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NEHARI MANIFOLD AND MULTIPLICITY RESULTS FOR A CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN

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Abstract. In this work, we investigate the following fractional boundary value problems

$$\begin{cases} {}_tD_T^\alpha\left(|{}_0D_t^\alpha(u(t))|^{p-2}{}_0D_t^\alpha u(t)\right) \\ = \nabla W(t,u(t))+\lambda g(t)|u(t)|^{q-2}u(t),\ t\in(0,T), \\ u(0)=u(T)=0, \end{cases}$$

where $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u and $W\in C([0,T]\times\mathbb{R}^n,\mathbb{R})$ is homogeneous of degree r, λ is a positive parameter, $g\in C([0,T])$, $1<r<p<q$ and $\frac{1}{p}<\alpha<1$. Using the Fibering map and Nehari manifold, for some positive constant λ_0 such that $0<\lambda<\lambda_0$, we prove the existence of at least two non-trivial solutions.

1. Introduction

Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [2, 11, 15, 18, 19, 21] and the papers [1, 6, 10, 14, 29].

Recently, equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives is an interesting and new field in fractional differential equations

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theory. In this topic, many results are obtained in dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory (including Leray-Schauder nonlinear alternative), topological degree theory (including coincidence degree theory) and comparison method (including upper and lower solutions and monotone iterative method), see [4, 12, 30] and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [17], Rabinowitz [20], Schechter [22] and the references listed therein.

Motivated by the above classical works, in recent paper [13], the authors showed that critical point theory is an effective approach to deal with the existence of solutions for the following fractional boundary value problem

$$\text{(FBVP)} \qquad \begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla W(t,u(t)), \quad t\in[0,T], \\ u(0)=u(T), \end{cases}$$

where $\alpha \in (\frac{1}{2},1)$, ${}_tD_T^{\alpha}u$ is the so called Riemann-Liouville fractional derivatives which is given by Definition 2.2, $u\in\mathbb{R}^n$, $W\in C^1([0,T]\times\mathbb{R}^n,\mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u, which has been generalized in recent papers [16, 26–28, 31–33].

Note that the α-order Riemann-Liouville fractional derivatives at time t is not defined locally, it relies on the total effects of the commonly used integer derivative on the interval $[0,t]$. So it can be used to describe the variation of a system in which the instantaneous change rate depends on the past state, which is called the "memory" effect in a visualized manner [2]. In addition, as indicated in [3, 23–25] the fractional theory has been applied to almost all fields of science including viscoelasticity and rheology, medicine and biology.

In this paper we want to contribute with the development of this new area on fractional differential equations theory. More precisely, the purpose of this work is to investigate the following fractional nonlinear Dirichlet problem

$$(P_\lambda) \qquad \begin{cases} {}_tD_T^{\alpha}\left(|{}_0D_t^{\alpha}(u(t))|^{p-2}\,{}_0D_t^{\alpha}u(t)\right) \\ = \nabla W(t,u(t)) + \lambda g(t)|u(t)|^{q-2}u(t),\ t\in(0,T) \\ u(0)=u(T)=0, \end{cases}$$

where $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u and $W\in C([0,T]\times\mathbb{R}^n,\mathbb{R})$ is homogeneous of degree r, λ is a positive parameter, $g\in C([0,T])$, $1<r<p<q$ and $\frac{1}{p}<\alpha<1$. We assume the following hypothesis:

(H$_1$) $W : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is homogeneous of degree r that is

$$W(t, su) = s^r W(t,u) \ (s > 0) \ \text{ for all } \ t \in [0,T], \ u \in \mathbb{R}^n;$$

(H$_2$) $W^{\pm}(t,u) = \max(\pm W(t,u), 0) \neq 0$ for all $u \neq 0$.

Note that, from (H$_1$), $W(t,u)$ leads to the so-called Euler identity

$$u \nabla W(x,u) = rW(x,u), \tag{1.1}$$

and

$$|W(x,u)| \leq K|u|^r \quad \text{for some constant } K > 0. \tag{1.2}$$

Our main result is the following.

Theorem 1.1. *Let* $\frac{1}{p} < \alpha < 1$, $1 < r < p < q$ *and assume that* $W(t,u)$ *satisfies the conditions* (H$_1$)-(H$_2$). Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, (P_λ) has at least two nontrivial solutions.

Remark 1.2. Recently, for $\lambda = 0$, the authors in [7, 8] studied the existence and multiplicity of solutions for (P_λ) (i.e., (P_0)) when the potential $W(t,u)$ is superquadratic or subquadratic at infinity. In our Theorem 1.1, we focus our attention on the case that the potential is of the form a combination of superquadratic term and subquadratic term. In addition, we do not need any assumption on the sign of the potential. Therefore, the recent related results are generalized and improved significantly.

2. Preliminaries

In this section, we give some background theory on the fractional calculus, in particular the Riemann-Liouville operators and results which will used throughout this paper. Let us start by introducing the definition of the fractional integral in the sense of Riemann-Liouville. We refer the reader to [15,19] or other texts on basic fraction calculus.

Definition 2.1. Let $\alpha > 0$ and u be a function defined a.e. on $(a,b) \subset \mathbb{R}$ with values in $\mathbb{R}$. The left (resp. right) fractional integral in the sense of Riemann-Liouville with inferior limit a (resp. superior limit b) of order α of u is given by

$${}_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [a,b]$$

respectively

$${}_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} u(s) ds, \quad t \in [a,b],$$

provided the right side is point-wise defined on $[a,b]$, where Γ denotes Euler's Gamma function. If $u \in L^1(a,b)$, then ${}_aI_t^\alpha u$ and ${}_tI_b^\alpha u$ are defined a.e. on (a,b).

Now, we define the fractional derivative in the sense of Riemann-Liouville as follows.

Definition 2.2. Let $0 < \alpha < 1$. Then, the left (resp. right) fractional derivative in the sense of Riemann-Liouville with inferior limit a (resp. superior limit b) of order α of u is given by

$$ {}_aD_t^\alpha u(t) = \frac{d}{dt}\left({}_aI_t^{1-\alpha}u\right)(t), \ \forall\, t \in [a,b], $$

respectively

$$ {}_tD_b^\alpha u(t) = \frac{d}{dt}\left({}_tI_b^{1-\alpha}u\right)(t), \ \forall\, t \in [a,b], $$

provided that the right-hand side is point-wise defined.

Remark 2.3. From [15], if u is an absolutely continuous function in $[a,b]$, then ${}_aD_t^\alpha u$ and ${}_tD_b^\alpha u$ are defined a.e. on (a,b) and satisfy

$$ {}_aD_t^\alpha u(t) = {}_aI_t^{1-\alpha}u'(t) + \frac{u(a)}{(t-a)^\alpha\Gamma(1-\alpha)} \tag{2.1} $$

and

$$ {}_tD_b^\alpha u(t) = -{}_tI_b^{1-\alpha}u'(t) + \frac{u(b)}{(b-t)^\alpha\Gamma(1-\alpha)}. \tag{2.2} $$

Moreover, if $u(a) = u(b) = 0$, then ${}_aD_t^\alpha u(t) = {}_aI_t^{1-\alpha}u'(t)$ and ${}_tD_b^\alpha u(t) = -{}_tI_b^{1-\alpha}u'(t)$. So in this case we have the equality of Riemann-Liouville fractional derivative and Caputo derivative defined by

$$ {}_a^cD_t^\alpha u(t) = {}_aI_t^{1-\alpha}u'(t) $$

and

$$ {}_t^cD_b^\alpha u(t) = -{}_tI_b^{1-\alpha}u'(t). $$

Consequently, one gets

$$ {}_aD_t^\alpha u(t) = {}_a^cD_t^\alpha u(t) + \frac{u(a)}{(t-a)^\alpha\Gamma(1-\alpha)}, $$

and

$$ {}_tD_b^\alpha u(t) = {}_t^cD_b^\alpha u(t) + \frac{u(b)}{(b-t)^\alpha\Gamma(1-\alpha)}. $$

Next, we provide some properties concerning the left fractional operators of Riemann-Liouville. For more details we refer the reader to [5].

Proposition 2.4. *For any α, $\beta > 0$ and any $u \in L^1(a,b)$, the following equality holds*

$$ {}_aI_t^\alpha \circ {}_aI_t^\beta u = {}_aI_t^{\alpha+\beta}. $$

From Proposition 2.4 and the equations (2.1) and (2.2), it is simple to deduce the following results concerning the composition between fractional integral and fractional derivative. That is, for any $0 < \alpha < 1$, if $u \in L^1(a,b)$ we have

$$ {}_aD_t^\alpha \circ {}_aI_t^\alpha u = u, $$

and if u is absolutely continuous such that $u(a) = 0$, then, one has

$$ {}_aI_t^\alpha \circ {}_aD_t^\alpha u = u. $$

Now, we presented an important result on the boundedness of the left fractional integral from $L^p(a,b)$ to $L^p(a,b)$.

Proposition 2.5. *For any $\alpha > 0$ and $p \geq 1$, ${}_aI_t^\alpha$ is linear and continuous from $L^p(a,b)$ to $L^p(a,b)$. Moreover for all $u \in L^p(a,b)$, we have*

$$ \|{}_aI_t^\alpha u\|_p \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|u\|_p. $$

In the same way, we give another classical result on the boundedness of the left fractional integral from $L^p(a,b)$ to $C_a(a,b)$ which completes Proposition 2.5 in the case $\frac{1}{p} < \alpha < 1$, where $C_a(a,b) := \{u \in C(a,b) : \lim_{t \to a^+} u(t) = 0\}$.

Proposition 2.6. *Let $0 < \frac{1}{p} < \alpha < 1$ and $p' = \frac{p}{p-1}$ (the conjugate exponent of p). Then, for any $u \in L^p(a,b)$, ${}_aI_t^\alpha u$ is Hölder continuous on $(a,b]$ with exponent $\alpha - \frac{1}{p} > 0$, moreover, $\lim_{t \to a^+} {}_aI_t{}^\alpha u(t) = 0$. Consequently, ${}_aI_t^\alpha u$ can be continuously extended by 0 in $t = a$. Finally, ${}_aI_t^\alpha u \in C_a(a,b)$, and*

$$ \|{}_aI_t^\alpha u\|_\infty \leq \frac{(b-a)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)\left((a-1)p'+1\right)^{\frac{1}{p'}}} \|u\|_p. \tag{2.3} $$

Also, we will need the following formula for integration by parts, see (7) and (8) in [13].

Proposition 2.7. *Let $0 < \alpha < 1$ and p, q are such that*

$$ p \geq 1,\ q \geq 1 \ \ \text{and} \ \ \frac{1}{p} + \frac{1}{q} < 1 + \alpha \ \ \text{or} \ \ p \neq 1,\ q \neq 1 \ \ \text{and} \ \ \frac{1}{p} + \frac{1}{q} = 1 + \alpha. $$

Then, for all $u \in L^p(a,b)$ and all $v \in L^q(a,b)$, one has

$$ \int_a^b v(t)_a I_t^\alpha u(t) dt = \int_a^b u(t)_a I_t^\alpha v(t) dt, \tag{2.4} $$

and

$$ \int_a^b u(t)_a^c D_t^\alpha v(t) dt = v(t)_t I_b^{1-\alpha} u(t)\big|_{t=a}^{t=b} + \int_a^b v(t)_a D_t^\alpha u(t) dt. \tag{2.5} $$

Moreover, if $v(a) = v(b) = 0$, then, one gets

$$ \int_a^b u(t)_a D_t^\alpha v(t) dt = \int_a^b v(t)_a^c D_t^\alpha u(t) dt. \tag{2.6} $$

3. Variational setting and main result

To show the existence of solutions to (P_λ), we will use Nehari manifold and fibering maps theory. For this purpose we introduce some basic notations and results which are used in the proof of our main result.

As we say in Section 1, to deal with the existence of solutions for (P_λ) or some simpler case for the nonlinear terms (for example, (FBVP)), the pioneer work is completed on some function space using the variational methods and critical point theory in [14]. Unfortunately, the authors in the recent work [8] pointed out that the fractional derivatives space $E_0^{\alpha,p}$ defined in [14] is *problematic*. Therefore, we choose the fractional Sobolev space $E_0^{\alpha,p}$ constructed in [8] (the same symbol as usual) in the sense of weak fractional derivatives.

For this purpose, we recall the definitions of left and right weak fractional derivatives.

Definition 3.1. Let $0 < \alpha \leq 1$, u, $v \in L^1([0,T],\mathbb{R})$, if

$$\int_0^T \psi(t)v(t)dt = \int_0^T u(t)({}_tD_T^\alpha \psi)(t)dt, \quad \forall \psi \in C_0^\infty([0,T],\mathbb{R}),$$

then v is called as the left weak fractional derivative, and denoted by ${}_0\dot{D}_t^\alpha u$; if

$$\int_0^T \psi(t)v(t)dt = \int_0^T u(t)({}_tD_t^\alpha \psi)(t)dt, \quad \forall \psi \in C_0^\infty([0,T],\mathbb{R}),$$

then v is called as the right weak fractional derivative, and denoted by ${}_0\dot{D}_T^\alpha u$.

Based on Definition 3.1, we can introduce the appropriate function space corresponding to (P_λ). In fact, we only use the definition of left weak fractional derivative. For the simplicity and to be consistent with the classical notation, we still denote by ${}_0D_t^\alpha u$ the left weak fractional derivative of u in what follows. The set of all functions $u \in C^\infty([0,T],\mathbb{R})$ with $u(0) = u(T) = 0$ is denoted by $C_0^\infty([0,T],\mathbb{R})$. For $\alpha > 0$ we define the weak fractional derivative space $E_0^{\alpha,p}$ as the closure of $C_0^\infty([0,T],\mathbb{R})$ under the norm

$$\|u\|_{\alpha,p} = \left(\|u\|_p^p + \|{}_0D_t^\alpha u\|_p^p\right)^{\frac{1}{p}}. \tag{3.1}$$

Note that, as pointed out in [8], if we define the space $E_0^{\alpha,p}$ as in [14], taking a Cauchy sequence $\{u_n\} \subset C_0^\infty([0,T],\mathbb{R})$ with respect to the norm $\|\cdot\|_{\alpha,p}$, one has

$$u_n \to u_0, \quad {}_0D_t^\alpha u_n \to v_0 \quad \text{in} \quad L^p([0,T],\mathbb{R}).$$

Unfortunately, ${}_0D_t^\alpha u_0$ may not exist. Even if ${}_0D_t^\alpha u_0$ exists, ${}_0D_t^\alpha u_0$ may not be equal to v_0. That is, if we define the space as in [14], the space is not complete. Therefore, we could not choose it as our variational space. To fix this gap, the authors in [8] introduced the definition of weak fractional derivative operator (see [8] for the details). In other words, they optimized the completeness of the fractional derivative space. Thus, we can define the variational framework in the space chosen in our paper for the problem (P_λ).

Lemma 3.2 ([8, Corollary 3.6 and Remark 3.10])**.** *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*. For all* $u \in E_0^{\alpha,p}$*, we have*

$$\|u\|_p \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_p. \tag{3.2}$$

Moreover, if $\alpha > \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$*, then*

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)p'+1\right)^{\frac{1}{p'}}} \|{}_0D_t^\alpha u\|_p. \tag{3.3}$$

According to (3.2), we can consider $E_0^{\alpha,p}$ with respect to the equivalent norm

$$\|u\| = \|{}_0D_t^\alpha u\|_p. \tag{3.4}$$

Lemma 3.3 ([8, Theorem 3.11])**.** *Let* $0 < \alpha \leq 1$*, and* $1 < p < \infty$*. Assume that* $\alpha > \frac{1}{p}$ *and the sequence* $\{u_n\} \rightharpoonup u$ *weakly in* $E_0^{\alpha,p}$*. Then,* $\{u_n\} \to u$ *in* $C([0,T],\mathbb{R})$*, that is*

$$\|u_n - u\|_\infty \to 0 \quad as \quad n \to \infty.$$

We say that $u \in E_0^{\alpha,p}$ is a solution to the problem (P_λ), if u satisfies the following equality

$$\int_0^T |{}_0D_t^\alpha u(t)|^{p-2}({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t))dt - \int_0^T (\nabla W(t,u(t)), v(t))\,dt$$
$$-\lambda \int_0^T g(t)|u(t)|^{q-2}(u(t),v(t))dt = 0 \quad \text{for any } v \in E_0^{\alpha,p}.$$

Therefore, associated to the problem (P_λ), we define the functional

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{1}{r}\int_0^T W(t,u(t))dt - \frac{\lambda}{q}\int_0^T g(t)|u|^q dt. \tag{3.5}$$

We need to show that the following lemma holds.

Lemma 3.4. (i) *The functional* J_λ *is well defined on* $E_0^{\alpha,p}$*.*

(ii) *The functional* J_λ *is of class* $C^1(E_0^{\alpha,p},\mathbb{R})$ *and for all* $u, v \in E_0^{\alpha,p}$ *we have*

$$\langle J_\lambda'(u), v\rangle = \int_0^T |{}_0D_t^\alpha u(t)|^{p-2}({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t))dt$$
$$- \int_0^T (\nabla W(t,u(t)), v(t))\,dt - \lambda \int_0^T g(t)|u(t)|^{q-2}(u(t),v(t))dt. \tag{3.6}$$

Proof. (i) From the continuous embedding and the Hölder inequality, we have

$$\begin{aligned} J_\lambda(u) &\leq \frac{1}{p}\|u\|^p + \frac{1}{r}\int_0^T W(t,u(t))dt + \frac{\lambda}{q}\int_0^T g(t)|u|^q dt \\ &\leq \frac{1}{p}\|u\|^p - \frac{K}{r}\|u\|_r^r - \frac{\lambda\|g\|_\infty}{q}\|u\|_q^q \\ &\leq \frac{1}{p}\|u\|^p + c_1\|u\|^r + c_2\|u\|^q, \end{aligned}$$

which implies that J_λ is well defined on $E_0^{\alpha,p}$.

(ii) Put $G(u) = \frac{1}{p}|{}_0D_t^\alpha u|^p - \frac{1}{r}W(t,u(t)) - \frac{\lambda}{q}g(t)|u|^q$. Then, we can easily show that for all $u, v \in E_0^{\alpha,p}$ and for almost every $t \in [0,T]$

$$\begin{aligned}
&\lim_{s\to 0}\frac{G(u(t)+sv(t)) - G(u(t))}{s}\\
&= |{}_0D_t^\alpha u(t)|^{p-2}({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t)) - (\nabla W(t,u(t)), v(t))\\
&\quad - \lambda g(t)|u(t)|^{q-2}(u(t), v(t)).
\end{aligned}$$

So, from the Lagrange mean value theorem, (1.1) and (1.2), there exists a real number θ such that $|\theta| \leq |s|$ and

$$\begin{aligned}
&\frac{G(u(t)+sv(t)) - G(u(t))}{s}\\
&= |{}_0D_t^\alpha (u(t)+\theta v(t))|^{p-2}({}_0D_t^\alpha (u(t)+\theta v(t)), {}_0D_t^\alpha v(t))\\
&\quad -(\nabla W(t,(u(t)+\theta v(t))), v(t)) - \lambda g(t)|(u(t)+\theta v(t))|^{q-2}(u(t)+\theta v(t), v(t))\\
&\leq |{}_0D_t^\alpha (u(t)+\theta v(t))|^{p-1}|{}_0D_t^\alpha v(t)|\\
&\quad + rK|u(t)+\theta v(t)|^{r-1}|v(t)| + \lambda|g(t)||(u(t)+\theta v(t))|^{q-1}|v(t)|\\
&\leq |{}_0D_t^\alpha u(t)|^{p-1}|{}_0D_t^\alpha v(t)| + |{}_0D_t^\alpha v(t)|^p\\
&\quad + rK|u(t)|^{r-1}|v(t)| + rK|v(t)|^r + \lambda|g(t)||u(t)|^{q-1}|v(t)| + \lambda|g(t)||v(t)|^{q-1}.
\end{aligned} \tag{3.7}$$

On the other hand, from the Hölder inequality, we get

$$\int_0^T |{}_0D_t^\alpha u(t)|^{p-1}|{}_0D_t^\alpha v(t)|dt \leq \||{}_0D_t^\alpha u|^{p-1}\|_{\frac{p}{p-1}}\|{}_0D_t^\alpha v\|_p,$$

and

$$\int_0^T |u(t)|^{\sigma-1}\;|v(t)|dt \leq \||u|^{\sigma-1}\|_{\frac{\sigma}{\sigma-1}}\|v\|_\sigma \quad \text{for } \sigma = r \text{ or } q.$$

Since g is bounded, then, from the above inequalities, one concludes that the expression 3.7 is in $L^1([0,T])$. Therefore, by the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{s\to 0}\frac{J_\lambda(u+sv) - J_\lambda(u)}{s}\\
&= \int_0^T |{}_0D_t^\alpha (u(t)|^{p-2}({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t))dt\\
&\quad - \int_0^T (\nabla W(t,u(t)), v(t)) - \lambda g(t)|u(t)|^{q-2}(u(t), v(t))dt.
\end{aligned}$$

That is, J_λ is Gâteaux differentiable.

In what follows, it is sufficient to prove that the Gâteaux derivative of J_λ is continuous. It is similar to the one in Avci et al. [3], therefore we omit it. □

We deduce from Lemma 3.4 and equation (1.1) that

$$\langle J'_\lambda(u), u\rangle = \int_0^T |{}_0D_t^\alpha u(t)|^p dt - \int_0^T W(t,u(t))dt - \lambda \int_0^T g(t)|u(t)|^q dt.$$

It is easy to see that the energy functional J_λ is not bounded below on the space $E_0^{\alpha,p}$, but it is bounded below on a suitable subset of $E_0^{\alpha,p}$. In order to investigate the problem (P_λ), we define the constraint set

$$\mathcal{N}_\lambda := \left\{u \in E_0^{\alpha,p} \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\right\}.$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of (P_λ), and $u \in \mathcal{N}_\lambda$ if and only if

$$\|u\|^p - \int_0^T W(t,u(t))dt - \lambda \int_0^T g(t)|u(t)|^q dt = 0. \tag{3.8}$$

To obtain the existence of solutions, we split $\mathcal{N}_\lambda$ into three parts: corresponding to local minima, local maxima and points of inflection, are measurable sets defined as follows:

$$\mathcal{N}_\lambda^+ = \left\{u \in \mathcal{N}_\lambda : (p-1)\|u\|^p - (r-1)\int_0^T W(t,u(t))dt - \lambda(q-1)\int_0^T g(t)|u|^q dt > 0\right\},$$

$$\mathcal{N}_\lambda^- = \left\{u \in \mathcal{N}_\lambda : (p-1)\|u\|^p - (r-1)\int_0^T W(t,u(t))dt - \lambda(q-1)\int_0^T g(t)|u|^q dt < 0\right\},$$

$$\mathcal{N}_\lambda^0 = \left\{u \in \mathcal{N}_\lambda : (p-1)\|u\|^p - (r-1)\int_0^T W(t,u(t))dt - \lambda(q-1)\int_0^T g(t)|u|^q dt = 0\right\}.$$

Next, we present some important properties of $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$. Let $\overline{p}$ be such that $\frac{1}{p} + \frac{1}{\overline{p}} = 1$ and put

$$\mu_0 = \frac{(p-r)\left(\Gamma(\alpha)\right)^q \left((\alpha-1)\overline{p}+1\right)^{\frac{q}{\overline{p}}}}{(q-r)\|g\|_\infty T^{1+q(\alpha-\frac{1}{p})}} \left(\frac{(q-p)\left(\Gamma(\alpha)\right)^r \left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}{K(q-r)T^{1+r(\alpha-\frac{1}{p})}}\right)^{\frac{q-p}{p-r}}.$$

Then, we have the following crucial result.

Lemma 3.5. *If $\lambda \in (0,\mu_0)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. We proceed by contradiction to prove that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0,\mu_0)$. Let us suppose that there exists $u_0 \in \mathcal{N}_\lambda^0$. Then, from (3.8) we obtain

$$(p-r)\|u_0\|^p - \lambda(q-r)\int_0^T g(t)|u_0|^q dt = 0, \tag{3.9}$$

and

$$(q-p)\|u_0\|^p - (q-r)\int_0^T W(t,u_0(t))dt = 0. \tag{3.10}$$

From (3.3) and (3.9), one has

$$\|u_0\| \geq \left(\frac{(p-r)\left(\Gamma(\alpha)\right)^q \left((\alpha-1)\overline{p}+1\right)^{\frac{q}{\overline{p}}}}{\lambda(q-r)\|g\|_\infty T^{1+q(\alpha-\frac{1}{p})}} \right)^{\frac{1}{q-p}}. \tag{3.11}$$

On the other hand, from (1.2), (3.3) and (3.10), one has

$$\|u_0\| \leq \left(\frac{K(q-r)T^{1+r(\alpha-\frac{1}{p})}}{(q-p)\left(\Gamma(\alpha)\right)^r \left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}} \right)^{\frac{1}{p-r}}. \tag{3.12}$$

Combining (3.11) and (3.12) we obtain $\lambda \geq \mu_0$, which gives a contradiction. This completes the proof of Lemma 3.5. □

Lemma 3.6. *If $\lambda \in (0, \mu_0)$, then J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Let $u \in \mathcal{N}_\lambda$. Then, using (1.2) and (3.3), we obtain

$$\int_0^T W(t,u(t))dt \leq K \int_0^T |u(t)|^r dt \leq \frac{KT^{1+r(\alpha-\frac{1}{p})}}{\left(\Gamma(\alpha)\right)^r \left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}} \|u\|^r. \tag{3.13}$$

Consequently, from (3.8), we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{q-p}{qp}\|u\|^p - \frac{q-r}{rq}\int_0^T W(t,u(t))dt \\ &\geq \frac{q-p}{qp}\|u\|^p - \frac{K(q-r)T^{1+r(\alpha-\frac{1}{p})}}{qr\left(\Gamma(\alpha)\right)^r \left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}\|u\|^r. \end{aligned}$$

Since $r < p < q$, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. The proof of Lemma 3.6 is now completed. □

Now as we know that the Nehari manifold is closely related to the behavior of the functions $\Phi_u : [0,\infty) \to \mathbb{R}$ defined as

$$\Phi_u(s) = J_\lambda(su).$$

Such maps are called fibering maps and were introduced by Drabek and Pohozaev in [9]. For $u \in E_0^{\alpha,p}$, we define

$$\Phi_u(s) = \frac{s^p}{p}\|u\|^p - \frac{s^r}{r}\int_0^T W(t,u(t))dt - \lambda\frac{s^q}{q}\int_0^T g(t)|u(t)|^q dt,$$

then, we have

$$\Phi_u'(s) = s^{p-1}\|u\|^p - s^{r-1}\int_0^T W(t,u(t))dt - \lambda s^{q-1}\int_0^T g(t)|u(t)|^q dt,$$

and

$$\begin{aligned} \Phi_u''(s) &= (p-1)s^{p-2}\|u\|^p - (r-1)s^{r-2}\int_0^T W(t,u(t))dt \\ &\quad - \lambda(q-1)s^{q-2}\int_0^T g(t)|u(t)|^q dt. \end{aligned} \tag{3.14}$$

Then, it is easy to see that $su \in \mathcal{N}_\lambda$ if and only if $\Phi_u'(s) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi_u'(1) = 0$.

Before studying the behavior of Nehari manifold using fibering maps, we introduce some notations

$$\mathcal{W}^{\pm} = \{u \in E_0^{\alpha,p} \setminus \{0\} : \int_0^T W(t,u(t))dt \gtrless 0\},$$

$$\mathcal{W}^{0} = \{u \in E_0^{\alpha,p} \setminus \{0\} : \int_0^T W(t,u(t))dt = 0\},$$

$$\mathcal{G}^{\pm} = \{u \in E_0^{\alpha,p} \setminus \{0\} : \int_0^T g(t)|u(t)|^q dt \gtrless 0\},$$

and

$$\mathcal{G}^{0} = \{u \in E_0^{\alpha,p} \setminus \{0\} : \int_0^T g(t)|u(t)|^q dt = 0\}.$$

In what follows, we study the fibering map Φ_u according to the sign of $\int_0^T g(t)|u(t)|^q dt$ and $\int_0^T \nabla W(t,u(t))dt$. For this purpose, we define $m_u : [0,\infty) \to \mathbb{R}$ by

$$m_u(s) = s^{p-r}\|u\|^p - \lambda s^{q-r}\int_0^T g(t)|u(t)|^q dt. \tag{3.15}$$

Then, for $s > 0$ we have

$$\begin{aligned} \Phi_u'(s) &= s^{p-1}\|u\|^p - s^{r-1}\int_0^T W(t,u(t))dt - \lambda s^{q-1}\int_0^T g(t)|u(t)|^q dt, \\ &= s^{r-1}\left(m_u(s) - \int_0^T W(t,u(t))dt\right), \end{aligned} \tag{3.16}$$

which implies that $su \in \mathcal{N}_\lambda$ if and only if s is a solution of the following equation

$$m_u(s) = \int_0^T W(t,u(t))dt.$$

Moreover, it is obvious that $m_u(0) = 0$ and

$$m_u'(s) = (p-r)s^{p-r-1}\|u\|^p - \lambda(q-r)s^{q-r-1}\int_0^T g(t)|u(t)|^q dt. \tag{3.17}$$

Lemma 3.7. *If $u \in \mathcal{W}^{\underline{0}} \cap \mathcal{G}^{\underline{0}}$, then Φ_u has no critical point.*

Proof. In this case $\Phi_u(0) = 0$ and $\Phi_u'(s) > 0$, $\forall s > 0$ which implies that Φ_u is strictly increasing and hence has no critical point. $\square$

Lemma 3.8. *If $u \in \mathcal{W}^{\underline{0}} \cap \mathcal{G}^{+}$, then Φ_u has a unique critical point which corresponds to a global maximum point. Moreover, there exists $s_0 > 0$ such that $s_0 u \in \mathcal{N}_\lambda^+$ and $J_\lambda(s_0 u) < 0$.*

Proof. In this case, there exists a unique $\bar{s} \in (0, \infty)$ such that $m'_u(\bar{s}) = 0$. In addition, $m'_u(s) > 0$ for $s \in (0, \bar{s})$ and $m'_u(s) < 0$ for $s \in (\bar{s}, \infty)$. Note that $m_u(0) = 0$ and $m_u(s) \to -\infty$ as $s \to \infty$. So, for $u \in \mathcal{W}^-$, there exists a unique s_0 such that $m_u(s_0) = \int_0^T W(t, u(t))dt$. Consequently, according to (3.16), we have $\Phi'_u(s) > 0$ for $0 < s < s_0$, and $\Phi'_u(s) < 0$ for $s > s_0$. That is, Φ_u is increasing on $(0, s_0)$, decreasing on (s_0, ∞). Therefore, Φ_u has exactly one critical point at s_0, which is a global maximum point. Thus, by (3.14), $s_0 u \in \mathcal{N}_\lambda^-$. □

Lemma 3.9. *If $u \in \mathcal{W}^+ \cap \mathcal{G}^0_-$, then Φ_u has a unique critical point which correspond to a global minimum point. Moreover, there exists $s_1 > 0$ such that $s_1 u \in \mathcal{N}_\lambda^+$ and $J_\lambda(s_1 u) < 0$.*

Proof. In this case, it is easy to see that $m_u(0) = 0$ and $m'_u(s) > 0$, $\forall s > 0$, which implies that m_u is strictly increasing. Since $u \in \mathcal{W}^+$, there exists a unique $s_1 > 0$ such that $m_u(s_1) = \int_0^T W(t, u(t))dt$. This implies that Φ_u is decreasing on $(0, s_1)$, increasing on (s_1, ∞) and $\Phi'_u(s_1) = 0$. Thus, Φ_u has exactly one critical point corresponding to global minimum point. Hence $s_1 u \in \mathcal{N}_\lambda^+$. Moreover, since $J_\lambda(0) = 0$, then we have $J_\lambda(s_1 u) < 0$. □

Lemma 3.10. *If $u \in \mathcal{W}^+ \cap \mathcal{G}^+$, then there exists $\mu_1 > 0$ such that for $\lambda \in (0, \mu_1)$, Φ_u have a positive value and Φ_u has exactly two critical points which correspond to the local minimum and local maximum. Moreover, there exists $s_2 > 0$ such that $s_2 u \in \mathcal{N}_\lambda^+$ and $J_\lambda(s_2 u) < 0$.*

Proof. Let $u \in E_0^{\alpha,p}$. As in above, we define

$$M_u(s) = \frac{s^p}{p}\|u\|^p - \lambda \frac{s^q}{q} \int_0^T g(t)|u(t)|^q dt.$$

Then,

$$M'_u(s) = s^{p-1}\|u\|^p - \lambda s^{q-1} \int_0^T g(t)|u(t)|^q dt.$$

It is clear that M_u attains its maximum value at $\widetilde{s} = \left(\frac{\|u\|^p}{\lambda \int_0^T g(t)|u(t)|^q dt}\right)^{\frac{1}{q-p}}$. Moreover,

$$M_u(\widetilde{s}) = \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{\|u\|^q}{\lambda \int_0^T g(t)|u(t)|^q dt}\right)^{\frac{p}{q-p}}$$

and

$$M''_u(\widetilde{s}) = (p-q)\frac{\|u\|^{\frac{p(q-2)}{q-p}}}{\left(\lambda \int_0^T g(t)|u(t)|^q dt\right)^{\frac{p-2}{q-p}}} < 0.$$

From (3.3), we deduce that

$$M_u(\widetilde{s}) \geq \frac{q-p}{qp}\left(\frac{(\Gamma(\alpha))^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}{\lambda\|g\|_\infty T^{1+q(\alpha-\alpha\frac{1}{p})}}\right)^{\frac{p}{q-p}} := \delta, \tag{3.18}$$

which is independent of u. We now show that there exists $\mu_1 > 0$ such that $\Phi_u(\widetilde{s}) > 0$. Using (1.2) and (3.3), we get

$$\begin{aligned}
\frac{\widetilde{s}^r}{r}\int_0^1 W(t,u(t))dt &\le \frac{KT^{1+r(\alpha-\frac{1}{p})}||u||^r}{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}\widetilde{s}^r \\
&= \frac{KT^{1+r(\alpha-\frac{1}{p})}}{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}||u||^r\left(\frac{||u||^p}{\lambda\int_0^T g(t)|u(t)|^q dt}\right)^{\frac{r}{q-p}} \\
&= \frac{KT^{1+r(\alpha-\frac{1}{p})}}{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}\left(\frac{||u||^q}{\lambda\int_0^T g(t)|u(t)|^q dt}\right)^{\frac{r}{q-p}} \\
&= \frac{KT^{1+r(\alpha-\frac{1}{p})}}{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}\left(\frac{pq}{q-p}\right)^{\frac{r}{p}}\left(M_u(\widetilde{s})\right)^{\frac{r}{p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\Phi_u(\widetilde{s}) &= M_u(\widetilde{s}) - \frac{\widetilde{s}^r}{r}\int_0^T W(t,u(t))dt \\
&\ge M_u(\widetilde{s}) - \frac{KT^{1+r(\alpha-\frac{1}{p})}}{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}\left(\frac{pq}{q-p}\right)^{\frac{r}{p}}\left(M_u(\widetilde{s})\right)^{\frac{r}{p}} \\
&\ge \delta^{\frac{r}{p}}\left(\delta^{\frac{p-r}{p}} - \frac{KT^{1+r(\alpha-\frac{1}{p})}}{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}\left(\frac{pq}{q-p}\right)^{\frac{r}{p}}\right) > 0
\end{aligned}$$

for $0 < \lambda < \mu_1$, where δ is the constant given in (3.18) and

$$\mu_1 = \frac{(q-p)\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}{qp\|g\|_\infty T^{1+q(\alpha-\frac{1}{p})}}\left(\frac{r\left(\Gamma(\alpha)\right)^r\left((\alpha-1)\overline{p}+1\right)^{\frac{r}{\overline{p}}}}{KT^{1+r(\alpha-\frac{1}{p})}}\left(\frac{q-p}{qp}\right)^{\frac{r}{p}}\right)^{\frac{q-p}{p-r}}.$$

The same arguments used in the proof of Lemma 3.8 show that Φ_u has exactly two critical points which correspond to the local minimum and local maximum. Moreover, there exists $s_2 > 0$ such that $s_2u \in \mathcal{N}_\lambda^+$ and $J_\lambda(s_2u) < 0$. The proof of Lemma 3.10 is now completed. □

Remark 3.11. In what follows, let us define λ_0 as

(3.19) $$\lambda_0 = \min(\mu_0, \mu_1).$$

Note that if $0 < \lambda < \lambda_0$, then all the above Lemmas hold true.

Lemma 3.12. *Let u be a local minimizer for J_λ on subset $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$. Then u is a critical point of J_λ.*

Proof. Since u is a minimizer for J_λ under the constraint

$$I_\lambda(u) := \langle J'_\lambda(u), u\rangle = 0.$$

Then, applying the theory of Lagrange multipliers, we get the existence of $\mu \in \mathbb{R}$ such that

$$J'_\lambda(u) = \mu I'_\lambda(u).$$

So, we get

$$\langle J'_\lambda(u), u\rangle = \mu\langle I'_\lambda(u), u\rangle = \mu\Phi''_u(1) = 0,$$

but $u \notin \mathcal{N}_\lambda^0$ and so $\Phi''_u(1) \neq 0$. Hence, $\mu = 0$, which gives the proof of Lemma 3.12. □

4. Proof of Theorem 1.1

Throughout this section, we assume that $\frac{1}{2} < \alpha < 1$ and $1 < r < p < q$. Let λ_0 be the constant given by (3.19). Then the proof of Theorem 1.1 is based on the following two Propositions.

Proposition 4.1. *Assume that hypothesis of Theorem 1.1 are satisfied. Then, for all $0 < \lambda < \lambda_0$, J_λ achieves its minimum on $\mathcal{N}_\lambda^+$.*

Proof. Since J_λ is bounded below on $\mathcal{N}_\lambda$ and also on $\mathcal{N}_\lambda^+$, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u).$$

As J_λ is coercive on $\mathcal{N}_\lambda$, $\{u_k\}$ is a bounded sequence in $E_0^{\alpha,p}$ up to a subsequence, there exists $u_\lambda \in E_0^{\alpha,p}$ such that

$$u_k \rightharpoonup u_\lambda \quad \text{weakly in } E_0^{\alpha,p}.$$

Let $u \in E_0^{\alpha,p}$ such that $\int_0^T W(t, u(t))dt > 0$. Then, from Lemma 3.9 and Lemma 3.10, there exists $s_1 > 0$ such that $s_1 u \in \mathcal{N}_\lambda^+$ and $J_\lambda(s_1 u) < 0$. Hence,

$$\inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0.$$

Since $\{u_k\} \subset \mathcal{N}_\lambda$, we get

$$J_\lambda(u_k) = (\frac{1}{p} - \frac{1}{r})||u_k||^p - (\frac{1}{r} - \frac{1}{q})\int_0^T W(t, u_k(t))dt,$$

which yields that

$$(\frac{1}{r} - \frac{1}{q})\int_0^T W(t, u_k(t))dt = (\frac{1}{p} - \frac{1}{q})||u_k||^p - J_\lambda(u_k).$$

Let k go to infinity in the above equation, we get

$$\int_0^T W(t, u_\lambda(t))dt > 0. \tag{4.1}$$

Now, we claim that $u_k \to u_\lambda$ strongly in $E_0^{\alpha,p}$. Otherwise, we have

$$||u_\lambda||^p < \lim\inf_{k\to\infty} ||u_k||^p. \tag{4.2}$$

Since $\Phi'_{u_\lambda}(s_1) = 0$ it follows from (4.2) that $\Phi'_{u_k}(s_1) > 0$ for sufficiently large k. So, we must have $s_1 > 1$. However, $s_1 u_\lambda \in \mathcal{N}_\lambda^+$ and so

$$J_\lambda(s_1 u_\lambda) < J_\lambda(u_\lambda) \leq \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),$$

which gives a contradiction. Thus,

$$u_k \to u_\lambda \ \text{ strongly in } \ E_0^{\alpha,p},$$

which implies that $u_\lambda \in \mathbb{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$. In addition, it is easy to check by contradiction that $u_\lambda \in \mathcal{N}_\lambda^+$. Consequently, from (4.1), u_λ is a nontrivial solution of (P_λ). $\square$

Proposition 4.2. *Assume that hypothesis of Theorem 1.1 are satisfied. Then, for all* $0 < \lambda < \lambda_0$*,* J_λ *achieves its minimum on* $\mathcal{N}_\lambda^-$.

Proof. Let $u \in \mathcal{N}_\lambda^-$. Therefore, using the result in Lemma 3.10, we have the existence of $\mu_1 > 0$ such that $J_\lambda(u) \geq \mu_1$. So, there exists a minimizing sequence $\{v_k\} \subset \mathcal{N}_\lambda^-$ such that

$$\lim_{k\to\infty} J_\lambda(v_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) > 0. \tag{4.3}$$

Moreover, since J_λ is coercive, $\{v_k\}$ is a bounded sequence in $E_0^{\alpha,p}$ up to a sub-sequence, there exists $v_\lambda \in E_0^{\alpha,p}$ such that

$$v_k \rightharpoonup v_\lambda \quad \text{weakly in } \ E_0^{\alpha,p}.$$

Since $v_k \in \mathcal{N}_\lambda$, then we have

$$J_\lambda(v_k) + (\frac{1}{q} - \frac{1}{p})||v_k||^p + \lambda(\frac{1}{q} - \frac{1}{r})\int_0^1 g(t)|v_k(t)|^q dt. \tag{4.4}$$

Let k go to infinity in (4.4), it follows from (4.3) that

$$\int_0^1 g(t)|v_\lambda(t)|^q dt > 0. \tag{4.5}$$

Hence, $v_\lambda \in \mathcal{G}^+$ and so Φ_{v_λ} has a global maximum at some point $\widetilde{s}$. Consequently, $\widetilde{s}v_\lambda \in \mathcal{N}_\lambda^-$. On the other hand, $v_k \in \mathcal{N}_\lambda^-$ implies that 1 is a global maximum point for Φ_{v_k}, i.e.,

$$J_\lambda(\widetilde{s}v_k) = \Phi_{v_k}(\widetilde{s}) \leq \Phi_{v_k}(1) = J_\lambda(v_k). \tag{4.6}$$

Now, as in the step 1, we claim that $v_k \to v_\lambda$. Assume it is not true, then

$$||v_\lambda||^p < \lim\inf_{k\to\infty} ||v_k||^p,$$

it follows from (4.6) that

$$\begin{aligned} J_\lambda(\widetilde{s}v_\lambda) &= \frac{\widetilde{s}^p}{p}||v_\lambda||^p - \frac{\widetilde{s}^r}{r}\int_0^1 W(t, v_\lambda(t))dt - \lambda\frac{\widetilde{s}^q}{q}\int_0^1 g(t)|v_\lambda(t)|^q dt \\ &< \inf_{k\to\infty}\left(\frac{\widetilde{s}^p}{p}||v_k||^p - \frac{\widetilde{s}^r}{r}\int_0^1 W(t, v_k(t))dt - \lambda\frac{\widetilde{s}^q}{q}\int_0^1 g(t)|v_k(t)|^q dt\right) \\ &\leq \lim_{k\to\infty} J_\lambda(\widetilde{s}v_k) \leq \lim_{k\to\infty} J_\lambda(v_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u), \end{aligned}$$

which gives a contradiction. Hence, $v_k \to v_\lambda$ and so $v_\lambda \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$, since $\mathcal{N}_\lambda^0 = \emptyset$, then, v_λ is a minimizer for J_λ on $\mathcal{N}_\lambda^-$. On the other hand, from (4.5),

v_λ is a nontrivial solution of problem (P_λ). Finally, since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$, u_λ and v_λ are distinct. That is the result of Theorem 1.1 holds true. $\square$

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