Bull. Korean Math. Soc. **56** (2019), No. 5, pp. 1315–1325
https://doi.org/10.4134/BKMS.b181175
pISSN: 1015-8634 / eISSN: 2234-3016

RICCI SOLITONS AND RICCI ALMOST SOLITONS ON PARA-KENMOTSU MANIFOLD

Dhriti Sundar Patra

Abstract. The purpose of this article is to study the Ricci solitons and Ricci almost solitons on para-Kenmotsu manifold. First, we prove that if a para-Kenmotsu metric represents a Ricci soliton with the soliton vector field V is contact, then it is Einstein and the soliton is shrinking. Next, we prove that if a η-Einstein para-Kenmotsu metric represents a Ricci soliton, then it is Einstein with constant scalar curvature and the soliton is shrinking. Further, we prove that if a para-Kenmotsu metric represents a gradient Ricci almost soliton, then it is η-Einstein. This result is also hold for Ricci almost soliton if the potential vector field V is pointwise collinear with the Reeb vector field ξ.

1. Introduction

A pseudo-Riemannian metric g, defined on a manifold M^n, is called a Ricci soliton metric, or in short a Ricci soliton if there exist a constant $\lambda \in \mathbb{R}$ and a vector field $V \in \chi(M)$ such that

$$\frac{1}{2}\pounds_V g + Ric = \lambda g, \tag{1.1}$$

where $\pounds_V$ denotes the Lie-derivative in the direction of V and Ric is the Ricci tensor of g. A Ricci soliton is said to be trivial if V is either zero or Killing on M. Ricci soliton is considered as a generalization of Einstein metric and often arises as a fixed point of Hamilton's Ricci flow. In [19], Pigoli-Rigoli-Rimoldi-Setti generalized the notion of Ricci soliton to Ricci almost soliton by allowing the soliton constant λ to be a smooth function. We denote it by (M^n, g, V, λ). The Ricci almost soliton is said to be shrinking, steady, and expanding accordingly as λ is positive, zero, and negative respectively. Moreover, if the potential vector field V is the gradient of some smooth function u on M^n, i.e., $V = Du$, where D is the gradient operator of g on M^n, then the Ricci soliton is called a gradient Ricci soliton and the soliton Eq. (1.1) becomes

$$Hess\ u + Ric = \lambda g, \tag{1.2}$$

Received December 3, 2018; Revised April 8, 2019; Accepted April 25, 2019.
2010 *Mathematics Subject Classification.* 53C15, 53C25, 53D10, 53D15.
Key words and phrases. Ricci soliton, Ricci almost soliton, Einstein manifold, paracontact metric manifold, para-Kenmotsu manifold.

where $Hess\ u$ denotes the Hessian of u. The function u is known as the potential function.

As a generalization of Einstein metric, Ricci solitons grow interest on a new class of pseudo-Riemannian geometry called paracontact geometry which is introduced by Kaneyuki and Williams [15]. The importance of paracontact manifolds comes from the theory of para-Kähler manifolds. Since then many authors studied the paracontact geometry (see [1, 3, 5, 6, 9, 15, 16, 18, 25, 26]). Specially, Calvaruso-Perrone [4] explicitly studied the Ricci solitons on almost paracontact metric three-manifolds and describe more examples and Bejan-Crasmareanu [1] studied Ricci solitons on 3-dimensional normal paracontact manifolds. Further, Blaga [2] studied the η-Ricci soliton on para-Kenmotsu manifolds. On the other hand, studies on Ricci solitons in the frame work of contact geometry are very interesting and therefore many authors have been developed (see [7, 8, 10–14, 17, 20] and references therein). Among these many contexts: on Kenmotsu manifolds [10, 11], on K-contact and (κ, μ)-contact manifolds [20], on Sasakian manifolds [14], on Kähler manifolds [8] etc. Recently, the present author and Ghosh explicitly studied the Ricci solitons and $*$-Ricci solitons in the frame-work of Sasakian and (κ, μ)-contact manifolds (see [12, 13]). Further, the study of Ricci solitons on almost Kenmotsu manifolds was started by the Wang and Liu [23] and explicitly studied by Wang (see [21, 22]). Motivated by the above results we study the Ricci solitons and Ricci almost solitons on para-Kenmotsu manifolds.

This paper is organized as follows. In Section 2, the basic information about paracontact metric manifolds and para-Kenmotsu manifolds are given. In Section 3, we consider Ricci solitons on para-Kenmotsu manifold and prove that if a para-Kenmotsu metric g represents a Ricci soliton where the soliton vector field V is contact, then it becomes a shrinking soliton which is Einstein. In Section 4, we prove that if a para-Kenmotsu metric g represents a gradient Ricci almost soliton, then it is η-Einstein. Also we prove this result for Ricci almost soliton with the potential vector field V is pointwise collinear with the Reeb vector field ξ.

2. Notes on paracontact metric manifolds

In this section, we recall some information about paracontact metric manifolds. We refer to [3, 5, 6, 9, 15, 16, 25, 26] for more details as well as some examples. A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (φ, ξ, η) if it admits a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying the following conditions:

$$\varphi^2 = I - \eta \circ \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

and there exists a distribution $\mathcal{D} : p \in M \to \mathcal{D}_p \subset T_pM : \mathcal{D}_p = \mathrm{Ker}(\eta) = \{x \in T_pM : \eta(x) = 0\}$, called paracontact distribution generated by η. If an almost paracontact manifold M^{2n+1} with a structure (φ, ξ, η) admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y on M, then we say that M has an almost paracontact metric structure and g is called a compatible metric. The fundamental 2-form Φ of an almost paracontact metric structure (φ, ξ, η, g) defined by $\Phi(X, Y) = g(X, \varphi Y)$ for all vector fields X, Y on M. If $\Phi = d\eta$, then the manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is called a paracontact metric manifold. In this case, η is a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$, ξ is its Reeb vector field and M is a contact manifold (see [6, 16, 18]). An almost paracontact metric manifold is said to be para-Kenmotsu manifold if

$$(\nabla_X \varphi)Y = \eta(Y)\varphi X + g(X, \varphi Y)\xi \tag{2.3}$$

for all vector fields X, Y on M. On para-Kenmotsu manifold [25]:

$$\nabla_X \xi = -X + \eta(X)\xi, \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.5}$$

$$Q\xi = -2n\xi, \tag{2.6}$$

for all vector fields X, Y on M, where ∇ is the operator of covariant differentiation of g and Q denotes the Ricci operator associated with the Ricci tensor given by $Ric(X, Y) = g(QX, Y)$ for all vector fields X, Y on M.

3. Para-Kenmotsu metric as a Ricci soliton

In this section, we study the Ricci Solitons on para-Kenmotsu manifold. First we recall the following.

Lemma 3.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. Then we have*

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{3.1}$$

$$(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y), \tag{3.2}$$

$$(\pounds_\xi g)(Y, Z) = -2\{g(Y, Z) - \eta(Y)\eta(Z)\}, \tag{3.3}$$

for all vector fields Y, Z on M.

We can prove Lemma 3.1 by simple routine calculation. Using these results now we prove the following lemma for later use.

Lemma 3.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. Then we have*

$$(\pounds_\xi Q)Y = 2QY + 4nY = (\nabla_\xi Q)Y \tag{3.4}$$

for all vector fields Y on M.

Proof. First taking the covariant derivative of (3.3) along an arbitrary vector field X on M and using (3.2) we obtain

$$(3.5)\quad (\nabla_X \pounds_\xi g)(Y,Z) = -2\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\}$$

for all vector fields X, Y on M. Now, we recalling the following commutation formula (see Yano [24], p. 23):

$$(\pounds_V \nabla_Z g - \nabla_Z \pounds_V g - \nabla_{[V,Z]} g)(X,Y) = -g((\pounds_V \nabla)(Z,X),Y) - g((\pounds_V \nabla)(Z,Y),X)$$

for all vector fields X, Y, Z on M. By virtue of parallelism of the pseudo-Riemannian metric g, this formula reduces to

$$(3.6)\qquad (\nabla_Z \pounds_V g)(X,Y) = g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X)$$

for all vector fields X, Y, Z on M. Using (3.5) in (3.6) we have

$$\begin{aligned} &g((\pounds_\xi \nabla)(X,Y),Z) + g((\pounds_\xi \nabla)(X,Z),Y) \\ &= -2\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\} \end{aligned}$$

for all vector fields X, Y, Z on M. By a straightforward combinatorial computation the foregoing equation yields

$$(3.7)\qquad ((\pounds_\xi \nabla)(Y,Z) = 2\{\eta(Y)\eta(Z)\xi - g(Y,Z)\xi\}$$

for all vector fields Y, Z on M. Taking covariant differentiation of (3.7) along X and using (2.2), we find

$$\begin{aligned} (\nabla_X \pounds_\xi \nabla)(Y,Z) = &-2\{g(X,Y)\eta(Z)\xi + g(Y,Z)\eta(X)\xi + g(Z,X)\eta(Y)\xi \\ &- g(Y,Z)X + \eta(Y)\eta(Z)X - 3\eta(X)\eta(Y)\eta(Z)\xi\} \end{aligned}$$

for all vector fields Y, Z on M. Using this in the following commutation formula (see Yano [24], p. 23)

$$(3.8)\qquad (\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

we can compute

$$(\pounds_\xi R)(X,Y)Z = -2\{g(X,Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}$$

for all vector fields X, Y, Z on M. Now, contracting the foregoing Eq. over X, we find

$$(3.9)\qquad (\pounds_\xi Ric)(Y,Z) = -4n\{\eta(Y)\eta(Z) - g(Y,Z)\}$$

for all vector fields Y, Z on M. On the other hand, taking Lie derivative of $Ric(Y,Z) = g(QY,Z)$ with respect to ξ, we get

$$(3.10)\qquad (\pounds_\xi Ric)(Y,Z) = (\pounds_\xi g)(QY,Z) + g((\pounds_\xi Q)Y,Z)$$

for all vector fields Y, Z on M. Now, replacing Y by QY in (3.3) and using (2.6), we find

$$(3.11)\qquad (\pounds_\xi g)(QY,Z) = -2\{g(QY,Z) + 2n\eta(Y)\eta(Z)\}$$

for all vector fields Y, Z on M. By virtue of (3.11) and (3.10), Eq. (3.9) reduces to $(\pounds_\xi Q)Y = 2QY + 4nY$ for all vector fields Y on M. Further, it is well known that

$$\begin{aligned}(\pounds_\xi Q)Y &= \pounds_\xi QY - Q(\pounds_\xi Y)\\ &= \nabla_\xi QY - \nabla_{QY}\xi - Q(\nabla_\xi Y) + Q\nabla_Y \xi\\ &= (\nabla_\xi Q)Y - \nabla_{QY}\xi + Q\nabla_Y\xi\end{aligned}$$

for all vector fields Y on M. Thus, by virtue of (2.4) and (2.6) we see that $(\pounds_\xi Q)Y = (\nabla_\xi Q)Y$ for all vector fields Y on M. This completes the proof. □

Now we consider a para-Kenmotsu metric as a Ricci soliton where the soliton vector field V is contact and proof the following.

Theorem 3.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$, be a para-Kenmotsu manifold. If g represents a Ricci soliton, then the soliton is shrinking. Further, if the soliton vector field V is contact, then V is strict and g is Einstein with Einstein constant $-2n$.*

Proof. First, from (2.4) we get $R(X,\xi)\xi = -X + \eta(X)\xi$ and the Lie derivative of this along V provides

$$\begin{aligned}&(\pounds_V R)(X,\xi)\xi + R(X,\pounds_V\xi)\xi + R(X,\xi)\pounds_V\xi\\ &= \{(\pounds_V\eta)X\}\xi + \eta(X)\pounds_V\xi\end{aligned} \tag{3.12}$$

for all vector fields X on M. Now, taking covariant derivative of (1.1) along an arbitrary vector field Z on M and using (3.6) we have

$$g((\pounds_V\nabla)(Z,X),Y) + g((\pounds_V\nabla)(Z,Y),X) = -2(\nabla_Z Ric)(X,Y)$$

for all vector fields X, Y on M. By a straightforward combinatorial combination of the last equation one can deduce

$$\begin{aligned}g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Z Ric)(X,Y) &- (\nabla_X Ric)(Y,Z)\\ &- (\nabla_Y Ric)(Z,X)\end{aligned} \tag{3.13}$$

for all vector fields X, Y, Z on M. Next, differentiating (2.6) along an arbitrary vector field X on M and recalling (2.2) we get

$$(\nabla_X Q)\xi = QX + 2nX \tag{3.14}$$

for all vector fields X on M. Taking into account of this, (3.4) and replacing Y by ξ in (3.13) we deduce

$$(\pounds_V\nabla)(X,\xi) = -2QX - 4nX \tag{3.15}$$

for all vector fields X on M. Taking covariant derivative of (3.15) along Y and using (2.2), (3.15) we obtain

$$(\nabla_Y \pounds_V\nabla)(X,\xi) - (\pounds_V\nabla)(X,Y) - 2\eta(Y)(QX + 2nX) = -2(\nabla_Y Q)X$$

for all vector fields X on M. Making use of this in (3.8) yields

$$(\pounds_V R)(X,Y)\xi = 2[\eta(X)QY - \eta(Y)QX + 2n\{\eta(X)Y - \eta(Y)X\}$$

$$- \{(\nabla_X Q)Y - (\nabla_Y Q)X\}] \tag{3.16}$$

for all vector fields X, Y on M. Now, replacing Y by ξ in (3.16) and using (3.14) and (3.4) we have $(\pounds_V R)(X,\xi)\xi = 0$. Making use of this along with (2.4), (3.1) in (3.12), one can deduce

$$g(X, \pounds_V \xi) - 2\eta(\pounds_V \xi)X = \{(\pounds_V \eta)X\}\xi \tag{3.17}$$

for all vector fields X on M. Next, taking into account (1.1), (2.6) in the Lie differentiation $g(\xi,\xi) = 1$ along V leads to

$$\eta(\pounds_V \xi) = \lambda - 2n. \tag{3.18}$$

Further, by virtue of (2.6), the soliton equation (1.1) reduces to

$$(\pounds_V \eta)X = g(X, \pounds_V \xi) - 2(\lambda - 2n)\eta(X) \tag{3.19}$$

for all vector fields X on M. By the help of (3.19) and (3.18), Eq. (3.17) provides $\lambda = 2n$ and therefore the soliton is shrinking. Further, Eq. (3.18) together with (3.19) yields

$$\pounds_V \xi = 0 = \pounds_V \eta. \tag{3.20}$$

Also by our assumption, V is a contact vector field, i.e., $\pounds_V \xi = f\xi$. Making use of this in (3.18) gives $f = \lambda - 2n$ and therefore $f = 0$. Thus, $\pounds_V \xi = 0$, and hence V is strict. Now, recall the well known formula (see [24, p. 23]):

$$\pounds_V \nabla_X Y - \nabla_X \pounds_V Y - \nabla_{[V,X]} Y = (\pounds_V \nabla)(X, Y) \tag{3.21}$$

for all vector fields X, Y, V on M. Next, taking ξ instead of Y in the preceding equation and using (3.20) we get

$$(\pounds_V \nabla)(X, \xi) = \pounds_V \nabla_X \xi - \pounds_V X + \eta(\pounds_V X)$$

for all vector fields X, V on M. Taking into account (2.2) and (3.20), the last equation provides $(\pounds_V \nabla)(X, \xi) = 0$ for all vector fields X, V on M. By virtue of this, Eq. (3.15) proves that g is Einstein. This completes the proof. □

A pseudo-Riemannian manifold is called η-Einstein, if the Ricci tensor Ric is of the form

$$Ric = ag + b\eta \otimes \eta, \tag{3.22}$$

where a, b are smooth functions on M. For a para-Sasakian manifold of dimension > 3, the functions a, b are constant (see [25]).

Lemma 3.3. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If M is an η-Einstein manifold, we have*

$$Ric(Y, Z) = (1 + \frac{r}{2n})g(Y, Z) - \{(2n + 1) + \frac{r}{2n}\}\eta(Y)\eta(Z) \tag{3.23}$$

for all vector fields Y, Z on M.

Proof. Equations (3.22) and (2.6) gives $r = (2n + 1)a + b$ and $a + b = -2n$. Thus, we have $a = 1 + \frac{r}{2n}$ and $b = -\{(2n + 1) + \frac{r}{2n}\}$. Then the Eq. (3.22) can be written as the required form. This completes the proof. □

Theorem 3.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$, be a η-Einstein para-Kenmotsu manifold. If g represents a Ricci soliton, then the soliton is shrinking and g is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. By the help of (3.23), the soliton Eq. (1.1) becomes

$$\pounds_V g(Y,Z) = \{2(\lambda - 1) - \frac{r}{n}\}g(Y,Z) + \{2(2n+1) + \frac{r}{n}\}\eta(Y)\eta(Z) \tag{3.24}$$

for all vector fields Y, Z on M. Differentiating this along an arbitrary vector field X on M and using (2.4), (3.6) we have

$$\begin{aligned} &g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X) \\ = &-\frac{(Xr)}{n}g(Y,Z) + \frac{(Xr)}{n}\eta(Y)\eta(Z) - \{2(2n+1) + \frac{r}{n}\}\{g(X,Y)\eta(Z) \\ &+ g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\} \end{aligned} \tag{3.25}$$

for all vector fields Y, Z on M. By straightforward combinatorial computation of the last equation provides

$$\begin{aligned} &2ng((\pounds_V \nabla)(X,Y),Z) \\ = &-(Xr)g(Y,Z) - (Yr)g(X,Z) + (Zr)g(X,Y) \\ &+ (Xr)\eta(Y)\eta(Z) + (Yr)\eta(X)\eta(Z) - (Zr)\eta(X)\eta(Y) \\ &- 2\{2n(2n+1) + r\}\{g(X,Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\} \end{aligned} \tag{3.26}$$

for all vector fields X, Y, Z on M. Consider a local orthonormal basis $\{e_i : i = 1, 2, \dots, 2n+1\}$ of tangent space at each point of M. Next, setting $X = Z = e_i$ in (3.13) and summing over $i : 1 \leq i \leq 2n+1$, we have $(\pounds_V \nabla)(e_i, e_i) = 0$. Now, putting $X = Y = e_i$ in (3.26) gives

$$(\xi r)\eta(Z) + (n-1)(Zr) = 2n\{2n(2n+1) + r\}\eta(Z) \tag{3.27}$$

for all vector fields Z on M. Taking $Z = \xi$ in the last equation we get $(\xi r) = 2\{2n(2n+1) + r\}$. By virtue of this, Eq. (3.27) yields $Dr = (\xi r)\xi$. Next, substituting X by ξ in (3.26) we obtain

$$2n(\pounds_V \nabla)(\xi, Y) = -(\xi r)\{Y - \eta(Y)\xi\} \tag{3.28}$$

for all vector fields Y on M. Taking covariant derivative of this along X and using (2.6) and (3.15) we get

$$\begin{aligned} 2n(\nabla_X \pounds_V \nabla)(\xi, Y) = &\ 2n(\pounds_V \nabla)(X,Y) - X(\xi r)\{Y - \eta(Y)\xi\} \\ &- (\xi r)\{g(X,Y)\xi + \eta(Y)X - \eta(X)Y - \eta(X)\eta(Y)\xi\} \end{aligned} \tag{3.29}$$

for all vector fields Y on M. Next, interchanging X, Y in (3.29) and using the well known formula (see [24, p. 23]):

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

it follows that

$$2n(\pounds_V R)(X,Y)\xi = Y(\xi r)\{X - \eta(X)\xi\} - X(\xi r)\{Y - \eta(Y)\xi\} \tag{3.30}$$

$$- 2(\xi r)\{\eta(Y)X - \eta(X)Y\}$$

for all vector fields X, Y on M. Contracting this over X we have $(\pounds_V Ric)(Y, \xi) = 0$, where we use $Dr = (\xi r)\xi$. Further, using (3.23), (3.30) in the Lie derivative of $Ric(Y, \xi) = -2n\eta(Y)$ along V yields

$$\begin{aligned} &(1 + \frac{r}{2n})g(Y, \pounds_V\xi) - \{(2n+1) + \frac{r}{2n}\}\eta(Y)\eta(\pounds_V\xi) \\ &= -4n(\lambda - 2n)\eta(Y) - 2ng(Y, \pounds_V\xi) \end{aligned} \tag{3.31}$$

for all vector fields Y on M. Taking $Y = \xi$ in the last equation we get $\lambda = 2n$ and therefore the soliton is shrinking. Again, setting $Y = Z = \xi$ in (3.24) we obtain $\eta(\pounds_V\xi) = 0$. Using this in (3.31) yields

$$\{r + 2n(2n+1)\}\pounds_V\xi = 0. \tag{3.32}$$

Suppose $r \neq -2n(2n+1)$ on some open set $\mathcal{O}$ of M. Then from (3.32) it follows that $\pounds_V\xi = 0$. Thus, from (2.4) we deduce that $\nabla_\xi V = V - \eta(V)\xi$. Using this, (2.4), (2.5) and (3.28) in the identity (see [24, p. 39]):

$$(\pounds_V\nabla)(X, Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V + R(V, X)Y,$$

we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so the scalar curvature r is constant. This shows from (3.27) that $r = -2n(2n+1)$ on $\mathcal{O}$, which is a contradiction on $\mathcal{O}$. Thus, Eq. (3.32) gives $r = -2n(2n+1)$ and therefore we can conclude from (3.23) that M is Einstein. This completes the proof. □

4. Para-Kenmotsu metric as a Ricci almost soliton

In this section, we study the Ricci almost solitons on para-Kenmotsu manifold. First, we consider a para-Kenmotsu metric as a gradient Ricci almost soliton. Thus, the equation (1.1) and (1.2) holds for a smooth function λ.

Theorem 4.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold. If g represents a gradient Ricci almost soliton, then it is η-Einstein. Moreover, if the Reeb vector field ξ leaves the scalar curvature r invariant, then g is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. Making use of (1.2) in the well known expression of the curvature tensor $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_[X, Y]$, one can obtain

$$R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + (X\lambda)Y - (Y\lambda)X \tag{4.1}$$

for all vector fields X, Y on M. Now, replacing Y by ξ in (4.1) and using (3.4) and (3.14) we deduce

$$R(X, \xi)Df = QX + 2nX + (X\lambda)\xi - (\xi\lambda)X$$

for all vector fields X on M. By virtue of (3.1), the preceding equation reduces to

$$g(X, Df - D\lambda)\xi = QX + 2nX + \{(\xi f) - (\xi\lambda)\}X \tag{4.2}$$

for all vector fields X on M. Taking scalar product of (4.2) with ξ and using (2.6) yields

$$Df - D\lambda = \{(\xi f) - (\xi\lambda)\}\xi. \tag{4.3}$$

Using this in (4.2) we have

$$Ric(X,Y) = -\{2n + (\xi f) - (\xi\lambda)\}g(X,Y) + \{(\xi f) - (\xi\lambda)\}\eta(X)\eta(Y) \tag{4.4}$$

for all vector fields X, Y on M. Consider a local orthonormal basis $\{e_i : i = 1, 2, \dots, 2n+1\}$ of tangent space at each point of M. Next, taking the inner product of (4.1) with Z and then setting $X = Z = e_i$ and summing over $i : 1 \leq i \leq 2n+1$, we have

$$Ric(Y, D\lambda) = \{\sum_{i=1}^{2n+1} g(g((\nabla_Y Q)e_i, e_i) - (\nabla_{e_i} Q)Y, e_i)\} - 2n(Y\lambda) \tag{4.5}$$

for all vector fields Y on M. Contraction of Bianchi's second identity gives $divQ = \frac{1}{2}Dr$ and therefore Eq. (4.5) yields

$$Ric(Y, D\lambda) = \frac{1}{2}Yr - 2nY\lambda \tag{4.6}$$

for all vector fields Y on M. Replacing ξ by Y and using (2.6) we have $(\xi r) = 4n\{(\xi\lambda) - (\xi f)\}$. Again, tracing (3.4) gives $(\xi r) = 2(r + 2n(2n+1))$ and therefore $2n\{(\xi f) - (\xi\lambda)\} = (r + 2n(2n+1))$. Using this in (4.4) we have

$$Ric(X,Y) = -(\frac{r}{2n} + 4n + 1)g(X,Y) + (\frac{r}{2n} + 2n + 1)\eta(X)\eta(Y) \tag{4.7}$$

for all vector fields X, Y on M. Thus, M is η-Einstein. Moreover, if ξ leaves the scalar curvature r invariant, i.e., $\xi r = 0$ and therefore, $r = -2n(2n+1)$. This transform the Eq. (4.7) into $Ric = -2ng$, i.e., g is Einstein. This complete the proof. □

Next, we extend the above Theorem from gradient Ricci almost soliton to Ricci almost soliton and consider para-Kenmotsu metric as a Ricci almost soliton and the potential vector field V is pointwise collinear with the Reeb vector field ξ and prove:

Theorem 4.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If g represents a non-trivial Ricci almost soliton such that the potential vector field V is pointwise collinear with the Reeb vector field ξ, then it is η-Einstein.*

Proof. By hypothesis: $V = \rho\xi$ for some smooth function ρ on M. Taking covariant derivative of this along an arbitrary vector field X on M and using (2.2) provides

$$\nabla_X V = (X\rho)\xi - \rho(X + \eta(X)\xi). \tag{4.8}$$

Then the soliton equation (2.1) reduces to

$$2Ric(X,Y) = 2(\rho - \lambda)g(X,Y) - (X\rho)\eta(Y) - (Y\rho)\eta(X) - 2\rho\eta(X)\eta(Y) \tag{4.9}$$

for all vector fields X, Y on M. Now, replacing $X = Y = \xi$ in the foregoing equation and using (2.6), we have $\xi\rho = 2n - \lambda$. Taking into account of this, (2.6) and putting $Y = \xi$ in (4.9) gives $X\rho = (2n - \lambda)\eta(X)$. using this in (4.9), we have

$$Ric(X,Y) = (\rho - \lambda)g(X,Y) - (2n + \rho - \lambda)\eta(X)\eta(Y) \tag{4.10}$$

for all vector fields X, Y on M. Tracing the preceding equation gives

$$\rho - \lambda = \frac{r}{2n} + 1. \tag{4.11}$$

This transform the Eq. (4.10) into

$$Ric(X,Y) = (\frac{r}{2n} + 1)g(X,Y) - (\frac{r}{2n} + 2n + 1)\eta(X)\eta(Y)$$

for all vector fields X, Y on M. This implies that M is η-Einstein. This complete the proof. □

Next, if we take ρ a constant instead of a function, then from $X\rho = (2n - \lambda)\eta(X)$, we have $\lambda = 2n$, which is constant. Thus from (4.11) follows that $\xi r = 0$. Again, tracing (3.4) gives $(\xi r) = 2(r + 2n(2n+1))$. Hence $r = -2n(2n+1)$. Making use of this in (4.11) we see that $\rho = 0$, and therefore from the soliton Eq. we conclude that g is Einstein. Thus, we have the following.

Corollary 4.1. *If a para-Kenmotsu metric g represents a non-trivial Ricci almost soliton with $V = \rho\xi$ for some constant ρ, then it is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

In particular, we can also say that if a para-Kenmotsu metric g represents a non-trivial Ricci almost soliton where the potential vector field V is ξ, then it is Einstein with constant scalar curvature $r = -2n(2n+1)$.

References

[1] C. L. Bejan and M. Crasmareanu, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Ann. Global Anal. Geom. **46** (2014), no. 2, 117–127. https://doi.org/10.1007/s10455-014-9414-4

[2] A. M. Blaga, *Eta-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), no. 1, 1–13.

[3] G. Calvaruso, *Homogeneous paracontact metric three-manifolds*, Illinois J. Math. **55** (2011), no. 2, 697–718 (2012). http://projecteuclid.org/euclid.ijm/1359762409

[4] G. Calvaruso and A. Perrone, *Ricci solitons in three-dimensional paracontact geometry*, J. Geom. Phys. **98** (2015), 1–12. https://doi.org/10.1016/j.geomphys.2015.07.021

[5] G. Calvaruso and D. Perrone, *Geometry of H-paracontact metric manifolds*, Publ. Math. Debrecen **86** (2015), no. 3-4, 325–346.

[6] B. Cappelletti-Montano, A. Carriazo, and V. Martín-Molina, *Sasaki-Einstein and paraSasaki-Einstein metrics from (κ,μ)-structures*, J. Geom. Phys. **73** (2013), 20–36. https://doi.org/10.1016/j.geomphys.2013.05.001

[7] J. T. Cho and R. Sharma, *Contact geometry and Ricci solitons*, Int. J. Geom. Methods Mod. Phys. **7** (2010), no. 6, 951–960. https://doi.org/10.1142/S0219887810004646

[8] O. Chodosh and F. T.-H. Fong, *Rotational symmetry of conical Kähler-Ricci solitons*, Math. Ann. **364** (2016), no. 3-4, 777–792. https://doi.org/10.1007/s00208-015-1240-x

[9] K. Erken and C. Murathan, *A study of three-dimensional paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-spaces*, Int. J. Geom. Methods Mod. Phys. **14** (2017), no. 7, 1750106, 35 pp. https://doi.org/10.1142/S0219887817501067

[10] A. Ghosh, *Kenmotsu 3-metric as a Ricci soliton*, Chaos Solitons Fractals **44** (2011), no. 8, 647–650. https://doi.org/10.1016/j.chaos.2011.05.015

[11] ———, *An η-Einstein Kenmotsu metric as a Ricci soliton*, Publ. Math. Debrecen **82** (2013), no. 3-4, 591–598. https://doi.org/10.5486/PMD.2013.5344

[12] A. Ghosh and D. S. Patra, *The k-almost Ricci solitons and contact geometry*, J. Korean Math. Soc. **55** (2018), no. 1, 161–174. https://doi.org/10.4134/JKMS.j170103

[13] ———, *$*$-Ricci soliton within the frame-work of Sasakian and (κ, μ)-contact manifold*, Int. J. Geom. Methods Mod. Phys. **15** (2018), no. 7, 1850120, 21 pp. https://doi.org/10.1142/S0219887818501207

[14] A. Ghosh and R. Sharma, *Sasakian metric as a Ricci soliton and related results*, J. Geom. Phys. **75** (2014), 1–6. https://doi.org/10.1016/j.geomphys.2013.08.016

[15] S. Kaneyuki and F. L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173–187. https://doi.org/10.1017/S0027763000021565

[16] V. Martín-Molina, *Local classification and examples of an important class of paracontact metric manifolds*, Filomat **29** (2015), no. 3, 507–515. https://doi.org/10.2298/FIL1503507M

[17] D. S. Patra, *Ricci solitons and paracontact geometry*, accepted in Mediterr. J. Math. (2019).

[18] A. Perrone, *Some results on almost paracontact metric manifolds*, Mediterr. J. Math. **13** (2016), no. 5, 3311–3326. https://doi.org/10.1007/s00009-016-0687-7

[19] S. Pigola, M. Rigoli, M. Rimoldi, and A. Setti, *Ricci almost solitons*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), no. 4, 757–799.

[20] R. Sharma, *Certain results on K-contact and (k, μ)-contact manifolds*, J. Geom. **89** (2008), no. 1-2, 138–147. https://doi.org/10.1007/s00022-008-2004-5

[21] Y. Wang, *Gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds*, J. Korean Math. Soc. **53** (2016), no. 5, 1101–1114. https://doi.org/10.4134/JKMS.j150416

[22] ———, *Ricci solitons on almost Kenmotsu 3-manifolds*, Open Math. **15** (2017), no. 1, 1236–1243. https://doi.org/10.1515/math-2017-0103

[23] Y. Wang and X. Liu, *Ricci solitons on three-dimensional η-Einstein almost Kenmotsu manifolds*, Taiwanese J. Math. **19** (2015), no. 1, 91–100. https://doi.org/10.11650/tjm.19.2015.4094

[24] K. Yano, *Integral Formulas in Riemannian Geometry*, Pure and Applied Mathematics, No. 1, Marcel Dekker, Inc., New York, 1970.

[25] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom. **36** (2009), no. 1, 37–60. https://doi.org/10.1007/s10455-008-9147-3

[26] S. Zamkovoy and V. Tzanov, *Non-existence of flat paracontact metric structures in dimension greater than or equal to five*, Annuaire Univ. Sofia Fac. Math. Inform. **100** (2011), 27–34.

Dhriti Sundar Patra
Department of Mathematics
Birla Institute of Technology Mesra
Ranchi: 835 215, India
Email address: dhritimath@gmail.com