Bull. Korean Math. Soc. **56** (2019), No. 5, pp. 1273–1283
https://doi.org/10.4134/BKMS.b181065
pISSN: 1015-8634 / eISSN: 2234-3016

ON WIELANDT-MIRSKY'S CONJECTURE FOR MATRIX POLYNOMIALS

Công-Trình Lê

Abstract. In matrix analysis, the *Wielandt-Mirsky conjecture* states that

$$\mathrm{dist}(\sigma(A),\sigma(B)) \leq \|A-B\|$$

for any normal matrices $A, B \in \mathbb{C}^{n\times n}$ and any operator norm $\|\cdot\|$ on $C^{n\times n}$. Here $\mathrm{dist}(\sigma(A),\sigma(B))$ denotes the optimal matching distance between the spectra of the matrices A and B. It was proved by A. J. Holbrook (1992) that this conjecture is *false* in general. However it is true for the Frobenius distance and the Frobenius norm (the *Hoffman-Wielandt inequality*). The main aim of this paper is to study the Hoffman-Wielandt inequality and some weaker versions of the Wielandt-Mirsky conjecture for matrix polynomials.

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n\times n$ matrices whose entries in $\mathbb{C}$. Let $A, B \in \mathbb{C}^{n\times n}$ be complex matrices whose spectra are $\sigma(A) = \{\alpha_1,\ldots,\alpha_n\}$ and $\sigma(B) = \{\beta_1,\ldots,\beta_n\}$, respectively. The *optimal matching distance* between $\sigma(A)$ and $\sigma(B)$ is defined by

$$\mathrm{dist}(\sigma(A),\sigma(B)) := \min_{\theta} \max_{j=1,\ldots,n} |\alpha_j - \beta_{\theta(j)}|,$$

where the minimum is taken over all permutations θ on the set $\{1,\ldots,n\}$.

One of the interesting conjectures in matrix analysis is the *Wielandt-Mirsky conjecture* [2] which states that for any normal matrices $A, B \in \mathbb{C}^{n\times n}$,

$$\mathrm{dist}(\sigma(A),\sigma(B)) \leq \|A-B\|, \tag{1}$$

where $\|\cdot\|$ denotes the operator bound norm.

This conjecture has been proved to be true in the following special cases (cf. [10]):

(1) A and B are Hermitian (Weyl, 1912);
(2) A, B and $A-B$ are normal (Bhatia, 1982);

Received November 6, 2018; Revised January 27, 2019; Accepted February 27, 2019.
2010 *Mathematics Subject Classification.* Primary 15A18, 15A42, 15A60, 65F15.
Key words and phrases. Wielandt-Mirsky's conjecture, Hoffman-Wielandt's inequality, Wielandt's inequality, matrix polynomial, spectral variation.

(3) A is Hermitian and B is skew-Hermitian (Sunder, 1982);
(4) A and B are constant multiples of unitaries (Bhatia and Holbrook, 1985).

It has been proven by Holbrook (1992, [10]) that this conjecture is *false* in general. However, if we replace the optimal matching distance $\mathrm{dist}(\sigma(A),\sigma(B))$ by the *Frobenius distance* between the spectra of the matrices A and B,

$$\mathrm{dist}_F(\sigma(A),\sigma(B)) := \min_\theta \big[\sum_{j=1}^n |\alpha_j - \beta_{\theta(j)}|^2\big]^{\frac{1}{2}},$$

then we have the following *Hoffman-Wielandt inequality* [9]

$$\mathrm{dist}_F(\sigma(A),\sigma(B)) \le |A-B|_F \tag{2}$$

for any normal matrices A and B. Here $|A-B|_F$ denotes the *Frobenius norm*[1] of the matrix $A-B$.

It is clear that

$$\mathrm{dist}(\sigma(A),\sigma(B)) \le \mathrm{dist}_F(\sigma(A),\sigma(B)) \le \sqrt{n}\cdot \mathrm{dist}(\sigma(A),\sigma(B)).$$

Therefore it follows from the Hoffman-Wielandt inequality that *the Wielandt-Mirsky conjecture is true for the Frobenius norm.*

A weaker version of the Wielandt-Mirsky conjecture was proved by R. Bhatia, C. Davis and A. Mcintosh (1983, [3]) that there exists a universal constant c such that for any normal matrices $A, B \in \mathbb{C}^{n\times n}$ we have

$$\mathrm{dist}(\sigma(A),\sigma(B)) \le c\|A-B\|. \tag{3}$$

If we don't require the universality of the constant c in the above inequality, for any normal matrix $A\in\mathbb{C}^{n\times n}$ and for any $B\in\mathbb{C}^{n\times n}$, we have (cf. [2, p. 4])

$$\mathrm{dist}(\sigma(A),\sigma(B)) \le (2n-1)\|A-B\|. \tag{4}$$

If A is Hermitian and B is arbitrary, we have the following inequality due to W. Kahan ([11, p. 166]):

$$\mathrm{dist}(\sigma(A),\sigma(B)) \le (\gamma_n+2)\|A-B\|, \tag{5}$$

where γ_n is a constant depending on the size n of the matrices.

For a *matrix polynomial* we mean the matrix-valued function of a complex variable of the form

$$P(z) = A_m z^m + \cdots + A_1 z + A_0, \tag{6}$$

where $A_i \in \mathbb{C}^{n\times n}$ for all $i = 0,\ldots,m$. If $A_m \neq 0$, $P(z)$ is called a *matrix polynomial of degree m*. When $A_m = I$, the identity matrix in $\mathbb{C}^{n\times n}$, the matrix polynomial $P(z)$ is called a *monic*.

[1]For a matrix $A=(a_{ij})\in\mathbb{C}^{n\times n}$, the *Frobenius norm* of A is defined by

$$|A|_F := \sqrt{trace(AA^*)} = \big(\sum_{i,j=1}^n |a_{ij}|^2\big)^{\frac{1}{2}}.$$

It is easy to see that $|A|_F^2 = |AA^*|_F$.

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix polynomial $P(z)$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. Then the vector x is called, as usual, an *eigenvector* associated to the eigenvalue λ. Note that each eigenvalue of $P(z)$ is a root of the *characteristic polynomial* $\det(P(z))$.

For an $(m+1)$-tuple $\mathbf{A} = (A_0, \dots, A_m)$ of matrices $A_i \in \mathbb{C}^{n\times n}$, the matrix polynomial

$$P_{\mathbf{A}}(z) := A_m z^m + \cdots + A_1 z + A_0$$

is called the *matrix polynomial associated to* $\mathbf{A}$.

The *spectrum* of the matrix polynomial $P_{\mathbf{A}}(z)$ is defined by

$$\sigma(\mathbf{A}) := \sigma\big(P_{\mathbf{A}}(z)\big) = \{\lambda \in \mathbb{C} \mid \det(P_{\mathbf{A}}(\lambda)) = 0\},$$

which is the set of all its eigenvalues. We should observe that for a matrix $A \in \mathbb{C}^{n\times n}$, its usual spectrum $\sigma(A)$ is actually the spectrum of the monic matrix polynomial $Iz - A$. Interested readers may refer to the book of I. Gohberg, P. Lancaster and L. Rodman [5] for the theory of matrix polynomials and applications.

The main goal of this paper is to give some versions of the Wielandt-Mirsky conjecture for matrix polynomials.

The paper is organized as follows. In Section 2 we establish some Wielandt's inequalities for matrix polynomials. In Section 3 we give some weaker versions of the Wielandt-Mirsky conjecture for monic matrix polynomials and for matrix polynomials whose leading coefficients are non-singular.

Notation and conventions. Throughout this paper, by a positive integer p we mean $p \geq 1$ or $p = \infty$.

For a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, a positive integer p, and a vector p-norm $|\cdot|_p$ on $\mathbb{C}^n$, the *matrix p-norm* of A is defined by

$$|A|_p := \begin{cases} \Big(\displaystyle\sum_{i,j=1}^{n} |a_{ij}|^p\Big)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \displaystyle\max_{i,j=1,\dots,n} |a_{ij}| & (p = \infty). \end{cases}$$

In particular, $|A|_2 = |A|_F$, the Frobenius norm.

The *operator p-norm* of A is defined by

$$\|A\|_p := \max\{|Ax|_p : |x|_p = 1\}.$$

Note that

$$\|A\|_1 := \max_{j=1,\dots,n} \sum_{i=1}^{n} a_{ij}, \quad \|A\|_\infty := \max_{i=1,\dots,n} \sum_{j=1}^{n} a_{ij}.$$

There are many relations between operator and matrix p-norms. Interested readers may refer to the paper of A. Tonge [13] and the references therein for more details.

2. Some Wielandt's inequalities for matrix polynomials

In the first part of this section we give some versions of the Hoffman-Wielandt inequality (2) for monic matrix polynomials.

For a monic matrix polynomial $P_{\mathbf{A}}(z) = I \cdot z^m + A_{m-1}z^{m-1} + \cdots + A_1 z + A_0$ with $A_i \in \mathbb{C}^{n\times n}$, the $(mn \times mn)$-matrix

$$C_{\mathbf{A}} := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{m-1} \end{bmatrix}$$

is called the *companion matrix* of the matrix polynomial $P_{\mathbf{A}}(z)$ or of the tuple $(A_0, \ldots, A_{m-1}, I)$.

Note that the spectrum $\sigma(\mathbf{A})$ of $P_{\mathbf{A}}(z)$ coincides to the spectrum $\sigma(C_{\mathbf{A}})$ of $C_{\mathbf{A}}$ (cf. [5]).

For two $(m+1)$-tuples $\mathbf{A} = (A_0, \ldots, A_{m-1}, I)$ and $\bar{\mathbf{A}} = (\bar{A}_0, \ldots, \bar{A}_{m-1}, I)$, the relation between the operator norms of their difference and those of their companion matrices is given in the following key lemma.

Lemma 2.1. *Let* $\mathbf{A} = (A_0, \ldots, A_{m-1}, I)$ *and* $\bar{\mathbf{A}} = (\bar{A}_0, \ldots, \bar{A}_{m-1}, I)$ *be* $(m+1)$*-tuples. Then for any integer* $p > 0$*, we have*

(1) $|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}|_p = |\mathbf{A} - \bar{\mathbf{A}}|_p = \begin{cases} \left(\sum_{i=0}^{m-1} |A_i - \bar{A}_i|_p^p\right)^{\frac{1}{p}} & (1 \le p < \infty), \\ \max\limits_{i=0,\ldots,m} |A_i|_\infty & (p = \infty). \end{cases}$

(2) $\|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_p = \|\mathbf{A} - \bar{\mathbf{A}}\|_p \le \sum\limits_{i=0}^{m-1} \|A_i - \bar{A}_i\|_p.$

(3) $\|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_1 = \max\limits_{i=0,\ldots,m-1} \|A_i - \bar{A}_i\|_1.$

Proof. We have the following expression of the difference of companion matrices

$$C_{\mathbf{A}} - C_{\bar{\mathbf{A}}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \bar{A}_0 - A_0 & \bar{A}_1 - A_1 & \bar{A}_2 - A_2 & \cdots & \bar{A}_{m-1} - A_{m-1} \end{bmatrix}.$$

This implies that the matrix (resp. operator) norm of $C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}$ is the same of that of the m-tuple $(\bar{A}_0 - A_0, \ldots, \bar{A}_{m-1} - A_{m-1})$, i.e., we have the first equalities in (1) and (2).

The second equality in (2) follows from the subadditivity of the operator p-norm. On the other hand, for an $(m+1)$-tuple $\mathbf{A} = (A_0, \ldots, A_m)$ of matrices in $\mathbb{C}^{n\times n}$, by a direct computation, we have

$$|\mathbf{A}|_p := \begin{cases} \left(\sum_{i=0}^m |A_i|_p^p\right)^{\frac{1}{p}} & (1 \le p < \infty), \\ \max\limits_{i=0,\dots,m} |A_i|_\infty & (p = \infty), \end{cases} \tag{7}$$

thus we get the second equality of (1). Moreover, we have

$$\|\mathbf{A}\|_1 = \max_{i=0,\dots,m} \|A_i\|_1. \tag{8}$$

Thus, we get (3). □

As a consequence of Lemma 2.1, we obtain the following Hoffman-Wielandt inequality for matrix polynomials.

Proposition 2.2. *Let $P_{\mathbf{A}}(z) = I \cdot z^m + A_{m-1}z^{m-1} + \dots + A_1 z + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_{m-1}z^{m-1} + \dots + \bar{A}_1 z + \bar{A}_0$ be monic matrix polynomials whose corresponding companion matrices $C_{\mathbf{A}}$ and $C_{\bar{\mathbf{A}}}$ are normal. Then we have*

$$dist_F(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \le |\mathbf{A} - \bar{\mathbf{A}}|_F.$$

Proof. Applying the Hoffman-Wielandt inequality (2) for two normal matrices $C_{\mathbf{A}}$ and $C_{\bar{\mathbf{A}}}$ we get

$$\mathrm{dist}_F(\sigma(C_{\mathbf{A}}), \sigma(C_{\bar{\mathbf{A}}})) \le |C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}|_F.$$

Then the theorem follows from Lemma 2.1 (applying for $p = 2$) with the observation that $\sigma(\mathbf{A}) = \sigma(C_{\mathbf{A}})$ and $\sigma(\bar{\mathbf{A}}) = \sigma(C_{\bar{\mathbf{A}}})$. □

We should observe that

$$C_{\mathbf{A}} \text{ is normal if and only if } A_0 \text{ is unitary and } A_1 = \dots = A_{m-1} = 0. \tag{9}$$

Therefore, Theorem 2.2 yields the following consequence.

Corollary 2.3. *Let $P_{\mathbf{A}}(z) = I \cdot z^m + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_0$ be monic matrix polynomials with A_0 and $\bar{A}_0$ unitary. Then we have*

$$dist_F(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \le |A_0 - \bar{A}_0|_F.$$

One more interesting inequality was established by Wielandt for scalar matrices which is stated as follows.

Theorem 2.4 (cf. [8, Theorem 1.45])**.** *Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix such that for some numbers $a, b > 0$ the inequalities $bI \le A \le aI$ hold. Then for any orthogonal unit vectors $x, y \in \mathbb{C}^n$, the inequality*

$$|x^* A y|^2 \le \left(\frac{a-b}{a+b}\right)^2 (x^* A x) \cdot (y^* A y) \tag{10}$$

holds.

In the following we establish a version of this inequality for matrix polynomials.

Theorem 2.5. *Let $P_A(\lambda) = \sum_{i=0}^{d} A_i\lambda^i$ be a matrix polynomial whose coefficient matrices satisfy*

$$bI \le A_i \le aI$$

for some numbers $a, b > 0$ and for all $i = 0, \dots, d$. Then for any orthogonal unit vectors $x, y \in \mathbb{C}^n$,

$$|x^* P_A(\lambda)y|^2 \le \big(\frac{a-b}{a+b}\big)^2 (x^* P_A(|\lambda|)x) \cdot (y^* P_A(|\lambda|)y).$$

Proof. It follows from (10) that for each $i = 0, \dots, d$,

$$|x^* A_i y| \le \big(\frac{a-b}{a+b}\big)(x^* A_i x)^{\frac{1}{2}} \cdot (y^* A_i y)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}
|x^* P_A(\lambda)y|^2 &= \big(\big|\sum_{i=0}^{d}(x^* A_i y)\lambda^i\big|\big)^2 \le \big(\sum_{i=0}^{d}|x^* A_i y| \cdot |\lambda|^i\big)^2 \\
&\le \big(\frac{a-b}{a+b}\big)^2 \big(\sum_{i=0}^{d}(x^* A_i x)^{\frac{1}{2}}|\lambda|^{\frac{i}{2}} \cdot (y^* A_i y)^{\frac{1}{2}} \cdot |\lambda|^{\frac{i}{2}}\big)^2 \\
&\le \big(\frac{a-b}{a+b}\big)^2 \big(\sum_{i=0}^{d}(x^* A_i x)|\lambda|^i\big)\big(\sum_{i=0}^{d}(y^* A_i y)|\lambda|^i\big) \\
&= \big(\frac{a-b}{a+b}\big)^2 \big(x^*(\sum_{i=0}^{d} A_i|\lambda|^i)x\big)\big(y^*(\sum_{i=0}^{d} A_i|\lambda|^i)y\big) \\
&= \big(\frac{a-b}{a+b}\big)^2 \big(x^* P_A(|\lambda|)x\big) \cdot \big(y^* P_A(|\lambda|)y\big),
\end{aligned}$$

where the last inequality follows from the classical Cauchy-Schwarz inequality. The proof is complete. □

3. Some weaker versions of the Wielandt-Mirsky conjecture for matrix polynomials

In this section we give some estimations for the optimal matching distance between the spectra of matrix polynomials.

Proposition 3.1. *There exists a constant $c > 0$ such that for every positive integer p and monic matrix polynomials $P_{\mathbf{A}}(z) = I \cdot z^m + A_{m-1}z^{m-1} + \dots + A_1 z + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_{m-1}z^{m-1} + \dots + \bar{A}_1 z + \bar{A}_0$ whose corresponding companion matrices $C_{\mathbf{A}}$ and $C_{\bar{\mathbf{A}}}$ are normal we have*

$$dist(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \le c\|\mathbf{A} - \bar{\mathbf{A}}\|_p \le c\sum_{i=0}^{m-1}\|A_i - \bar{A}_i\|_p.$$

Proof. It follows from the result of R. Bhatia, C. Davis and A. Mcintosh (see the inequality (3)) that there exists a constant $c > 0$ such that for every monic matrix polynomials $P_{\mathbf{A}}(z)$ and $P_{\bar{\mathbf{A}}}(z)$ whose corresponding companion matrices $C_{\mathbf{A}}$ and $C_{\bar{\mathbf{A}}}$ are normal, we have

$$\operatorname{dist}(\sigma(C_{\mathbf{A}}), \sigma(C_{\bar{\mathbf{A}}})) \leq c\|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_p.$$

Then the theorem follows from Lemma 2.1(2) with the observation that $\sigma(\mathbf{A}) = \sigma(C_{\mathbf{A}})$ and $\sigma(\bar{\mathbf{A}}) = \sigma(C_{\bar{\mathbf{A}}})$. □

Using again the observation (9) we obtain the following consequence.

Corollary 3.2. *There exists a constant $c > 0$ such that for every positive integer p and monic matrix polynomials $P_{\mathbf{A}}(z) = I \cdot z^m + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_0$ with A_0 and $\bar{A}_0$ unitary, we have*

$$dist(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \leq c\|A_0 - \bar{A}_0\|_p.$$

Similarly, applying the inequality (4) and Lemma 2.1(2) with the observation (9) we obtain the following results.

Corollary 3.3. *Let $P_{\mathbf{A}}(z) = I \cdot z^m + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_0$ be matrix polynomials with A_0 and $\bar{A}_0$ unitary. Then for every positive integer p we have*

$$dist(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \leq (2mn - 1)\|A_0 - \bar{A}_0\|_p.$$

A similar version of (5) for monic matrix polynomials is given as follows, whose proof is similar to that of Proposition 2.2.

Proposition 3.4. *Let $P_{\mathbf{A}}(z) = I \cdot z^m + A_{m-1}z^{m-1} + \cdots + A_1 z + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_{m-1}z^{m-1} + \cdots + \bar{A}_1 z + \bar{A}_0$ be monic matrix polynomials. Assume that $C_{\mathbf{A}}$ is Hermitian. Then for every positive integer p we have*

$$dist(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \leq (\gamma_{m,n} + 2)\|\mathbf{A} - \bar{\mathbf{A}}\|_p \leq (\gamma_{m,n} + 2)\sum_{i=0}^{m-1} \|A_i - \bar{A}_i\|_p,$$

where $\gamma_{m,n}$ is a constant depending on m and n.

Remark 3.5. The constant $\gamma_{m,n}$ have the following properties (cf. [2, p. 3]):

(1) $\frac{2}{\pi}\ln(mn) - 0(1) \leq \gamma_{m,n} \leq \log_2(mn) + 0.038$.
(2) A. Pokrzywa (1981, [12]) proved that

$$\gamma_{m,n} = \frac{2}{mn}\sum_{j=1}^{[mn/2]} \cot\frac{2j-1}{2mn}\pi.$$

One of the condition for the companion matrix $C_{\mathbf{A}}$ to be Hermitian is given as follows.

Corollary 3.6. *Let $P_{\mathbf{A}}(z) = I \cdot z^m + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I \cdot z^m + \bar{A}_0$ be matrix polynomials with A_0 unitary. Assume that $P_{\mathbf{A}}(z)$ has only real eigenvalues. Then for every positive integer p we have*

$$dist(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \leq (\gamma_{m,n} + 2)\|A_0 - \bar{A}_0\|_p.$$

Proof. It is well-known that a normal matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if it has only real eigenvalues. Therefore, the normal matrix $C_{\mathbf{A}}$ is Hermitian if and only if it, whence the matrix polynomial $P_{\mathbf{A}}(z)$, has only real eigenvalues. Note also that in this case, $C_{\mathbf{A}}$ is normal if and only if A_0 is unitary. Then the result follows from Proposition 3.4. □

Remark 3.7. There are some characterization for matrix polynomials to have only real eigenvalues. These kinds of matrix polynomials are sometimes called *weakly hyperbolic*. The readers can refer to the work of M. Al-Ammari and F. Tisseur (2012, [1, Theorem 3.4]) and the references therein for more details characterization.

In the following we will establish an estimation for the optimal matching distance between spectra of two arbitrary monic matrix polynomials. For the proof, we need the following estimation given by Bhatia and Friedland (1981), Elsner (1982, 1985).

Lemma 3.8 ([2, Theorem 20.4])**.** *Let A and B be any $k \times k$-matrices. Then the optimal matching distance between their eigenvalues are bounded as*

$$dist(\sigma(A), \sigma(B)) \leq c(k) \cdot k^{\frac{1}{k}} \cdot (2M)^{1-\frac{1}{k}} \cdot \|A - B\|^{\frac{1}{k}},$$

where $M = \max\{\|A\|, \|B\|\}$, $\|\cdot\|$ any operator norm, and $c(k) = k$ or $k-1$ according to whether k is odd or even.

Theorem 3.9. *Let N be any positive number and p any positive integer. Let $P_{\mathbf{A}}(z) = I\cdot z^m + A_{m-1}z^{m-1} + \cdots + A_1 z + A_0$ and $P_{\bar{\mathbf{A}}}(z) = I\cdot z^m + \bar{A}_{m-1}z^{m-1} + \cdots + \bar{A}_1 z + \bar{A}_0$ be monic matrix polynomials such that $|\mathbf{A}|_p \leq N$ and $|\bar{\mathbf{A}}|_p \leq N$. Then there exists a constant c such that*

$$dist(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \leq c\|\mathbf{A} - \bar{\mathbf{A}}\|_p^{\frac{1}{mn}}.$$

Proof. Applying Lemma 3.8 for the companion matrices $C_{\mathbf{A}}$ and $C_{\bar{\mathbf{A}}}$ with the operator norm $\|\cdot\|_p$ we obtain

$$\mathrm{dist}(\sigma(C_{\mathbf{A}}), \sigma(C_{\bar{\mathbf{A}}})) \leq c(mn)\cdot(mn)^{\frac{1}{mn}} \cdot (2M)^{1-\frac{1}{mn}} \cdot \|C_{\mathbf{A}} - C_{\bar{\mathbf{A}}}\|_p^{\frac{1}{mn}},$$

where $M = \max\{\|C_{\mathbf{A}}\|_p, \|C_{\bar{\mathbf{A}}}\|_p\}$. Then it follows from Lemma 2.1(2) and the equalities $\sigma(\mathbf{A}) = \sigma(C_{\mathbf{A}})$ and $\sigma(\bar{\mathbf{A}}) = \sigma(C_{\bar{\mathbf{A}}})$ that

$$\mathrm{dist}(\sigma(\mathbf{A}), \sigma(\bar{\mathbf{A}})) \leq c(mn)\cdot(mn)^{\frac{1}{mn}} \cdot (2M)^{1-\frac{1}{mn}} \cdot \|\mathbf{A} - \bar{\mathbf{A}}\|_p^{\frac{1}{mn}}.$$

Now we need to estimate $\|C_{\mathbf{A}}\|_p$ and $\|C_{\bar{\mathbf{A}}}\|_p$. By a comparison of the operator p-norm $\|C_{\mathbf{A}}\|_p$ and the matrix p-norm $|C_{\mathbf{A}}|_p$ given by M. Gohberg [6] (see also in [13]), we have

$$\|C_{\mathbf{A}}\|_p \leq (mn)^{\max\{1/p'-1/p,0\}}|C_{\mathbf{A}}|_p. \tag{11}$$

It is easy to see that for $1 \leq p < \infty$,

$$|C_{\mathbf{A}}|_p^p = \sum_{i=0}^{m-1}|A_i|_p^p + (m-1)n = |\mathbf{A}|_p^p + (m-1)n \leq N^p + (m-1)n, \tag{12}$$

and for $p = \infty$,

$$|C_{\mathbf{A}}|_\infty = \max\{|\mathbf{A}|_\infty, 1\} \le \max\{N, 1\}. \tag{13}$$

It follows from the inequalities (11), (12) and (13) that

$$\|C_{\mathbf{A}}\|_p \le M' := \begin{cases} (mn)^{\max\{1/p'-1/p,0\}}\big(N^p + (m-1)n\big)^{\frac{1}{p}} & (1 \le p < \infty), \\ mn\max\{N, 1\} & (p = \infty). \end{cases}$$

Similarly,

$$\|C_{\bar{\mathbf{A}}}\|_p \le M'.$$

Then the number

$$c := c(mn) \cdot (mn)^{\frac{1}{mn}} \cdot (2M')^{1-\frac{1}{mn}}$$

satisfies the requirement. □

For matrix polynomials which are not necessarily monic, we have the following estimation, only for operator 1-norm.

Theorem 3.10. *Let $\bar{\mathbf{A}} = (\bar{A}_0, \ldots, \bar{A}_m)$ be a fixed $(m+1)$-tuple such that $\bar{A}_m$ non-singular. Then there exists a constant $c > 0$ such that for every pair of matrix polynomials $P_{\mathbf{A}}(z) = A_m \cdot z^m + A_{m-1}z^{m-1} + \cdots + A_1 z + A_0$ and $P_{\mathbf{A}'}(z) = A'_m \cdot z^m + A'_{m-1}z^{m-1} + \cdots + A'_1 z + A'_0$ with $\|\mathbf{A} - \bar{\mathbf{A}}\|_1 < \frac{1}{\|\bar{A}_m^{-1}\|_1}$ and $\|\mathbf{A}' - \bar{\mathbf{A}}\|_1 < \frac{1}{\|\bar{A}_m^{-1}\|_1}$ we have*

$$\mathit{dist}(\sigma(\mathbf{A}), \sigma(\mathbf{A}')) < c\|\mathbf{A} - \mathbf{A}'\|_1^{\frac{1}{mn}}.$$

Proof. It follows from the sub-multiplicative property of operator 1-norm and the formula (8) that

$$\|\bar{A}_m^{-1}(A_m - \bar{A}_m)\|_1 \le \|\bar{A}_m^{-1}\|_1 \cdot \|A_m - \bar{A}_m\|_1 \le \|\bar{A}_m^{-1}\|_1 \cdot \|\mathbf{A} - \bar{\mathbf{A}}\|_1 < 1.$$

Then A_m is also non-singular (cf. [7, Theorem 2.3.4]), thus each element of $\sigma(\mathbf{A})$ is finite. Similarly, each element of $\sigma(\mathbf{A}')$ is also finite.

Since $\sigma(\mathbf{A}')$ is the set of roots of the polynomial $\det(P_{\mathbf{A}'}(z)) \in \mathbb{C}[z]$ whose degree is mn, it is easy to see that

$$\mathrm{dist}(x, \sigma(\mathbf{A}'))^{mn} \le |\det(P_{\mathbf{A}'}(x))| \quad \text{for all } x \in \mathbb{C}.$$

In particular, for any $\lambda \in \sigma(\mathbf{A})$, we have

$$\mathrm{dist}(\lambda, \sigma(\mathbf{A}'))^{mn} \le |\det(P_{\mathbf{A}'}(\lambda))|. \tag{14}$$

Since $\det(P_{\mathbf{A}}(\lambda)) = 0$, we have

$$|\det(P_{\mathbf{A}'}(\lambda))| = |\det(P_{\mathbf{A}'}(\lambda)) - \det(P_{\mathbf{A}}(\lambda))|. \tag{15}$$

The following inequality is useful for the later estimation (cf. [2, 2007, p. 107]): For any matrices $A, B \in \mathbb{C}^{n\times n}$ and for any integer $p > 0$, we have

$$|\det(A) - \det(B)| \le n\max\{\|A\|_p, \|B\|_p\}^{n-1} \cdot \|A - B\|_p. \tag{16}$$

Applying the inequality (16), we get

$$\begin{aligned} &|\det(P_{\mathbf{A}'}(\lambda)) - \det(P_{\mathbf{A}}(\lambda))| \\ &\le n \max\{\|P_{\mathbf{A}'}(\lambda)\|_1, \|P_{\mathbf{A}}(\lambda)\|_1\}^{n-1} \cdot \|P_{\mathbf{A}}(\lambda) - P_{\mathbf{A}'}(\lambda)\|_1. \end{aligned} \tag{17}$$

Using again the sub-multiplicativity of operator 1-norm and the formula (8), we have

$$\|P_{\mathbf{A}}(\lambda) - P_{\mathbf{A}'}(\lambda)\|_1 = \|\sum_{i=0}^{m}(A_i - A'_i)\lambda^i\|_1 \le \sum_{i=0}^{m}|\lambda|^i \cdot \|\mathbf{A} - \mathbf{A}'\|_1.$$

It follows from a matrix version of Cauchy theorem (cf. [4, Theorem 3.4]) that for $\lambda \in \sigma(\mathbf{A})$, we have

$$|\lambda| < 1 + \|A_m^{-1}\|_1 \cdot \max\{\|A_i\|_1, i = 0, \dots, m-1\} \le 1 + \|A_m^{-1}\|_1 \cdot \|\mathbf{A}\|_1.$$

On the other hand, by the sub-additivity of operator norm, we have

$$\|\mathbf{A}\|_1 \le \|\mathbf{A} - \bar{\mathbf{A}}\|_1 + \|\bar{\mathbf{A}}\|_1 < \frac{1}{\|\bar{A}_m^{-1}\|_1} + \|\bar{\mathbf{A}}\|_1.$$

Hence for each $\lambda \in \sigma(\mathbf{A})$ we have

$$|\lambda| < 2 + \|\bar{A}_m^{-1}\|_1 \cdot \|\bar{\mathbf{A}}\|_1.$$

Then

$$\sum_{i=0}^{m}|\lambda|^i < L := \frac{(2 + \|\bar{A}_m^{-1}\|_1 \cdot \|\bar{\mathbf{A}}\|_1)^{m+1} - 1}{1 + \|\bar{A}_m^{-1}\|_1 \cdot \|\bar{\mathbf{A}}\|_1}.$$

This yields

$$\|P_{\mathbf{A}}(\lambda) - P_{\mathbf{A}'}(\lambda)\|_1 < L \cdot \|\mathbf{A} - \mathbf{A}'\|_1. \tag{18}$$

By a similar estimation, we get

$$\|P_{\mathbf{A}}(\lambda)\|_1 < L \cdot \|\mathbf{A}\|_1 \le L \cdot \big(\frac{1}{\|\bar{A}_m^{-1}\|_1} + \|\bar{\mathbf{A}}\|_1\big), \tag{19}$$

$$\|P_{\mathbf{A}'}(\lambda)\|_1 < L \cdot \|\mathbf{A}'\|_1 \le L \cdot \big(\frac{1}{\|\bar{A}_m^{-1}\|_1} + \|\bar{\mathbf{A}}\|_1\big). \tag{20}$$

It follows from the inequalities (14), (15), (17), (18), (19) and (20) that

$$\operatorname{dist}(\lambda, \sigma(\mathbf{A}')) < c\|\mathbf{A} - \mathbf{A}'\|_1^{\frac{1}{mn}},$$

where

$$c := \Big(n \cdot L^{n-1} \cdot \big(\frac{1}{\|\bar{A}_m^{-1}\|_1} + \|\bar{\mathbf{A}}\|_1\big)^{n-1}\Big)^{\frac{1}{mn}}.$$

Similarly, for every $\lambda' \in \sigma(\mathbf{A}')$, we have also

$$\operatorname{dist}(\lambda, \sigma(\mathbf{A})) < c\|\mathbf{A} - \mathbf{A}'\|_1^{\frac{1}{mn}}.$$

It follows that

$$\operatorname{dist}(\sigma(\mathbf{A}), \sigma(\mathbf{A}')) < c\|\mathbf{A} - \mathbf{A}'\|_1^{\frac{1}{mn}}. \qquad \square$$

Acknowledgements. The author would like to thank Dr. Minh Toan Ho for his suggestion to establish a Wielandt's inequality for matrix polynomials in Section 2. He would also like to express his sincere gratitude to the anonymous referees for their useful comments and suggestions to improve the results in the original version of this paper. The final version of this paper was finished during the visit of the author at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He thanks VIASM for financial support and hospitality.

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Công-Trình Lê
Division of Computational Mathematics and Engineering
Institute for Computational Science
Ton Duc Thang University
Ho Chi Minh City, Vietnam
and
Faculty of Mathematics and Statistics
Ton Duc Thang University
Ho Chi Minh City, Vietnam
Email address: lecongtrinh@tdtu.edu.vn