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RIGIDITY OF COMPLETE RIEMANNIAN MANIFOLDS WITH VANISHING BACH TENSOR

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Abstract. For complete Riemannian manifolds with vanishing Bach tensor and positive constant scalar curvature, we provide a rigidity theorem characterized by some pointwise inequalities. Furthermore, we prove some rigidity results under an inequality involving $L^{\frac{n}{2}}$-norm of the Weyl curvature, the traceless Ricci curvature and the Sobolev constant.

1. Introduction

In order to study conformal relativity, R. Bach [1] in early 1920s' introduced the following Bach tensor

$$B_{ij}=\frac{1}{n-3}W_{ikjl,lk}+\frac{1}{n-2}W_{ikjl}R_{kl}, \tag{1.1}$$

where $n\geq 4$, W_{ijkl} and R_{kl} denote the Weyl curvature and the Ricci curvature, respectively. A Bach tensor of the metric g is called a vanishing Bach tensor if $B_{ij}=0$. The authors in [3, 4, 10] consider complete noncompact Riemannian manifolds with vanishing Bach tensor and prove that M^n is of constant curvature if the L^2-norm of traceless Riemannian curvature tensor is small. In [11], Kim studied complete noncompact Riemannian manifolds with harmonic curvature and positive Sobolev constant, he obtained that M^n, $n\geq 5$, is Einstein if the L^2-norm of the Weyl curvature and the traceless Ricci curvature are small enough.

The aim of this paper is to achieve some rigidity results for complete Riemannian manifolds with vanishing Bach tensor. In order to state our results, throughout this paper, we always denote by R, $\mathring{R}_{ij}$ the scalar curvature and the traceless Ricci curvature of $M^n(n\geq 4)$, respectively.

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Theorem 1.1. *Let (M^n, g) be a complete manifold with vanishing Bach tensor, positive constant scalar curvature and*

$$\int_M |\mathring{R}_{ij}|^2 < \infty. \tag{1.2}$$

If

$$\Big|W + \frac{n}{\sqrt{8n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| \le \frac{R}{\sqrt{2(n-1)(n-2)}}, \tag{1.3}$$

then M^n is Einstein. In particular, when $n = 4, 5$, M^n is of constant positive sectional curvature.

Recall that the Sobolev constant $Q_g(M)$ is defined by

$$Q_g(M) = \inf_{0\neq u\in C_0^\infty(M)} \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)}R_g u^2\right)}{\left(\int_M |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}. \tag{1.4}$$

Moreover, there exist complete noncompact manifolds with negative scalar curvature which have positive Sobolev constant. For example, any simply connected complete manifold with $W_{ijkl} = 0$ has positive Sobolev constant (see [14]). Moreover, it is easy to see from (1.4), for any u,

$$Q_g(M)\Big(\int_M |u|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \le \int_M \Big(|\nabla u|^2 + \frac{n-2}{4(n-1)}R_g u^2\Big). \tag{1.5}$$

With the help of (1.5), we can achieve the following rigidity result:

Theorem 1.2. *Let (M^n, g) be a complete manifold with vanishing Bach tensor, constant scalar curvature, $Q_g(M) > 0$ and*

$$\int_M |\mathring{R}_{ij}|^2 < \infty. \tag{1.6}$$

(1) *If $n \ge 7$ and*

$$\Big(\int_M \Big|W + \frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < \frac{2}{n-2}\sqrt{\frac{2(n-1)}{n-2}}Q_g(M), \tag{1.7}$$

then M^n is Einstein;

(2) *If $4 \le n \le 6$, $R \ge 0$ and*

$$\Big(\int_M \Big|W + \frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < \sqrt{\frac{n-1}{2(n-2)}}Q_g(M), \tag{1.8}$$

then M^n is Einstein. In particular, for $M^n (n = 4, 5)$ with positive constant scalar curvature, it must be of constant sectional curvature.

It is well-known that there is no complete noncompact Einstein manifold with positive constant scalar curvature. Hence, the following results follow easily:

Corollary 1.3. *Suppose that (M^n, g) is a complete noncompact Riemannian manifold with vanishing Bach tensor and positive constant scalar curvature. If* (1.3) *holds, then we have*

$$\int_M |\mathring{\mathrm{Ric}}|^2 = \infty. \tag{1.9}$$

Corollary 1.4. *Let (M^n, g) be a complete noncompact manifold with vanishing Bach tensor and positive constant scalar curvature. If either $n \geq 7$ and* (1.7)*, or $4 \leq n \leq 6$ and* (1.8) *holds, then we have*

$$\int_M |\mathring{R}_{ij}|^2 = \infty. \tag{1.10}$$

Remark 1.1. The authors in [5,8] obtained some rigidity results similar to our theorems for compact manifolds with vanishing Bach tensor. Our theorems can be seen as a generalization to complete manifolds.

Remark 1.2. In [10, Theorem 2], Kim proved, for complete noncompact manifold M^4 with vanishing Bach tensor and nonnegative constant scalar curvature, that if there exists a constant c_0 small enough such that

$$\int_M (|W|^2 + |\mathring{R}_{ij}|^2) \leq c_0, \tag{1.11}$$

then M^4 is Einstein. When $n = 4$, our formula (1.8) becomes

$$\int_M \left| W + \frac{1}{2\sqrt{2}} \mathring{\mathrm{Ric}} \owedge g \right|^2 < \frac{3}{4} Q_g^2(M), \tag{1.12}$$

which, compared with (1.11), shows that our Theorem 1.2 gives an upper bound of c_0, in some sense. Moreover, we also prove that for the upper bound $\frac{3}{4}Q_g^2(M)$ given by (1.12), M^4 must be of constant sectional curvature if it has positive constant scalar curvature.

2. Some lemmas

Recall that the Weyl curvature W_{ijkl} is relate to the Riemannian curvature R_{ijkl} by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}). \tag{2.1}$$

By virtue of $\mathring{R}_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, (2.1) can be written as

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(\mathring{R}_{ik}g_{jl} - \mathring{R}_{il}g_{jk} + \mathring{R}_{jl}g_{ik} - \mathring{R}_{jk}g_{il}) \\ &- \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned} \tag{2.2}$$

Since the divergence of the Weyl curvature tensor is related to the Cotton tensor by

$$W_{ijkl,l} = -\frac{n-3}{n-2}C_{ijk}, \tag{2.3}$$

where the Cotton tensor is given by

$$\begin{aligned} C_{ijk} &= R_{kj,i} - R_{ki,j} - \frac{1}{2(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}) \\ &= \mathring{R}_{kj,i} - \mathring{R}_{ki,j} + \frac{n-2}{2n(n-1)}(R_{,i}g_{jk} - R_{,j}g_{ik}), \end{aligned} \tag{2.4}$$

the formula (1.1) reduces to

$$B_{ij} = \frac{1}{n-2}(C_{kij,k} + W_{ikjl}R^{kl}). \tag{2.5}$$

We denote by $p \in M^n$ and B_r a fixed point and the geodesic ball of M^n of radius r centered at p, respectively. Let ϕ_r be the nonnegative cut-off function defined on M^n satisfying

$$\phi_r = \begin{cases} 1, & \text{on } B_r; \\ 0, & \text{on } M^n \backslash B_{r+1}, \end{cases} \tag{2.6}$$

with $|\nabla \phi_r| \leq 2$ on $B_{r+1}\backslash B_r$.

Inspired by [5, Lemma 2.2], we give the following estimate with respect to $\mathring{R}_{ij}$ on a complete noncompact Riemannian manifold:

Lemma 2.1. *Let (M^n, g) be a complete noncompact Riemannian manifold with constant scalar curvature. Then for any $\theta \in \mathbb{R}$, we have*

$$\begin{aligned} &\int_M |\nabla \mathring{R}_{ij}|^2\phi_r^2 \\ &\geq \frac{2\theta}{\theta^2+1+\epsilon_1}\int_M \Big(W_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1}R|\mathring{R}_{ij}|^2\Big)\phi_r^2 \\ &\quad - \frac{4\theta^2}{\epsilon_1(\theta^2+1+\epsilon_1)}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2, \end{aligned} \tag{2.7}$$

where ϵ_1 is a positive constant.

Proof. By a direct calculation, we have

$$\begin{aligned} 0 &\leq \int_M |\mathring{R}_{kj,i} - \theta\mathring{R}_{ki,j}|^2\phi_r^2 \\ &= (\theta^2+1)\int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 - 2\theta\int_M \mathring{R}_{kj,i}\mathring{R}_{ki,j}\phi_r^2. \end{aligned} \tag{2.8}$$

Using the Ricci identity and (2.2), we have

$$\begin{aligned}\mathring{R}_{kj,ij}\mathring{R}_{ki} &= (\mathring{R}_{kj,ji} + \mathring{R}_{pj}R_{pkij} + \mathring{R}_{kp}R_{pjij})\mathring{R}_{ki} \\ &= \mathring{R}_{pj}\mathring{R}_{ki}R_{pkij} + \mathring{R}_{kp}\mathring{R}_{ki}\mathring{R}_{pi} + \frac{R}{n}|\mathring{R}_{ij}|^2 \\ &= -W_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl} + \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} + \frac{1}{n-1}R|\mathring{R}_{ij}|^2,\end{aligned} \tag{2.9}$$

where we used the second Bianchi identity and $\mathring{R}_{ij,j} = \frac{n-2}{2n}R_{,i} = 0$ from the fact that R is constant. Hence, we obtain

$$\begin{aligned}&-2\theta\int_M \mathring{R}_{kj,i}\mathring{R}_{ki,j}\phi_r^2 \\ =\ & 2\theta\int_M \mathring{R}_{kj,ij}\mathring{R}_{ki}\phi_r^2 + 2\theta\int_M \mathring{R}_{kj,i}\mathring{R}_{ki}(\phi_r^2)_j \\ =\ & -2\theta\int_M \Big(W_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1}R|\mathring{R}_{ij}|^2\Big)\phi_r^2 \\ & + 2\theta\int_M \mathring{R}_{kj,i}\mathring{R}_{ki}(\phi_r^2)_j \\ \le\ & -2\theta\int_M \Big(W_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1}R|\mathring{R}_{ij}|^2\Big)\phi_r^2 \\ & + \epsilon_1\int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 + \frac{4\theta^2}{\epsilon_1}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2.\end{aligned} \tag{2.10}$$

Applying (2.10) into (2.8) yields the desired estimate (2.7). □

Lemma 2.2. *Let (M^n, g) be a complete noncompact Riemannian manifold with constant scalar curvature. If the Bach tensor is flat, we have*

$$\begin{aligned}\int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 \le \frac{1}{1-\epsilon_2}\int_M \Big(2W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} \\ - \frac{1}{n-1}R|\mathring{R}_{ij}|^2\Big)\phi_r^2 + \frac{1}{\epsilon_2(1-\epsilon_2)}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2,\end{aligned} \tag{2.11}$$

where $\epsilon_2 \in (0, 1)$ is a constant.

Proof. Using the formula (2.2), we can derive

$$\mathring{R}_{kl}R_{ikjl} = \mathring{R}_{kl}W_{ikjl} + \frac{1}{n-2}(|\mathring{R}_{ij}|^2 g_{ij} - 2\mathring{R}_{ik}\mathring{R}_{jk}) - \frac{1}{n(n-1)}R\mathring{R}_{ij}, \tag{2.12}$$

which shows

$$\begin{aligned}\mathring{R}_{kj,ik} &= \mathring{R}_{kj,ki} + \mathring{R}_{lj}R_{lkik} + \mathring{R}_{kl}R_{ljik} \\ &= \mathring{R}_{ik}\mathring{R}_{jk} + \frac{1}{n}R\mathring{R}_{ij} - \Big[\mathring{R}_{kl}W_{ikjl} \\ &\quad + \frac{1}{n-2}(|\mathring{R}_{ij}|^2 g_{ij} - 2\mathring{R}_{ik}\mathring{R}_{jk}) - \frac{1}{n(n-1)}R\mathring{R}_{ij}\Big]\end{aligned} \tag{2.13}$$

$$= \frac{n}{n-2}\mathring{R}_{ik}\mathring{R}_{jk} + \frac{1}{n-1}R\mathring{R}_{ij} - \mathring{R}_{kl}W_{ikjl} - \frac{1}{n-2}|\mathring{R}_{ij}|^2 g_{ij}.$$

Thus, from (2.4) and (2.13), we have

$$\begin{aligned} C_{kij,k} &= \Delta\mathring{R}_{ij} - \mathring{R}_{kj,ik} \\ &= \Delta\mathring{R}_{ij} - \Big(\frac{n}{n-2}\mathring{R}_{ik}\mathring{R}_{jk} + \frac{1}{n-1}R\mathring{R}_{ij} - \mathring{R}_{kl}W_{ikjl} \\ &\quad - \frac{1}{n-2}|\mathring{R}_{ij}|^2 g_{ij}\Big) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} 0 &= (n-2)B_{ij}\mathring{R}_{ij} \\ &= C_{kij,k}\mathring{R}_{ij} + W_{ikjl}\mathring{R}_{ij}\mathring{R}_{kl} \\ &= \mathring{R}_{ij}\Delta\mathring{R}_{ij} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1}R|\mathring{R}_{ij}|^2 + 2W_{ikjl}\mathring{R}_{ij}\mathring{R}_{kl}, \end{aligned} \tag{2.15}$$

which gives

$$\mathring{R}_{ij}\Delta\mathring{R}_{ij} = \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} + \frac{1}{n-1}R|\mathring{R}_{ij}|^2 - 2W_{ikjl}\mathring{R}_{ij}\mathring{R}_{kl}. \tag{2.16}$$

Thus,

$$\begin{aligned} \int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 = &-\int_M \mathring{R}_{ij}\Delta\mathring{R}_{ij}\phi_r^2 - \int_M \mathring{R}_{ij}\mathring{R}_{ij,k}(\phi_r^2)_k \\ = &\int_M \Big(2W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1}R|\mathring{R}_{ij}|^2\Big)\phi_r^2 \\ &-\int_M \mathring{R}_{ij}\mathring{R}_{ij,k}(\phi_r^2)_k \\ \le &\int_M \Big(2W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} - \frac{1}{n-1}R|\mathring{R}_{ij}|^2\Big)\phi_r^2 \\ &+ \epsilon_2\int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 + \frac{1}{\epsilon_2}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2. \end{aligned} \tag{2.17}$$

We complete the proof of Lemma 2.2. □

The following two lemmas come from [8] (for more details, see [2,6,7,12,13]. For a proof of Lemma 2.4, we refer to [9]):

Lemma 2.3. *On every Riemannian manifold (M^n, g), for any $\lambda \in \mathbb{R}$, the following estimate holds*

$$\begin{aligned} &\Big| - W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} + \lambda\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\Big| \\ &\le \sqrt{\frac{n-2}{2(n-1)}}\Big(|W|^2 + \frac{2(n-2)\lambda^2}{n}|\mathring{R}_{ij}|^2\Big)^{\frac{1}{2}}|\mathring{R}_{ij}|^2 \end{aligned} \tag{2.18}$$

$$= \sqrt{\frac{n-2}{2(n-1)}}\Big|W + \frac{\lambda}{\sqrt{2n}}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2.$$

Lemma 2.4. *On every Einstein manifold (M^n, g), we have*

$$\frac{1}{2}\Delta|W|^2 \geq \frac{n+1}{n-1}|\nabla|W||^2 + \frac{2}{n}R|W|^2 - 2C_n|W|^3, \tag{2.19}$$

where C_n is defined by

$$C_n = \begin{cases} \frac{\sqrt{6}}{4}, & \text{if } n = 4; \\ \frac{4\sqrt{10}}{15}, & \text{if } n = 5; \\ \frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}, & \text{if } n \geq 6. \end{cases} \tag{2.20}$$

In particular, if the scalar curvature of Einstein metric g is positive, then it is of constant positive sectional curvature, provided either

$$C_n|W| < \frac{1}{n}R, \tag{2.21}$$

or

(1) *for $n \neq 5$,*

$$\Big(\int_M |W|^{\frac{n}{2}}\Big)^{\frac{2}{n}} < E_n\, Q_g(M), \tag{2.22}$$

where E_n is given by

$$E_n = \begin{cases} \sqrt{6}, & \text{if } n = 4; \\ \frac{4(n-1)}{n(n-2)}\Big(\frac{n-2}{\sqrt{n(n-1)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}}\Big)^{-1}, & \text{if } n \geq 6; \end{cases} \tag{2.23}$$

(2) *for $n = 5$,*

$$\Big(\int_M |W|^{\frac{5}{2}}\Big)^{\frac{2}{5}} \leq \frac{2\sqrt{15}-4}{\sqrt{10}}\, Q_g(M). \tag{2.24}$$

3. Proof of theorems

3.1. Proof of Theorem 1.1

By combining the estimates (2.7) with (2.11), we derive

$$\begin{aligned} &\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2 + \epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \\ \geq\; & -2\frac{\theta^2-(1-\epsilon_2)\theta+(1+\epsilon_1)}{(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\int_M W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik}\phi_r^2 \\ & + \frac{\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)}{(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\frac{n}{n-2}\int_M \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\phi_r^2 \\ & + \frac{\theta^2-2(1-\epsilon_2)\theta+(1+\epsilon_1)}{(1-\epsilon_2)(\theta^2+1+\epsilon_1)}\frac{1}{n-1}\int_M R|\mathring{R}_{ij}|^2\phi_r^2. \end{aligned} \tag{3.1}$$

For all $\epsilon_2 \in (0,1)$, we have

$$\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1) = [\theta - (1-\epsilon_2)]^2 + (1+\epsilon_1) - (1-\epsilon_2)^2 > 0, \tag{3.2}$$

and hence (3.1) reduces to

$$\begin{aligned} &\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2 + \epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1)]} \int_M |\mathring{R}_{ij}|^2 |\nabla\phi_r|^2 \\ \geq\ & -\frac{2[\theta^2 - (1-\epsilon_2)\theta + (1+\epsilon_1)]}{\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1)} \int_M W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik}\phi_r^2 \\ &+ \frac{n}{n-2}\int_M \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki}\phi_r^2 + \frac{1}{n-1}\int_M R|\mathring{R}_{ij}|^2\phi_r^2. \end{aligned} \tag{3.3}$$

Using (2.18), we have

$$\begin{aligned} &-\frac{2[\theta^2 - (1-\epsilon_2)\theta + (1+\epsilon_1)]}{\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1)} W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} + \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} \\ \geq\ & -\sqrt{\frac{n-2}{2(n-1)}}\Big|\frac{2[\theta^2 - (1-\epsilon_2)\theta + (1+\epsilon_1)]}{\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1)} W \\ &+ \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| |\mathring{R}_{ij}|^2. \end{aligned} \tag{3.4}$$

Applying (3.4) into (3.3) gives

$$\begin{aligned} &\frac{[4\epsilon_2(1-\epsilon_2)+\epsilon_1]\theta^2 + \epsilon_1(1+\epsilon_1)}{\epsilon_1\epsilon_2[\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1)]} \int_M |\mathring{R}_{ij}|^2 |\nabla\phi_r|^2 \\ \geq\ & \int_M \Bigg[-\sqrt{\frac{n-2}{2(n-1)}}\Big|\frac{2[\theta^2 - (1-\epsilon_2)\theta + (1+\epsilon_1)]}{\theta^2 - 2(1-\epsilon_2)\theta + (1+\epsilon_1)} W \\ &+ \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| + \frac{1}{n-1}R\Bigg] |\mathring{R}_{ij}|^2\phi_r^2. \end{aligned} \tag{3.5}$$

Now, we fixed ϵ_1 and ϵ_2 and minimize the coefficient of W with respect to the function θ by taking

$$\theta = -\sqrt{1+\epsilon_1}, \tag{3.6}$$

then (3.5) becomes

$$\begin{aligned} &\frac{[2\epsilon_2(1-\epsilon_2)+\epsilon_1]\sqrt{1+\epsilon_1}}{\epsilon_1\epsilon_2[\sqrt{1+\epsilon_1} + (1-\epsilon_2)]} \int_M |\mathring{R}_{ij}|^2 |\nabla\phi_r|^2 \\ \geq\ & \int_M \Bigg[-\sqrt{\frac{n-2}{2(n-1)}}\Big|\Big(1 + \frac{1}{1+\frac{1-\epsilon_2}{\sqrt{1+\epsilon_1}}}\Big) W \\ &+ \frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| + \frac{1}{n-1}R\Bigg] |\mathring{R}_{ij}|^2\phi_r^2. \end{aligned} \tag{3.7}$$

Since W is perpendicular to $\mathring{\mathrm{Ric}} \owedge g$, for any given $\check{\epsilon}_1, \check{\epsilon}_2$, we have

$$\begin{aligned}
&\Big|\Big(1+\frac{1}{1+\frac{1-\check{\epsilon}_2}{\sqrt{1+\check{\epsilon}_1}}}\Big)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^2 \\
&= \Big(1+\frac{1}{1+\frac{1-\check{\epsilon}_2}{\sqrt{1+\check{\epsilon}_1}}}\Big)^2|W|^2+\frac{n}{2(n-2)^2}|\mathring{\mathrm{Ric}} \owedge g|^2 \\
&< 4|W|^2+\frac{n}{2(n-2)^2}|\mathring{\mathrm{Ric}} \owedge g|^2 \\
&= \Big|2W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|^2,
\end{aligned} \tag{3.8}$$

which shows that

$$\begin{aligned}
&\Big|\Big(1+\frac{1}{1+\frac{1-\check{\epsilon}_2}{\sqrt{1+\check{\epsilon}_1}}}\Big)W+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| \\
&< 2\Big|W+\frac{n}{\sqrt{8n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|.
\end{aligned} \tag{3.9}$$

Therefore, under the condition (1.3), the estimate (3.7) gives

$$\begin{aligned}
0 \le &\int_M \Big[-\sqrt{\frac{n-2}{2(n-1)}}\Big|\Big(1+\frac{1}{1+\frac{1-\check{\epsilon}_2}{\sqrt{1+\check{\epsilon}_1}}}\Big)W \\
&+\frac{n}{\sqrt{2n}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big|+\frac{1}{n-1}R\Big]|\mathring{R}_{ij}|^2\phi_r^2 \\
&\le \frac{[2\check{\epsilon}_2(1-\check{\epsilon}_2)+\check{\epsilon}_1]\sqrt{1+\check{\epsilon}_1}}{\check{\epsilon}_1\check{\epsilon}_2[\sqrt{1+\check{\epsilon}_1}+(1-\check{\epsilon}_2)]}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2.
\end{aligned} \tag{3.10}$$

Since

$$\int_M |\mathring{R}_{ij}|^2 < \infty, \tag{3.11}$$

then we have

$$\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \to 0 \tag{3.12}$$

as $r \to \infty$. This, together with (3.10), shows that M^n is Einstein. In this case, (1.3) becomes

$$|W| \le \frac{R}{\sqrt{2(n-1)(n-2)}}, \tag{3.13}$$

which yields

$$C_n|W| \le \frac{C_n}{\sqrt{2(n-1)(n-2)}}R. \tag{3.14}$$

It is easy to check that for $n = 4, 5$, we have

$$\frac{C_n}{\sqrt{2(n-1)(n-2)}} R < \frac{1}{n} R, \tag{3.15}$$

which combining with (2.21) shows that M^n is of constant positive sectional curvature.

3.2. Proof of Theorem 1.2

From (2.18), it is easy to see

$$\begin{aligned} &2W_{ijkl}\mathring{R}_{jl}\mathring{R}_{ik} - \frac{n}{n-2}\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ki} \\ &\leq \sqrt{\frac{2(n-2)}{n-1}}\Big|W + \frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2. \end{aligned} \tag{3.16}$$

Applying (3.16) into (2.11) and using the Kato inequality, we obtain

$$\begin{aligned} \int_M |\nabla|\mathring{R}_{ij}||^2\phi_r^2 \leq& \int_M |\nabla\mathring{R}_{ij}|^2\phi_r^2 \\ \leq& \frac{1}{1-\epsilon_2}\int_M \Bigg[\sqrt{\frac{2(n-2)}{n-1}}\Big|W + \frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big| \\ &- \frac{1}{n-1}R\Bigg]|\mathring{R}_{ij}|^2\phi_r^2 + \frac{1}{\epsilon_2(1-\epsilon_2)}\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2. \end{aligned} \tag{3.17}$$

Taking $u = |\mathring{R}_{ij}|\phi_r$ in (1.5) and applying (3.17) yield

$$\begin{aligned} &Q_g(M)\Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \\ &\leq \int_M \Big(|\nabla(|\mathring{R}_{ij}|\phi_r)|^2 + \frac{n-2}{4(n-1)}R|\mathring{R}_{ij}|^2\phi_r^2\Big) \\ &\leq (1+\epsilon_3)\int_M |\nabla|\mathring{R}_{ij}||^2\phi_r^2 + \Big(1+\frac{1}{\epsilon_3}\Big)\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2 \\ &\quad + \frac{n-2}{4(n-1)}\int_M R|\mathring{R}_{ij}|^2\phi_r^2 \\ &\leq \frac{1+\epsilon_3}{1-\epsilon_2}\sqrt{\frac{2(n-2)}{n-1}}\int_M \Big|W + \frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}} \owedge g\Big||\mathring{R}_{ij}|^2\phi_r^2 \\ &\quad + \frac{1}{n-1}\Big[\frac{n-2}{4} - \frac{1+\epsilon_3}{1-\epsilon_2}\Big]\int_M R|\mathring{R}_{ij}|^2\phi_r^2 \\ &\quad + (1+\epsilon_3)\Big[\frac{1}{\epsilon_3} + \frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2. \end{aligned} \tag{3.18}$$

Inserting the following Hölder inequality

$$\begin{aligned}&\int_M \Big|W+\frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}}\bigcirc\!\!\!\!\wedge\, g\Big||\mathring{R}_{ij}|^2\phi_r^2\\ &\le \Big(\int_M \Big|W+\frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}}\bigcirc\!\!\!\!\wedge\, g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\end{aligned}$$

into (3.18) yields

$$\begin{aligned}&\Bigg[Q_g(M)-\frac{1+\epsilon_3}{1-\epsilon_2}\sqrt{\frac{2(n-2)}{n-1}}\Big(\int_M \Big|W\\ &+\frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}}\bigcirc\!\!\!\!\wedge\, g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Bigg]\Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\le \frac{1}{n-1}\Big[\frac{n-2}{4}-\frac{1+\epsilon_3}{1-\epsilon_2}\Big]\int_M R|\mathring{R}_{ij}|^2\phi_r^2\\ &+(1+\epsilon_3)\Big[\frac{1}{\epsilon_3}+\frac{1}{\epsilon_2(1-\epsilon_2)}\Big]\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2.\end{aligned}\tag{3.19}$$

Now, we consider the following two cases:

Case one: When $n\ge 7$, there exist $\tilde{\epsilon}_2,\tilde{\epsilon}_3$ depending only on the dimension n such that

$$\frac{1+\tilde{\epsilon}_3}{1-\tilde{\epsilon}_2}=\frac{n-2}{4}.\tag{3.20}$$

In this case, (3.19) becomes

$$\begin{aligned}&\Bigg[Q_g(M)-\frac{n-2}{4}\sqrt{\frac{2(n-2)}{n-1}}\Big(\int_M \Big|W\\ &+\frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}}\bigcirc\!\!\!\!\wedge\, g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Bigg]\Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\le (1+\tilde{\epsilon}_3)\Big[\frac{1}{\tilde{\epsilon}_3}+\frac{1}{\tilde{\epsilon}_2(1-\tilde{\epsilon}_2)}\Big]\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2.\end{aligned}\tag{3.21}$$

Under the assumption that (1.6) and (1.7), we can derive from (3.21)

$$\begin{aligned}0\le&\Bigg[Q_g(M)-\frac{n-2}{4}\sqrt{\frac{2(n-2)}{n-1}}\Big(\int_M \Big|W\\ &+\frac{\sqrt{n}}{\sqrt{8}(n-2)}\mathring{\mathrm{Ric}}\bigcirc\!\!\!\!\wedge\, g\Big|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Bigg]\Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\\ &\le (1+\tilde{\epsilon}_3)\Big[\frac{1}{\tilde{\epsilon}_3}+\frac{1}{\tilde{\epsilon}_2(1-\tilde{\epsilon}_2)}\Big]\int_M |\mathring{R}_{ij}|^2|\nabla\phi_r|^2\to 0\end{aligned}\tag{3.22}$$

as $r\to\infty$, which shows that M^n is Einstein.

Case two: When $4 \le n \le 6$ and $R \ge 0$, for all ϵ_2, ϵ_3, we always have

$$\frac{n-2}{4} - \frac{1+\epsilon_3}{1-\epsilon_2} < 0. \tag{3.23}$$

Therefore, under the condition (1.8), there are $\bar{\epsilon}_2, \bar{\epsilon}_3$ small enough such that

$$\begin{aligned} 0 \le & \Bigg[Q_g(M) - \frac{1+\overline{\epsilon_3}}{1-\bar{\epsilon}_2} \sqrt{\frac{2(n-2)}{n-1}} \Big(\int_M \Big| W \\ & + \frac{\sqrt{n}}{\sqrt{8}(n-2)} \mathring{\mathrm{Ric}} \owedge g \Big|^{\frac{n}{2}} \Big)^{\frac{2}{n}} \Bigg] \Big(\int_M (|\mathring{R}_{ij}|\phi_r)^{\frac{2n}{n-2}} \Big)^{\frac{n-2}{n}} \\ \le & \frac{1}{n-1} \Big[\frac{n-2}{4} - \frac{1+\bar{\epsilon}_3}{1-\bar{\epsilon}_2} \Big] \int_M R |\mathring{R}_{ij}|^2 \phi_r^2 \\ & + (1+\bar{\epsilon}_3) \Big[\frac{1}{\bar{\epsilon}_3} + \frac{1}{\bar{\epsilon}_2(1-\bar{\epsilon}_2)} \Big] \int_M |\mathring{R}_{ij}|^2 |\nabla \phi_r|^2 \to 0 \end{aligned} \tag{3.24}$$

as $r \to \infty$. Hence, M^n is Einstein. In this case, (1.8) becomes

$$\Big(\int_M |W|^{\frac{n}{2}} \Big)^{\frac{2}{n}} < \sqrt{\frac{n-1}{2(n-2)}} Q_g(M). \tag{3.25}$$

It is easy to check that for $n = 4$ we have $\sqrt{\frac{n-1}{2(n-2)}} < \frac{\sqrt{6}}{4}$, for $n = 5$ we have $\sqrt{\frac{n-1}{2(n-2)}} < \frac{2\sqrt{15}-4}{\sqrt{10}}$. This combining with Lemma 2.4 shows that $M^n (n = 4, 5)$ is of constant sectional curvature.

This completes the proof of Theorem 1.2.

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