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TWO NEW BLOW-UP CONDITIONS FOR A PSEUDO-PARABOLIC EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper deals with the blow-up phenomenon of solutions to a pseudo-parabolic equation with logarithmic nonlinearity, which was studied extensively in recent years. The previous result depends on the mountain-pass level d (see (1.6) for its definition). In this paper, we obtain two blow-up conditions which do not depend on d. Moreover, the upper bound of the blow-up time is obtained.

1. Introduction

In this paper, we consider the following model of a nonlinear pseudo-parabolic equation with logarithmic nonlinearity:

$$\begin{cases} u_t - \Delta u_t - \Delta u = |u|^{q-2}u \log|u|, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in H_0^1(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$. The parameter q satisfies

$$2 < q < +\infty \text{ if } n \leq 2;\ 2 < q < \frac{2n}{n-2} \text{ if } n > 2. \tag{1.2}$$

Pseudo-parabolic equation is used to describe a series of important physical processes, such as the unidirectional propagation of nonlinear, long waves (see [2, 10]) and the aggregation of population (see [9]). Problem (1.1) also can be referred to as Showalter equation [1]. When $q = 2$, problem (1.1) has been studied by Chen and Tian in [4], where the existence of global solution and infinite time blow-up solutions are obtained. To obtain the finite time blow-up solution, the following model was proposed:

$$u_t - \Delta u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-2}u \log|u|, \tag{1.3}$$

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which is a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity. In fact, problem (1.3) has been studied extensively (see for example [3, 6–8]). Especially, He et al. in [6] studied (1.3) in Ω with zero Dirichlet boundary condition for the case $2 < p < q < p(1+2/n)$, and they obtained the solution will blow up in finite time if the initial data u_0 satisfies $J(u_0) < d$ and $I(u_0) < 0$, where J is the energy functional, I is the Nehari functional and d is the mountain-pass level. The results in [6] were extended by Ding and Zhou [5] for the case $1 < p \leq q < p^*$ recently, where

$$p^* = +\infty \text{ if } n \leq p;\ p^* = \frac{np}{n-p} \text{ if } n > p.$$

However, all the previous results depends on d, whose exact value is difficult to computation. The main purpose of this paper is to give some blow-up conditions which do not depend on d.

Firstly, let's introduce some notations. Throughout this paper, we denote the norm of $L^\gamma(\Omega)$ for $1 \leq \gamma \leq +\infty$ by $\|\cdot\|_\gamma$ and the norm of $H_0^1(\Omega)$ by $\|\cdot\|_{H_0^1(\Omega)}$. That is, for any $u \in L^\gamma(\Omega)$,

$$\|u\|_\gamma = \left(\int_\Omega |u(x)|^\gamma dx\right)^{\frac{1}{\gamma}} \text{ if } 1 \leq \gamma < +\infty;$$
$$\|u\|_\infty = \text{ess}\sup_{x\in\Omega} |u(x)|,$$

and for any $u \in H_0^1(\Omega)$,

$$\|u\|_{H_0^1} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2}.$$

Secondly, let us introduce the energy functional J and the Nehari functional I as follows:

$$\begin{aligned} J(u) &:= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q}\int_\Omega |u|^q \log|u| dx + \frac{1}{q^2}\|u\|_q^q, \\ I(u) &:= \langle J'(u), u\rangle = \|\nabla u\|_2^2 - \int_\Omega |u|^q \log|u| dx, \end{aligned} \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. From (1.4), we have

$$J(u) = \frac{1}{q}I(u) + \frac{q-2}{2q}\|\nabla u\|_2^2 + \frac{1}{q^2}\|u\|_q^q. \tag{1.5}$$

Let

$$d := \inf_{u\in N} J(u) \tag{1.6}$$

denote the mountain-pass level, where N is the Nehari manifold, which is defined by

$$N := \{u \in H_0^1(\Omega) \setminus \{0\} \mid I(u) = 0\}. \tag{1.7}$$

Similar to the proof of [6, Lemma 2], we know that d is positive and is achieved by some $u \in N$.

Thirdly, we denote by μ_1 the principal eigenvalue of $-\Delta$ operator with zero Dirichlet boundary condition in Ω. Then,

$$\mu_1 = \inf_{u \in H_0^1 \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2},$$

which implies, for any $u \in H_0^1(\Omega)$,

$$\|u\|_{H_0^1}^2 = \|\nabla u\|_2^2 + \|u\|_2^2 \le \|\nabla u\|_2^2 + \frac{1}{\mu_1}\|\nabla u\|_2^2 = \frac{1}{\lambda_1}\|\nabla u\|_2^2, \tag{1.8}$$

where $\lambda_1 := \mu_1/(\mu_1 + 1)$.

Definition 1.1 (Weak solution)**.** Assume (1.2) holds. Let $u_0 \in H_0^1(\Omega)$ and $T > 0$. A function $u = u(t) \in L^\infty(0, T; H_0^1(\Omega))$ with $u_t \in L^2(0, T; H_0^1(\Omega))$ is called a weak solution of problem (1.1) if $u(0)(x) = u_0(x)$ a.e. in Ω and the follow equality

$$(u_t, v) + (\nabla u, \nabla v) + (\nabla u_t, \nabla v) = (|u|^{q-2} u \log |u|, v) \tag{1.9}$$

with any $v \in H_0^1(\Omega)$ holds for a.e. $t \in (0, T)$, where $(\cdot, \cdot)$ means the inner product of $L^2(\Omega)$, that is

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx.$$

The main results of this paper are the following two theorems.

Theorem 1.1. *Let q satisfy* (1.2), $u_0 \in H_0^1(\Omega)$. *Then the weak solution of problem* (1.1) *will blow up at some finite time T provided that*

$$J(u_0) < \frac{(q-2)\lambda_1}{2q}\|u_0\|_{H_0^1}^2. \tag{1.10}$$

Moreover, T can be estimated by

$$T \le \frac{8q\|u_0\|_{H_0^1}^2}{(q-2)^2[(q-2)\lambda_1\|u_0\|_{H_0^1}^2 - 2qJ(u_0)]}. \tag{1.11}$$

Theorem 1.2. *Let q satisfy* (1.2) *and* $u_0 \in H_0^1(\Omega) \setminus \{0\}$ *satisfy*

$$J(u_0) = \frac{(q-2)\lambda_1}{2q}\|u_0\|_{H_0^1}^2. \tag{1.12}$$

Then the weak solution of problem (1.1) *will blow up in finite time.*

We organize the rest of this paper as follows. In Section 2, we give some preliminaries, and the proofs of the theorems are given in Section 3.

2. Preliminaries

In this section, we give some lemmas, which will be needed in our proofs. Firstly, we introduce the following new functionals

$$J^*(u)=J(u)-\frac{(q-2)\lambda_1}{2q}\|u\|^2_{H_0^1}. \tag{2.1}$$

Then, we define some sets as follows:

$$\begin{aligned} N^* &= \{u\in H_0^1(\Omega)\setminus\{0\} : J^*(u)=0\},\\ N^*_- &= \{u\in H_0^1(\Omega)\setminus\{0\} : J^*(u)<0\},\\ N^*_+ &= \{u\in H_0^1(\Omega)\setminus\{0\} : J^*(u)>0\}. \end{aligned} \tag{2.2}$$

Lemma 2.1. *For $u\in H_0^1(\Omega)\setminus\{0\}$, there exists a unique $\lambda_*>0$ such that $\lambda_* u\in N^*$, $\lambda u\in N^*_-$ for $\lambda>\lambda_*$, $\lambda u\in N^*_+$ for $0<\lambda<\lambda_*$.*

Proof. For any $u\in H_0^1(\Omega)\setminus\{0\}$ and $\lambda>0$, by (1.4) and the definition of $J^*(u)$, we have

$$\begin{aligned} g(\lambda) := J^*(\lambda u) &= J(\lambda u)-\frac{(q-2)\lambda_1\lambda^2}{2q}\|u\|^2_{H_0^1}\\ &= \frac{\lambda^2}{2}\|\nabla u\|_2^2-\frac{\lambda^q}{q}\int_\Omega |u|^q\log|u|dx\\ &\quad -\frac{\lambda^q}{q}\log\lambda\|u\|_q^q+\frac{\lambda^q}{q^2}\|u\|_q^q-\frac{(q-2)\lambda_1\lambda^2}{2q}\|u\|^2_{H_0^1}. \end{aligned}$$

Then $g'(\lambda)=\lambda f(\lambda)$, where

$$f(\lambda):=\|\nabla u\|_2^2-\frac{(q-2)\lambda_1}{q}\|u\|^2_{H_0^1}-\lambda^{q-2}\int_\Omega |u|^q\log|u|dx-\lambda^{q-2}\log\lambda\|u\|_q^q.$$

Since

$$f'(\lambda)=\lambda^{q-3}\left[(2-q)\int_\Omega |u|^q\log|u|dx+(2-q)\log\lambda\|u\|_q^q-\|u\|_q^q\right],$$

if we choose

$$\lambda_{**}=\exp\left(\frac{(2-q)\int_\Omega|u|^q\log|u|dx-\|u\|_q^q}{(q-2)\|u\|_q^q}\right),$$

it follows $f'(\lambda_{**})=0$, $f'(\lambda)>0$ for $0<\lambda<\lambda_{**}$ and $f'(\lambda)<0$ for $\lambda>\lambda_{**}$.

Since (by (1.8))

$$\lim_{\lambda\to 0} f(\lambda)=\|\nabla u\|_2^2-\frac{(q-2)\lambda_1}{q}\|u\|^2_{H_0^1}\geq\frac{2\lambda_1}{q}\|u\|^2_{H_0^1}>0$$

and

$$\lim_{\lambda\to+\infty} f(\lambda)=-\infty,$$

we deduce there must exist a $\lambda_{***}>\lambda_{**}$ such that $f(\lambda_{***})=0$, $f(\lambda)>0$ for $\lambda\in(0,\lambda_{***})$ and $f(\lambda)<0$ for $\lambda\in(\lambda_{***},+\infty)$. Then we can get $g'(\lambda_{***})=$

$\lambda_{***}f(\lambda_{***}) = 0$, $g'(\lambda) = \lambda f(\lambda) > 0$ for $\lambda \in (0, \lambda_{***})$ and $g'(\lambda) = \lambda f(\lambda) < 0$ for $\lambda \in (\lambda_{***}, +\infty)$.

Moreover, since $\lim_{\lambda\to 0} g(\lambda) = 0$ and $\lim_{\lambda\to+\infty} g(\lambda) = -\infty$, there must exist a unique $\lambda_* > \lambda_{***}$ such that $g(\lambda_*) = 0$, $g(\lambda) > 0$ for $\lambda \in (0, \lambda_*)$ and $g(\lambda) < 0$ for $\lambda \in (\lambda_*, +\infty)$. □

Lemma 2.2. *Let N and N^* be the sets defined in* (1.7) *and* (2.2)*, respectively. Then $N^* \cap N = \emptyset$.*

Proof. We argue by contradiction. If there exists a $u \in N^* \cap N$, then we have $u \in H_0^1 \setminus \{0\}$ and (by $I(u) = 0$ and (1.5))

$$J(u) = \frac{q-2}{2q}\|\nabla u\|_2^2 + \frac{1}{q^2}\|u\|_q^q.$$

Then it follows from $J^*(u) = 0$ and (1.8) that

$$\begin{aligned} 0 = J^*(u) &= J(u) - \frac{(q-2)\lambda_1}{2q}\|u\|_{H_0^1}^2 \\ &= \frac{q-2}{2q}\|\nabla u\|_2^2 + \frac{1}{q^2}\|u\|_q^q - \frac{(q-2)\lambda_1}{2q}\|u\|_{H_0^1}^2 \\ &= \frac{q-2}{2q}(\|\nabla u\|_2^2 - \lambda_1\|u\|_{H_0^1}^2) + \frac{1}{q^2}\|u\|_q^q \geq \frac{1}{q^2}\|u\|_q^q > 0, \end{aligned}$$

which is a contradiction. □

Lemma 2.3. *Let q satisfy* (1.2) *and $u = u(t)$ be a weak solution of* (1.1)*. Then $J(u(t))$ is non-increasing with respect to t.*

Proof. By the definition of $J(u(t))$, we obtain

$$\frac{d}{dt}J(u(t)) = \int_\Omega \nabla u \cdot \nabla u_t dx - \int_\Omega |u|^{q-2}uu_t \log|u|dx. \tag{2.3}$$

Let $v = u_t$ in (1.9) of Definition 1.1, we get

$$\|u_t\|_2^2 + \int_\Omega \nabla u \cdot \nabla u_t dx + \|\nabla u_t\|_2^2 = \int_\Omega |u|^{q-2}uu_t \log|u|dx,$$

which, together with (2.3), implies

$$\frac{d}{dt}J(u(t)) = -\|u_t\|_2^2 - \|\nabla u_t\|_2^2 \leq 0. \tag{2.4}$$

Hence $J(u(t))$ is non-increasing with respect to t. □

3. Proof of the theorems

Proof of Theorem 1.1. Let $u = u(t)$ be a solution of problem (1.1) with initial value u_0 satisfying the assumptions. We denote by T the maximum existence time of u. Firstly, we prove the solution will blow up in finite time.

If (1.10) holds, from (1.5) and (1.8) we obtain (note $q > 2$)

$$I(u_0) = \frac{2-q}{2}\|\nabla u_0\|_2^2 - \frac{1}{q}\|u_0\|_q^q + qJ(u_0) \le \frac{(2-q)\lambda_1}{2}\|u_0\|_{H_0^1}^2 + qJ(u_0) < 0. \tag{3.1}$$

Moreover, it holds

$$I(u(t)) < 0,\ \forall t \in [0,T). \tag{3.2}$$

In fact if it is false, by the continuity of $I(u)$, there exists a $t_0 \in (0,T)$ such that

$$I(u(t_0)) = 0 \text{ and } I(u(t)) < 0,\ \forall t \in [0,t_0). \tag{3.3}$$

Since

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|_{H_0^1}^2\right) = \int_\Omega |u|^q \log|u| dx - \|\nabla u\|_2^2 = -I(u(t)) > 0,\ \forall t \in [0,t_0), \tag{3.4}$$

i.e., $\|u\|_{H_0^1}^2$ is strictly increasing with respect to t, then we get

$$J(u_0) < \frac{(q-2)\lambda_1}{2q}\|u_0\|_{H_0^1}^2 < \frac{(q-2)\lambda_1}{2q}\|u(t_0)\|_{H_0^1}^2. \tag{3.5}$$

On the other hand, by (1.5), (1.8), (3.3) and Lemma 2.3, we have

$$\frac{(q-2)\lambda_1}{2q}\|u(t_0)\|_{H_0^1}^2 \le \frac{(q-2)}{2q}\|\nabla u(t_0)\|_2^2 \le J(u(t_0)) \le J(u_0),$$

which contradicts (3.5), so we obtain $I(u(t)) < 0$ for all $t \in [0,T)$.

Furthermore, we only need to consider the case that $J(u(t)) \ge 0$ for all $t \in [0,T)$. Indeed, if there exists a t_0 such that $J(u(t_0)) < 0$, combining with (3.2), we have $J(u(t_0)) < d$ and $I(u(t_0)) < 0$. Then by taking t_0 as the initial time we get the solution blows up in finite time by a similar proof as in [6, Theorem 3].

Next, we will prove the solution $u(t)$ will blow up in finite time by contradiction. Assume $u(t)$ exists globally and let

$$\psi(t) := \frac{1}{2}\|u(t)\|_{H_0^1}^2 - \frac{q}{(q-2)\lambda_1}J(u(t)),\ t \ge 0$$

Then, it follows from (3.4), (2.4), (1.5) and (1.8) that

$$\begin{aligned}\psi'(t) &= -I(u(t)) + \frac{q}{(q-2)\lambda_1}\|u_t\|_{H_0^1}^2 \\ &\ge \frac{q-2}{2}\|\nabla u(t)\|_2^2 + \frac{1}{q}\|u(t)\|_q^q - qJ(u(t)) \\ &\ge \frac{q-2}{2}\|\nabla u(t)\|_2^2 - qJ(u(t))\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{(q-2)\lambda_1}{2}\|u(t)\|_{H_0^1}^2 - qJ(u(t)) \\
&= (q-2)\lambda_1\left[\frac{1}{2}\|u(t)\|_{H_0^1}^2 - \frac{q}{(q-2)\lambda_1}J(u(t))\right] \\
&= (q-2)\lambda_1\psi(t)
\end{aligned}
$$

and

$$\psi(0) = \frac{1}{2}\|u_0\|_{H_0^1}^2 - \frac{q}{(q-2)\lambda_1}J(u_0) > 0,$$

which, combining with $J(u(t)) \geq 0$, implies

$$\frac{1}{2}\|u\|_{H_0^1}^2 \geq \psi(t) \geq \psi(0)e^{(q-2)\lambda_1 t},\ t \geq 0. \tag{3.6}$$

On the other hand, by Hölder's inequality, (2.4) and $J(u_0) \geq J(u(t)) \geq 0$, we have

$$
\begin{aligned}
\|u(t)\|_{H_0^1} &= \left\|u_0 + \int_0^t u_\tau d\tau\right\|_{H_0^1} \\
&\leq \|u_0\|_{H_0^1} + \int_0^t \|u_\tau\|_{H_0^1} d\tau \\
&\leq \|u_0\|_{H_0^1} + t^{\frac{1}{2}}\left(\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau\right)^{\frac{1}{2}} \\
&\leq \|u_0\|_{H_0^1} + t^{\frac{1}{2}}(J(u_0) - J(u(t)))^{\frac{1}{2}} \\
&\leq \|u_0\|_{H_0^1} + t^{\frac{1}{2}}(J(u_0))^{\frac{1}{2}}.
\end{aligned}
$$

Combining with (3.6), we obtain

$$\sqrt{2\psi(0)}e^{\frac{(q-2)\lambda_1 t}{2}} \leq \|u_0\|_{H_0^1} + t^{\frac{1}{2}}(J(u_0))^{\frac{1}{2}},$$

which is impossible when t is sufficiently large. So the assumption is false and then $u(t)$ blows up in finite time.

The above arguments show that the maximal existence time $T < +\infty$. Next, we give an upper bound estimate of T. For any $\widetilde{T} \in (0, T)$, let

$$G(t) := \int_0^t \|u(\tau)\|_{H_0^1}^2 d\tau + (T-t)\|u_0\|_{H_0^1}^2 + \alpha(t+\beta)^2,\ t \in [0, \widetilde{T}], \tag{3.7}$$

where α, β are two positive constants which will be specified later.

Then, for any $t \in [0, \widetilde{T}]$, by a simply computation, we obtain

$$
\begin{cases}
G(0) = T\|u_0\|_{H_0^1}^2 + \alpha\beta^2 > 0, \\
G'(t) = \|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\alpha(t+\beta) > 2\alpha(t+\beta) > 0, \\
G'(0) \geq 2\alpha\beta > 0,
\end{cases} \tag{3.8}
$$

and

$$G''(t) = -2I(u(t)) + 2\alpha$$

$$
\begin{aligned}
&\geq (q-2)\|\nabla u(t)\|_2^2 + \frac{2}{q}\|u(t)\|_q^q - 2qJ(u(t)) \\
&\geq (q-2)\|\nabla u(t)\|_2^2 - 2qJ(u(t)) \\
&\geq (q-2)\lambda_1 \|u(t)\|_{H_0^1}^2 - 2qJ(u_0) + 2q\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau \\
&\geq (q-2)\lambda_1 \|u_0\|_{H_0^1}^2 - 2qJ(u_0) + 2q\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau \\
&> 0.
\end{aligned}
$$

Thus, we can get

$$G(t) \geq G(0) > 0, \ t \in [0, \widetilde{T}].$$

Let

$$\mu(t) := \left(\int_0^t \|u(\tau)\|_{H_0^1}^2 d\tau\right)^{\frac{1}{2}}, \ \nu(t) := \left(\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau\right)^{\frac{1}{2}}.$$

By using Hölder's inequality, we have

$$
\begin{aligned}
&\left[\int_0^t \|u(\tau)\|_{H_0^1}^2 d\tau + \alpha(t+\beta)^2\right]\left[\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + \alpha\right] \\
&\quad - \left[\frac{1}{2}(\|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2) + \alpha(t+\beta)\right]^2 \\
= &\ \left[\mu^2(t) + \alpha(t+\beta)^2\right]\left[\nu^2(t)+\alpha\right] - \left[\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|_{H_0^1}^2 d\tau + \alpha(t+\beta)\right]^2 \\
\geq &\ \left[\mu^2(t) + \alpha(t+\beta)^2\right]\left[\nu^2(t)+\alpha\right] - \left[\int_0^t \|u\|_{H_0^1}\|u_\tau\|_{H_0^1} d\tau + \alpha(t+\beta)\right]^2 \\
\geq &\ \left[\mu^2(t) + \alpha(t+\beta)^2\right]\left[\nu^2(t)+\alpha\right] - \left[\mu(t)\nu(t) + \alpha(t+\beta)\right]^2 \\
= &\ \left[\sqrt{\alpha}\mu(t)\right]^2 - 2\alpha(t+\beta)\mu(t)\nu(t) + \left[\sqrt{\alpha}(t+\beta)\nu(t)\right]^2 \\
= &\ \left[\sqrt{\alpha}\mu(t) - \sqrt{\alpha}(t+\beta)\nu(t)\right]^2 \geq 0.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
-(G'(t))^2 = &\ -4\left(\frac{1}{2}(\|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2) + \alpha(t+\beta)\right)^2 \\
= &\ 4\left(\int_0^t \|u(\tau)\|_{H_0^1}^2 d\tau + \alpha(t+\beta)^2\right)\left(\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + \alpha\right) \\
&\ - 4\left(\frac{1}{2}(\|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2) + \alpha(t+\beta)\right)^2 \\
&\ - 4\left(G(t) - (T-t)\|u_0\|_{H_0^1}^2\right)\left(\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + \alpha\right) \\
\geq &\ -4G(t)\left(\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + \alpha\right).
\end{aligned}
$$

The above calculations show that

$$\begin{aligned} G(t)G''(t) - \frac{q}{2}(G'(t))^2 &\geq G(t)\left(G''(t) - 2q\left(\int_0^t \|u_\tau\|_{H_0^1}^2 d\tau + \alpha\right)\right) \\ &\geq G(t)\left((q-2)\lambda_1\|u_0\|_{H_0^1}^2 - 2qJ(u_0) - 2q\alpha\right). \end{aligned}$$

We choose α small enough, such that

$$\alpha \in \left(0, \frac{\varrho}{2q}\right], \tag{3.9}$$

where $\varrho := (q-2)\lambda_1\|u_0\|_{H_0^1}^2 - 2qJ(u_0) > 0$, then it follows that $G(t)G''(t) - \frac{q}{2}(G'(t))^2 \geq 0$.

Let $F(t) := G^{\frac{2-q}{2}}(t)$ for $t \in [0, \widetilde{T}]$, then by $G(t) > 0$, $G'(t) > 0$, $q > 2$ and the above inequality, we get

$$\begin{aligned} F'(t) &= -\frac{q-2}{2}G^{-\frac{q}{2}}(t)G'(t) < 0, \\ F''(t) &= -\frac{q-2}{2}G^{-\frac{q+2}{2}}(t)\left(G(t)G''(t) - \frac{q}{2}(G'(t))^2\right) \leq 0. \end{aligned}$$

It follows from $F''(t) \leq 0$ that

$$F(\widetilde{T}) - F(0) = \widetilde{T}\int_0^1 F'(\theta\widetilde{T})d\theta \leq F'(0)\widetilde{T}. \tag{3.10}$$

By (3.8) and the definition of $F(t)$, we obtain

$$\begin{aligned} F(0) &= G^{\frac{2-q}{2}}(0) > 0, \\ F(\widetilde{T}) &= G^{\frac{2-q}{2}}(\widetilde{T}) > 0, \\ F'(0) &= -\frac{q-2}{2}G^{-\frac{q}{2}}(0)G'(0) = (2-q)\alpha\beta G^{-\frac{q}{2}}(0) < 0, \end{aligned}$$

which, combining with (3.10), implies

$$\widetilde{T} \leq \frac{F(\widetilde{T})}{F'(0)} - \frac{F(0)}{F'(0)} < -\frac{F(0)}{F'(0)} = \frac{G(0)}{(q-2)\alpha\beta}.$$

Then it follows from (3.8) that

$$\widetilde{T} \leq \frac{T\|u_0\|_{H_0^1}^2 + \alpha\beta^2}{(q-2)\alpha\beta} = \frac{\|u_0\|_{H_0^1}^2}{(q-2)\alpha\beta}T + \frac{\beta}{q-2}, \quad \forall \widetilde{T} \in [0, T).$$

Hence, letting $\widetilde{T} \to T$, we have

$$T \leq \frac{\|u_0\|_{H_0^1}^2}{(q-2)\alpha\beta}T + \frac{\beta}{q-2}. \tag{3.11}$$

Let β be large enough such that

$$\beta \in \left(\frac{\|u_0\|_{H_0^1}^2}{(q-2)\alpha}, +\infty\right), \tag{3.12}$$

then by (3.11), we can get

$$T \leq \frac{\alpha\beta^2}{(q-2)\alpha\beta - \|u_0\|_{H_0^1}^2}.$$

In view of (3.9) and (3.12), we define

$$\begin{aligned}\Lambda &:= \left\{(\alpha,\beta) : \alpha \in \left(0, \frac{\varrho}{2q}\right], \beta \in \left(\frac{\|u_0\|_{H_0^1}^2}{(q-2)\alpha}, \infty\right)\right\} \\ &= \left\{(\alpha,\beta) : \alpha \in \left(\frac{\|u_0\|_{H_0^1}^2}{(q-2)\beta}, \frac{\varrho}{2q}\right], \beta \in \left(\frac{2q\|u_0\|_{H_0^1}^2}{(q-2)\varrho}, \infty\right)\right\},\end{aligned}$$

then

$$T \leq \inf_{(\alpha,\beta)\in\Lambda} \frac{\alpha\beta^2}{(q-2)\alpha\beta - \|u_0\|_{H_0^1}^2}.$$

Let $\zeta = \alpha\beta$ and

$$f(\beta,\zeta) := \frac{\zeta\beta}{(q-2)\zeta - \|u_0\|_{H_0^1}^2}.$$

Since $f(\beta,\zeta)$ is decreasing with ζ and we obtain

$$\begin{aligned} T &\leq \inf_{\beta \in \left(\frac{2q\|u_0\|_{H_0^1}^2}{(q-2)\varrho}, \infty\right)} f\left(\beta, \frac{\varrho\beta}{2q}\right) \\ &= \inf_{\beta \in \left(\frac{2q\|u_0\|_{H_0^1}^2}{(q-2)\varrho}, \infty\right)} \frac{\varrho\beta^2}{(q-2)\varrho\beta - 2q\|u_0\|_{H_0^1}^2} \\ &= \left.\frac{\varrho\beta^2}{(q-2)\varrho\beta - 2q\|u_0\|_{H_0^1}^2}\right|_{\beta = \frac{4q\|u_0\|_{H_0^1}^2}{(q-2)\varrho}} \\ &= \frac{8q\|u_0\|_{H_0^1}^2}{(q-2)^2\varrho}. \end{aligned}$$

Hence, by the definition of ϱ, we get (1.11). □

Proof of Theorem 1.2. Let $u_0 \in H_0^1(\Omega) \setminus \{0\}$ satisfy (1.12), i.e., $u_0 \in N^*$ and $u = u(t)$ be the corresponding solution, whose maximal existence time is T.

By (2.4) and (3.4), we obtain

$$\frac{d}{dt}J(u(t)) = -\|u_t\|_{H_0^1}^2 \tag{3.13}$$

and

$$\frac{d}{dt}\|u\|_{H_0^1}^2 = -2I(u(t)). \tag{3.14}$$

By (1.5), (1.8) and (1.12), we have

$$\begin{aligned} I(u_0) &= \frac{2-q}{2}\|\nabla u_0\|_2^2 - \frac{1}{q}\|u_0\|_q^q + qJ(u_0) \\ &\le \frac{2-q}{2}\|\nabla u_0\|_2^2 + \frac{(q-2)\lambda_1}{2}\|u_0\|_{H_0^1}^2 \\ &\le \frac{(2-q)\lambda_1}{2}\|u_0\|_{H_0^1}^2 + \frac{(q-2)\lambda_1}{2}\|u_0\|_{H_0^1}^2 \\ &= 0. \end{aligned}$$

We assert that $I(u_0) < 0$. In fact, if $I(u_0) = 0$, since $u_0 \neq 0$, then $u_0 \in N$; hence, we get $u_0 \in N \cap N^*$. However, by Lemma 2.2, this is impossible.

Actually, we may claim that

$$I(u(t)) < 0, \ \forall t \in [0, T). \tag{3.15}$$

Otherwise, there is a $t_0 \in (0, T)$ such that

$$I(u(t_0)) = 0, \tag{3.16}$$

and

$$I(u(t)) < 0, \ \forall t \in [0, t_0). \tag{3.17}$$

Then it follows from (3.14) that

$$\frac{d}{dt}\|u\|_{H_0^1}^2 = -2I(u(t)) > 0, \ \forall t \in (0, t_0). \tag{3.18}$$

Thus, by (1.12) and (3.18), we obtain

$$J(u_0) = \frac{(q-2)\lambda_1}{2q}\|u_0\|_{H_0^1}^2 < \frac{(q-2)\lambda_1}{2q}\|u(t_0)\|_{H_0^1}^2. \tag{3.19}$$

On the other hand, it follows from (1.5), (1.8) and (3.13) that

$$\frac{(q-2)\lambda_1}{2q}\|u(t_0)\|_{H_0^1}^2 \le \frac{q-2}{2q}\|\nabla u(t_0)\|_2^2 \le J(u(t_0)) \le J(u_0),$$

which contradicts (3.19). So (3.15) holds.

Then by (3.13) and (3.14), we get

$$\begin{aligned} \frac{d}{dt}J^*(u(t)) &= \frac{d}{dt}\left[J(u) - \frac{(q-2)\lambda_1}{2q}\|u\|_{H_0^1}^2\right] \\ &= -\|u_t\|_{H_0^1}^2 + \frac{(q-2)\lambda_1}{q}I(u(t)) \\ &\le \frac{(q-2)\lambda_1}{q}I(u(t)) \\ &< 0. \end{aligned}$$

Since $J^*(u_0) = 0$, the above inequality implies that there must exist a $t_* > 0$ such that $J^*(u(t_*)) < 0$, i.e.,

$$J(u(t_*)) < \frac{(q-2)\lambda_1}{2q} \|u(t_*)\|^2_{H_0^1}.$$

If we set t_* as the initial time, then by Theorem 1.1, we know that u blows up in finite time. □

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