\,\mathbf{H}^* \\ &= \left(1, a_{11}, \frac{\theta_1\beta_{1,1}}{\beta_{1,1}+\eta}, \ldots, \frac{\theta_1\beta_{m,1}}{\beta_{m,1}+\eta}, \frac{a_{11}\theta_2\beta_{1,2}}{\beta_{1,2}+\eta}, \ldots, \frac{a_{11}\theta_2\beta_{m,2}}{\beta_{m,2}+\eta}\right), \end{aligned} \tag{4.11}$$

$d_1, \ldots, d_{2m+2}$ *are determined by the following linear system*

$$\begin{aligned} &(d_1, \ldots, d_{2m+2})\mathbf{G}^* \\ &= \left(1, a_{22}, \frac{\theta_2\beta_{1,2}}{\beta_{1,2}+\eta}, \ldots, \frac{\theta_2\beta_{m,2}}{\beta_{m,2}+\eta}, \frac{\theta_1 a_{22}\beta_{1,1}}{\beta_{1,1}+\eta}, \ldots, \frac{\theta_1 a_{22}\beta_{m,1}}{\beta_{m,1}+\eta}\right), \end{aligned} \tag{4.12}$$

with

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \hat{h}_1(\rho_1) & \hat{h}_1(\rho_2) & \cdots & \hat{h}_1(\rho_{2m+2}) \\ \frac{\beta_{1,1}}{\beta_{1,1}+\rho_1} & \frac{\beta_{1,1}}{\beta_{1,1}+\rho_2} & \cdots & \frac{\beta_{1,1}}{\beta_{1,1}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{i,1}}{\beta_{i,1}+\rho_1} & \frac{\beta_{i,1}}{\beta_{i,1}+\rho_2} & \cdots & \frac{\beta_{i,1}}{\beta_{i,1}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{m,1}}{\beta_{m,1}+\rho_1} & \frac{\beta_{m,1}}{\beta_{m,1}+\rho_2} & \cdots & \frac{\beta_{m,1}}{\beta_{m,1}+\rho_{2m+2}} \\ \frac{\beta_{1,2}\hat{h}_1(\rho_1)}{\beta_{1,2}+\rho_1} & \frac{\beta_{1,2}\hat{h}_1(\rho_2)}{\beta_{1,2}+\rho_2} & \cdots & \frac{\beta_{1,2}\hat{h}_1(\rho_{2m+2})}{\beta_{1,2}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{i,2}\hat{h}_1(\rho_1)}{\beta_{i,2}+\rho_1} & \frac{\beta_{i,2}\hat{h}_1(\rho_2)}{\beta_{i,2}+\rho_2} & \cdots & \frac{\beta_{i,2}\hat{h}_1(\rho_{2m+2})}{\beta_{i,2}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{m,2}\hat{h}_1(\rho_1)}{\beta_{m,2}+\rho_1} & \frac{\beta_{m,2}\hat{h}_1(\rho_2)}{\beta_{m,2}+\rho_2} & \cdots & \frac{\beta_{m,2}\hat{h}_1(\rho_{2m+2})}{\beta_{m,2}+\rho_{2m+2}} \end{pmatrix},$$

and

$$\mathbf{G}=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \hat{h}_2(\rho_1) & \hat{h}_2(\rho_2) & \cdots & \hat{h}_2(\rho_{2m+2}) \\ \frac{\beta_{1,2}}{\beta_{1,2}+\rho_1} & \frac{\beta_{1,2}}{\beta_{1,2}+\rho_2} & \cdots & \frac{\beta_{1,2}}{\beta_{1,2}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{i,2}}{\beta_{i,2}+\rho_1} & \frac{\beta_{i,2}}{\beta_{i,2}+\rho_2} & \cdots & \frac{\beta_{i,2}}{\beta_{i,2}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{m,2}}{\beta_{m,2}+\rho_1} & \frac{\beta_{m,2}}{\beta_{m,2}+\rho_2} & \cdots & \frac{\beta_{m,2}}{\beta_{m,2}+\rho_{2m+2}} \\ \frac{\beta_{1,1}\hat{h}_2(\rho_1)}{\beta_{1,1}+\rho_1} & \frac{\beta_{1,1}\hat{h}_2(\rho_2)}{\beta_{1,1}+\rho_2} & \cdots & \frac{\beta_{1,1}\hat{h}_2(\rho_{2m+2})}{\beta_{1,1}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{i,1}\hat{h}_2(\rho_1)}{\beta_{i,1}+\rho_1} & \frac{\beta_{i,1}\hat{h}_2(\rho_2)}{\beta_{i,1}+\rho_2} & \cdots & \frac{\beta_{i,1}\hat{h}_2(\rho_{2m+2})}{\beta_{i,1}+\rho_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_{m,1}\hat{h}_2(\rho_1)}{\beta_{m,1}+\rho_1} & \frac{\beta_{m,1}\hat{h}_2(\rho_2)}{\beta_{m,1}+\rho_2} & \cdots & \frac{\beta_{m,1}\hat{h}_2(\rho_{2m+2})}{\beta_{m,1}+\rho_{2m+2}} \end{pmatrix}.$$

Proof. Note that $\lim\limits_{u\to+\infty}\Xi_1(u)=0$. Then the Laplace transforms $\Xi_1(u)$ and $\Xi_2(u)$ have the forms

$$\Xi_1(u)=\sum_{j=1}^{2m+2}c_je^{\rho_ju},\quad \Xi_2(u)=\sum_{j=1}^{2m+2}d_je^{\rho_ju},u>0,$$

where c_j's, d_j's are undetermined constants. It is easy to see that to determine $c_1,\dots,c_{2m+2}$, $2m+2$ equations are required. The smooth pasting condition implies that

$$\sum_{j=1}^{2m+2}c_j=1. \tag{4.13}$$

To derive other $2m+1$ equations, we substituting $\Xi_1(u)=\sum_{j=1}^{2m+2}c_je^{\rho_ju}$, $u>0$ into (4.3) and obtain

$$\begin{aligned} a_{11}\Xi_2(u)=&\sum_{j=1}^{2m+2}c_je^{\rho_ju}\hat{h}_1(\rho_j)\\ &+\lambda_1\sum_{i=1}^{2m+2}q_{i1}e^{-\beta_{i,1}u}\left(\frac{\theta_1\beta_{i,1}}{\beta_{i,1}+\eta}-\sum_{j=1}^{2m+2}\frac{c_j\beta_{i,1}}{\beta_{i,1}+\rho_j}\right). \end{aligned} \tag{4.14}$$

Inserting Eq. (4.14) into Eq. (4.6) and equating the coefficients of $e^{-\beta_{i,j}u}$, $i=1,2,\dots,m$, $j=1,2$, yield

$$\begin{cases} \frac{a_{11}\theta_2\beta_{i,2}}{\beta_{i,2}+\eta}-\sum\limits_{j=1}^{2m+2}\frac{c_j\beta_{i,2}\hat{h}_1(\rho_j)}{\beta_{i,2}+\rho_j}-\sum\limits_{l=1}^{2m+2}\frac{\lambda_1\beta_{i,2}q_{l1}}{\beta_{i,2}-\beta_{l1}}\left(\frac{\theta_1\beta_{l1}}{\beta_{l1}+\eta}-\sum\limits_{j=1}^{2m+2}\frac{c_j\beta_{l1}}{\beta_{l1}+\rho_j}\right) & =0,\\ \left(\frac{\theta_1\beta_{i,1}}{\beta_{i,1}+\eta}-\sum\limits_{j=1}^{2m+2}\frac{c_j\beta_{i,1}}{\beta_{i,1}+\rho_j}\right)\hat{h}_2(-\beta_{i,1}) & =0, \end{cases}$$

Note that $\hat{h}_2(-\beta_{i,1}) \neq 0$. Therefore, we obtain that

$$\sum_{j=1}^{2m+2} \frac{c_j \beta_{i,1}}{\beta_{i,1} + \rho_j} = \frac{\theta_1 \beta_{i,1}}{\beta_{i,1} + \eta}, \ i = 1, \ldots, m, \tag{4.15}$$

and

$$\sum_{j=1}^{2m+2} \frac{c_j \beta_{i,2} \hat{h}_1(\rho_j)}{\beta_{i,2} + \rho_j} = \frac{a_{11} \theta_2 \beta_{i,2}}{\beta_{i,2} + \eta}, \ i = 1, \ldots, m. \tag{4.16}$$

Furthermore, letting $u = 0$ in Eq. (4.14) and using Eq. (4.15), with the boundary condition $\Xi_2(0) = 1$ yield that and

$$\sum_{j=1}^{2m+2} c_j \hat{h}_1(\rho_j) = a_{11}. \tag{4.17}$$

Combining (4.15)-(4.17) yields (4.9) and (4.11).

The proof for (4.10) and (4.12) is similar, so we omit it here. □

From Theorem 4.4, we can directly obtain the following result.

Corollary 4.5. *For any $\delta > 0$, $0 < \eta < \alpha_{11}$, we have*

$$E\left[e^{-\delta\tau + \eta(u+Y(\tau))} \theta(X_\tau) | u + Y(0) = u\right] = \langle \vartheta(u, \delta, \eta, \boldsymbol{\theta}), X_0 \rangle, \tag{4.18}$$

where

$$\begin{aligned} \vartheta(u, \delta, \eta, \boldsymbol{\theta}) &= (\Xi_1(u, \delta, \eta, \boldsymbol{\theta}), \Xi_2(u, \delta, \eta, \boldsymbol{\theta}))^* \\ &= \left(\sum_{i=1}^{2m+2} c_i e^{\rho_i u}, \sum_{i=1}^{2m+2} d_i e^{\rho_i u} \right)^*, \end{aligned} \tag{4.19}$$

with c_j's, d_j's given by (4.11) *and* (4.12), *respectively.*

Now, we are prepared to price the participating insurance contract. Using the standard machinery of arbitrage theory, the arbitrage free price of the life insurance contract $V_L(0)$ is given by:

$$V_L(0) = E\left[e^{-rT}(L_T^g + \beta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+) 1_{\{\tau \geq T\}} + e^{-r\tau} A_\tau 1_{\{\tau \leq T\}}\right].$$

This contract can be split up into four simpler subcontracts:

$$V_L(0) = V_1 + V_2 + V_3 + V_4,$$

where

$$\begin{cases} V_1 &= L_0 e^{-(r-r_g)T} E\left[1_{\{\tau \geq T\}}\right] \doteq L_0 e^{-(r-r_g)T} GF(T), \\ V_2 &= \beta e^{-rT} E\left[(\alpha A_T - L_T^g)^+ 1_{\{\tau \geq T\}}\right] \doteq \beta e^{-(r-r_g)T} BO(T, k), \\ V_3 &= E\left[e^{-rT}(L_T^g - A_T)^+ 1_{\{\tau \geq T\}}\right] \doteq e^{-(r-r_g)T} PO(T, k'), \\ V_4 &= E\left[e^{-r\tau} A_\tau 1_{\{\tau \leq T\}}\right] \doteq LR(T), \end{cases}$$

with

$$\begin{cases} GF(T) &= E\left[1_{\{\tau\geq T\}}\right], \\ BO(T,k) &= E\left[(\alpha A_T e^{-r_g T} - e^{-k})^+ 1_{\{\tau\geq T\}}\right], \ k = -\log(L_0), \\ PO(T,k') &= E\left[(e^{k'} - A_T e^{-r_g T})^+ 1_{\{\tau\geq T\}}\right], \ k' = \log(L_0). \end{cases}$$

Here V_1 stands for the final guarantee, V_2 corresponds to the bonus option, V_3 stands for the put option, and V_4 is the rebate paid to policyholders in case of default.

Note that, there are no exact results for the price of the participating contract. However, the price is associated with some integral transforms of $GF(T)$, $BO(T,k)$, $PO(T,k')$ and $LR(T)$. Numerical algorithms, such as, Laplace inversion, fast Fourier transform, are widely used to numerically evaluate some financial products, see Dong [8], Siu et al. [6] and Carr and Madan [5]. We aim at providing a pricing model in this paper and therefore we only focus on deriving the price by using Laplace inversion.

Theorem 4.6. *For $\delta > 0$, the Laplace transform with respect to T of $GF(T)$ is*:

$$\hat{L}_1(\delta) = \int_0^\infty e^{-\delta T} GF(T) dT = \frac{1 - \langle \vartheta(u,\delta,0,\mathbf{1}), X_0\rangle}{\delta}, \tag{4.20}$$

where $\vartheta(u,\delta,0,\mathbf{1})$ is defined by (4.19) *with $\eta, \boldsymbol{\theta}$ replaced by* 0 *and* $\mathbf{1}$*, respectively.*

Proof. Applying the Fubini theorem yields that

$$\hat{L}_1(\delta) = E\left[\int_0^\tau e^{-\delta T} dT\right] = \frac{1 - E\left[e^{-\delta\tau}\right]}{\delta},$$

then the result can be obtained from (4.18) by letting $\eta = 0$ and $\boldsymbol{\theta} = (1,\dots,1)^*$. □

Theorem 4.7. *For $\delta > 0$, the Laplace transform with respect to T of $LR(T)$ is*:

$$\hat{L}_2(\delta) = \int_0^\infty e^{-\delta T} LR(T) dT = \frac{A_0 e^{-u}\langle \vartheta(u,\delta + r - r_g, 1, \mathbf{1}), X_0\rangle}{\delta}, \tag{4.21}$$

where $\vartheta(u,\delta+r-r_g,1,\mathbf{1})$ is defined by (4.18) *with $\delta, \eta, \boldsymbol{\theta}$ replaced by $\delta + r - r_g$,* 1 *and* $\mathbf{1}$*, respectively.*

Proof. Applying the Fubini theorem yields that

$$\hat{L}_2(\delta) = E\left[\int_\tau^\infty e^{-\delta T - r\tau} A_\tau dT\right] = \frac{E\left[A_\tau e^{-(\delta+r)\tau}\right]}{\delta} = \frac{E\left[A_0 e^{-(\delta+r-r_g)\tau + Y(\tau)}\right]}{\delta}.$$

Then by using (4.18), we can obtain the result. □

Theorem 4.8. *Let* $\mathbf{p} = (p_1, \ldots, p_N)^*$ *where* $p_i = \int_0^\infty e^{-\delta t}\langle \exp\{(\mathbf{diag}(\tilde{\mathbf{G}}(\eta + 1)) + A)t\}\mathbf{1}, e_i\rangle dt$ *with* $0 < \eta < \alpha_{1,1} - 1$ *and* $\delta > 0$ *satisfying* $p_i < \infty$. *Then the Laplace transform with respect to* T *and* k *of* $BO(T, k)$ *is*:

$$\begin{aligned}\hat{L}_3(\delta, \eta) &= \int_0^\infty \int_{-\infty}^\infty e^{-\delta T - \eta k} BO(T, k) dT dk \\ &= \frac{(\alpha A_0)^{\eta+1}}{\eta(\eta+1)} \langle \mathbf{p} - e^{-(\eta+1)u}\vartheta(u, \delta, \eta + 1, \mathbf{p}), X_0\rangle,\end{aligned} \tag{4.22}$$

where $\vartheta(u, \delta, \eta + 1, \mathbf{p})$ *is defined by* (4.18) *with* $\eta, \boldsymbol{\theta}$ *replaced by* $\eta + 1$ *and* $\mathbf{p}$, *respectively.*

Proof. By Fubini theorem,

$$\begin{aligned}\hat{L}_3(\delta, \eta) &= E\left[\int_0^\infty \int_{r_g T - \log(\alpha A_T)}^\infty e^{-\delta T - \eta k} 1_{\{\tau \geq T\}} (\alpha A_T e^{-r_g T} - e^{-k}) dk dT\right] \\ &= E\left[\int_0^\infty (\alpha A_0)^{\eta+1} 1_{\{\tau \geq T\}} \frac{e^{-\delta T + (\eta+1)Y(T)}}{\eta(\eta+1)} dT\right] \\ &= \frac{(\alpha A_0)^{\eta+1}}{\eta(\eta+1)} \left(\langle \mathbf{p}, X_0\rangle - E\left[\int_0^\infty e^{-\delta(t+\tau) + (\eta+1)Y(t+\tau)} dt\right]\right),\end{aligned}$$

where the last equality follows from Lemma 3.3. The strong Markov property of Y implies that

$$\begin{aligned}&E\left[\int_0^\infty e^{-\delta(t+\tau)+(\eta+1)Y(t+\tau)} dt\right] \\ &= E\left[e^{-\delta\tau + (\eta+1)Y(\tau)} \int_0^\infty e^{-\delta t} E\left[e^{(\eta+1)(Y(t+\tau) - Y(\tau))} | \mathcal{F}_\tau\right] dt\right] \\ &= e^{-(\eta+1)u} E\left[e^{-\delta\tau + (\eta+1)(u + Y(\tau))} \langle \mathbf{p}, X_\tau\rangle\right] \\ &= e^{-(\eta+1)u} \langle \vartheta(u, \delta, \eta + 1, \mathbf{p}), X_0\rangle,\end{aligned}$$

which concludes the result. □

Theorem 4.9. *Let* $\mathbf{q} = (q_1, \ldots, q_N)^*$ *where* $q_i = \int_0^\infty e^{-\delta t}\langle \exp\{(\mathbf{diag}(\tilde{\mathbf{G}}(1 - \eta)) + A)t\}\mathbf{1}, e_i\rangle dt$ *with* $0 < \eta < \max\{\alpha_{1,1} - 1, 1\}$ *and* $\delta > 0$ *satisfying* $q_i < \infty$. *Then the Laplace transform with respect to* T *and* k *of* $PO(T, k')$ *is*:

$$\begin{aligned}\hat{L}_4(\delta, \eta) &= \int_0^\infty \int_{-\infty}^\infty e^{-\delta T - \eta k'} PO(T, k') dT dk' \\ &= \frac{A_0^{1-\eta}}{\eta(\eta-1)} \langle \mathbf{q} - e^{(\eta-1)u}\vartheta(u, \delta, 1 - \eta, \mathbf{q}), X_0\rangle,\end{aligned} \tag{4.23}$$

where $\vartheta(u, \delta, 1 - \eta, \mathbf{q})$ *is defined by* (4.18) *with* $\eta, \boldsymbol{\theta}$ *replaced by* $1 - \eta$ *and* $\mathbf{q}$, *respectively.*

Proof. By Fubini theorem,

$$\begin{aligned}\hat{L}_4(\delta,\eta) &= E\left[\int_0^\infty \int_{\log(A_T)-r_gT}^\infty e^{-\delta T-\eta k'}1_{\{\tau\geq T\}}(e^{k'}-A_Te^{-r_gT})dkdT\right]\\ &= E\left[\int_0^\infty A_0^{1-\eta}1_{\{\tau\geq T\}}\frac{e^{-\delta T-(\eta-1)Y(T)}}{\eta(\eta-1)}dT\right]\\ &= \frac{A_0^{1-\eta}}{\eta(\eta-1)}\left(\langle \mathbf{q}, X_0\rangle - E\left[\int_0^\infty e^{-\delta(t+\tau)-(\eta-1)Y(t+\tau)}dt\right]\right).\end{aligned}$$

The strong Markov property of Y implies that

$$\begin{aligned}&E\left[\int_0^\infty e^{-\delta(t+\tau)-(\eta-1)Y(t+\tau)}dt\right]\\ &= E\left[e^{-\delta\tau-(\eta-1)Y(\tau)}\int_0^\infty e^{-\delta t}E\left[e^{-(\eta-1)(Y(t+\tau)-Y(\tau))}|\mathcal{F}_\tau\right]dt\right]\\ &= e^{(\eta-1)u}E\left[e^{-\delta\tau-(\eta-1)(u+Y(\tau))}\langle \mathbf{q}, X_\tau\rangle\right]\\ &= e^{(\eta-1)u}\langle \vartheta(u,\delta,1-\eta,\mathbf{q}), X_0\rangle,\end{aligned}$$

which yields the result. □

By inverting the Laplace transforms, we can obtain the price of participating contract numerically.

5. Numerical examples

In this section, we intend to price the participating life insurance contract by numerically inverting the associated Laplace transforms (4.20)-(4.23). Since the impacts of the parameters on the price of the participating life insurance contract have been investigated in a lot of literature, we only check whether changes of market regimes significantly impact the price of the participating life insurance contract.

In (4.20) and (4.21), the Laplace transforms of $GF(T)$ and $LR(T)$ with respect to T are one-sided, so we can obtain numerical results for $GF(T)$ and $LR(T)$ by inverting (4.20) and (4.21) via the Gaver-Stehfest algorithm, which is used in Kou and Wang (2003) and Dong et al. [10]. For the details of the implementation of the Gaver-Stehfest algorithm, we refer to Section 5 in Kou and Wang (2003) or Section 4 in Dong et al. [10]. In (4.22) and (4.23), the Laplace transforms with respect to k or k' are two-sided, so we invert (4.22) and (4.23) via the Euler inversion algorithm, which is introduced by Abate and Whitt [1] and later extended to the two-sided case by Petrella [17]. Parameters for the Euler inversion algorithm (see Eq. (5) in Petrella [17]) are $A_1 = A_2 = 40$, which are used in Petrella [17]. All the computations are done on a laptop with a 3.00 GHz CPU.

For all the computations, the values of certain parameters are held fixed except otherwise indicated: we take $T = 10$, $A_0 = 100$, $K_0 = 70$, $a_{11} = a_{22} =$

$-0.1, r = 0.03,\ r_g = 0.02,\ m = 2, \sigma_1 = 0.6, \sigma_2 = 0.3,\ \lambda_{11} = \lambda_{21} = 2, \lambda_{12} = \lambda_{22} = 1,\ \alpha_{1,1} = \beta_{1,1} = 25, \alpha_{1,2} = \beta_{1,2} = 50, \alpha_{2,1} = \beta_{2,1} = 30, \alpha_{2,2} = \beta_{2,2} = 60, \alpha = 0.85, \gamma = 0.7, \beta = 0.9$.

Figure 2 presents the relationship between the default probability and t for different γ. From it we can see that the default probability is much larger if we start at the "bad" economy at time $t = 0$. We can also see that the default probability increases with γ. That is because the firm is more likely to default with a higher default barrier.

Figure 3 plots default probability versus $-a_{11}$ with $a_{11} = a_{22}$ for different α. From it we can conclude that a higher a_{12} results in a lower default probability if we start at the "bad" economy at time $t = 0$ ($X_0 = e_1$). This is due to the fact that the probability that the Markov chain switches to the "good" economy increases with a_{12}. On the other hand, if we start at $X_0 = e_2$, the default probability increases with a_{12} because of an increasing probability of switching to the "bad" economy. We can also see that the default probability increases with α. That is because a larger value of α implies a larger value of L_0 and a higher default barrier.

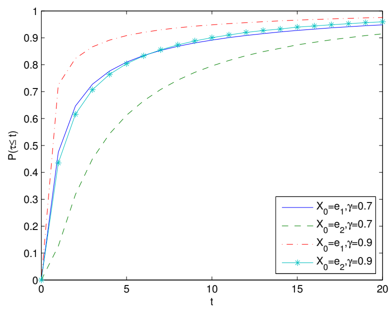

FIGURE 2. $P(\tau \le t)$ versus t

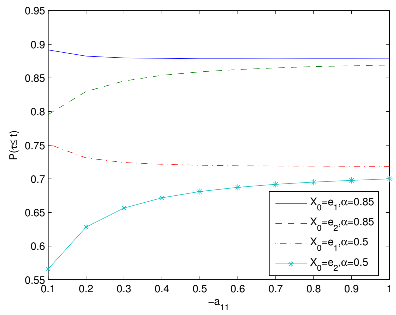

FIGURE 3. $P(\tau \le t)$ versus $-a_{11}$

Figure 4 presents the relationship between the contract value and the jump intensity λ_{21} with $\lambda_{21} = \lambda_{11}, \lambda_{12} = \lambda_{22} = 1/2\lambda_{21}$. From it we can see that the contract value is much larger if we start at the "good" economy. We can also see that the general shape of the curves is similar to the plots in Le Courtois and Quittard-Pinon [7] and we can find the existence of optima of the jump intensity where the contract value is maximized.

Figure 5 plots the impact of transition intensity on the contract value. From it we can conclude that the contract value is an increasing function of a_{12} if we start at the "bad" economy. This is because a higher a_{12} results in an increasing probability of switching to the "good" economy. On the other hand, if we start at the "good" economy, the contract value decreases with a_{12}. This is due to an increasing probability of switching to the "bad" economy. Note

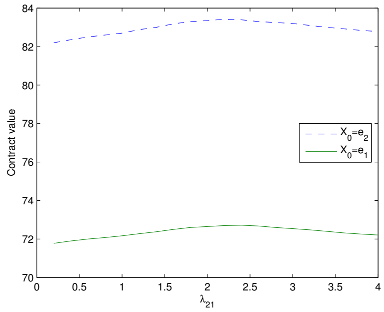

FIGURE 4. contract value versus λ_{21}

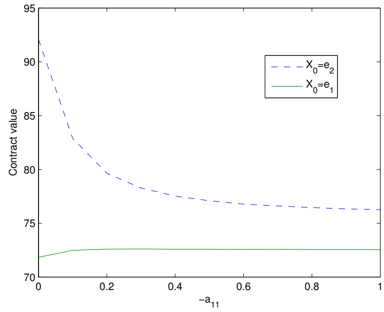

FIGURE 5. contract value versus a_{12}

that when $a_{12} \to 0$, then the regime-switching model becomes a model without regime-switching. Therefore, from Figure 5 we see that if we do not incorporate regime-switching into the model and assume the economy is always in a good state, then the contract will be overpriced; if we assume the economy is always in a bad state, then the contract will be underpriced.

To sum up, numerical results indicate that changes of market regimes have material effects on the default probability and the contract value.

6. Conclusions

In this paper, we investigate the valuation of participating contracts under a regime-switching jump-diffusion model. Since the Laplace transform for the price of the participating contract is associated with the Laplace transform of the first passage time, the firm's expected present market value at default and the distribution of the Markov chain at default, we derive integro-differential system for the joint Laplace transform of the default time, the firm's expected present market value at default and the distribution of Markov chain at default when the jumps follow a regime-switching hyper-exponential distribution. Based on the result, we give numerical calculations for the value of the participating contract by inverting one-sided and two-sided Laplace transforms via Gaver-Stehfest algorithm and Euler algorithm, respectively. Numerical results illustrate the regime-switching effects have a significant impact on the value of participating contracts. Therefore, we should incorporate changes of market regimes into models for pricing long term insurance products.

The present work might be extended by incorporating the mortality risk into the model. Then the valuation of the participating will be much more complex since the price will be dependent on the joint distribution of the date of death and the default time. We will study it in the future's research.

Appendix A. The derivation of the characteristic equation (4.7)

Similar to Dong et al. (2011), applying the differential operator polynomials $\prod_{i=1}^{m}(\mathbf{D}+\beta_{i,1}\mathbf{I})(\mathbf{D}-\alpha_{i,1}\mathbf{I})$ and $\prod_{i=1}^{m}(\mathbf{D}+\beta_{i,2}\mathbf{I})(\mathbf{D}-\alpha_{i,2}\mathbf{I})$ to the both sides of (4.5) and (4.6), respectively,

$$\begin{aligned}
&\prod_{i=1}^{m}\left[(\mathbf{D}+\beta_{i,1}\mathbf{I})(\mathbf{D}-\alpha_{i,1}\mathbf{I})\right]h_1(\mathbf{D})\Xi_1(u)\\
&= \lambda_1\sum_{i=1}^{m}p_{i1}\alpha_{i,1}\left((\mathbf{D}+\beta_{i,1}\mathbf{I})\prod_{j=1,j\neq i}^{m}\left[(\mathbf{D}+\beta_{j,1}\mathbf{I})(\mathbf{D}-\alpha_{j,1}\mathbf{I})\right]\right)\Xi_1(u)\\
&\quad -\lambda_1\sum_{i=1}^{m}q_{i1}\beta_{i,1}\left((\mathbf{D}-\beta_{i,1}\mathbf{I})\prod_{j=1,j\neq i}^{m}\left[(\mathbf{D}+\beta_{j,1}\mathbf{I})(\mathbf{D}-\alpha_{j,1}\mathbf{I})\right]\right)\Xi_1(u)\\
&\quad + a_{11}\prod_{i=1}^{m}\left[(\mathbf{D}+\beta_{i,1}\mathbf{I})(\mathbf{D}-\alpha_{i,1}\mathbf{I})\right]\Xi_2(u),
\end{aligned}\tag{A.1}$$

and

$$\begin{aligned}
&\prod_{i=1}^{m}\left[(\mathbf{D}+\beta_{i,2}\mathbf{I})(\mathbf{D}-\alpha_{i,2}\mathbf{I})\right]h_2(\mathbf{D})\Xi_2(u)\\
&= \lambda_2\sum_{i=1}^{m}p_{i2}\alpha_{i,2}\left((\mathbf{D}+\beta_{i,2}\mathbf{I})\prod_{j=1,j\neq i}^{m}\left[(\mathbf{D}+\beta_{j,2}\mathbf{I})(\mathbf{D}-\alpha_{j,2}\mathbf{I})\right]\right)\Xi_2(u)\\
&\quad -\lambda_2\sum_{i=1}^{m}q_{i2}\beta_{i,2}\left((\mathbf{D}-\beta_{i,2}\mathbf{I})\prod_{j=1,j\neq i}^{m}\left[(\mathbf{D}+\beta_{j,2}\mathbf{I})(\mathbf{D}-\alpha_{j,2}\mathbf{I})\right]\right)\Xi_2(u)\\
&\quad + a_{22}\prod_{i=1}^{m}\left[(\mathbf{D}+\beta_{i,2}\mathbf{I})(\mathbf{D}-\alpha_{i,2}\mathbf{I})\right]\Xi_1(u).
\end{aligned}\tag{A.2}$$

Define operators

$$\begin{aligned}
\tilde{h}_j(\mathbf{D}) = &\prod_{i=1}^{m}\left[(\mathbf{D}+\beta_{i,j}\mathbf{I})(\mathbf{D}-\alpha_{i,j}\mathbf{I})\right]h_j(\mathbf{D})\\
&-\lambda_j\Bigg(\sum_{i=1}^{m}p_{ij}\alpha_{i,j}((\mathbf{D}+\beta_{i,j}\mathbf{I})\prod_{l=1,l\neq i}^{m}[(\mathbf{D}+\beta_{l,j}\mathbf{I})(\mathbf{D}-\alpha_{l,j}\mathbf{I})])\\
&-\sum_{i=1}^{m}q_{ij}\beta_{i,j}\big((\mathbf{D}-\alpha_{i,j}\mathbf{I})\prod_{l=1,l\neq i}^{m}[(\mathbf{D}+\beta_{l,j}\mathbf{I})(\mathbf{D}-\alpha_{l,j}\mathbf{I})]\big)\Bigg),\ j=1,2.
\end{aligned}$$

Similarly, we define

$$\begin{aligned}\tilde{h}_j(x) = & \prod_{i=1}^{m}(x+\beta_{i,j})(x-\alpha_{i,j})h_j(x) \\ & -\lambda_j\bigg(\sum_{i=1}^{m}p_{ij}\alpha_{i,j}\big((x+\beta_{i,j})\prod_{l=1,l\neq i}^{m}[(x+\beta_{l,j})(x-\alpha_{l,j})]\big) \\ & -\sum_{i=1}^{m}q_{ij}\beta_{i,j}\big((x-\alpha_{i,j})\prod_{l=1,l\neq i}^{m}[(x+\beta_{l,j})(x-\alpha_{l,j})]\big)\bigg),\ j=1,2,\end{aligned}$$

where $h_i(x) = \frac{1}{2}\sigma_i^2 x^2 + \overline{\mu}_i x - (\delta - a_{ii} + \lambda_i)$.

By using $\tilde{h}_i(\mathbf{D})$, Eqs. (A.1) and (A.2) become

$$\tilde{h}_1(\mathbf{D})\Xi_1(u) = -a_{12}\prod_{i=1}^{m}(\mathbf{D}+\beta_{i,1}\mathbf{I})(\mathbf{D}-\alpha_{i,1}\mathbf{I})\Xi_2(u), \tag{A.3}$$

and

$$\tilde{h}_2(\mathbf{D})\Xi_2(u) = -a_{21}\prod_{i=1}^{m}(\mathbf{D}+\beta_{i,2}\mathbf{I})(\mathbf{D}-\alpha_{i,2}\mathbf{I})\Xi_1(u). \tag{A.4}$$

Then it follows from (A.3) and (A.4) that

$$\tilde{h}_2(\mathbf{D})\tilde{h}_1(\mathbf{D})\Xi_1(u) = a_{11}a_{22}\prod_{j=1}^{2}\prod_{i=1}^{m}(\mathbf{D}+\beta_{i,j}\mathbf{I})(\mathbf{D}-\alpha_{i,j}\mathbf{I})\Xi_1(u), \tag{A.5}$$

and

$$\tilde{h}_1(\mathbf{D})\tilde{h}_2(\mathbf{D})\Xi_2(u) = a_{11}a_{22}\prod_{j=1}^{2}\prod_{i=1}^{m}(\mathbf{D}+\beta_{i,j}\mathbf{I})(\mathbf{D}-\alpha_{i,j}\mathbf{I})\Xi_2(u). \tag{A.6}$$

Therefore, the characteristic equation of (A.5)-(A.6) is given by (4.7).

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Yinghui Dong
Department of Mathematics and Physics
Suzhou University of Science and Technology
Suzhou 215009, P. R. China
Email address: `dongyinghui1030@163.com`

Wenxin Lv
Department of Mathematics and Physics
Suzhou University of Science and Technology
Suzhou 215009, P. R. China
Email address: 15136627969@163.com

Sang Wu
Department of Mathematics and Physics
Suzhou University of Science and Technology
Suzhou 215009, P. R. China
Email address: 15715848990@163.com