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REVERSIBLE AND PSEUDO-REVERSIBLE RINGS

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Abstract. This article concerns the structure of idempotents in reversible and pseudo-reversible rings in relation with various sorts of ring extensions. It is known that a ring R is reversible if and only if $ab \in I(R)$ for $a, b \in R$ implies $ab = ba$; and a ring R shall be said to be *pseudo-reversible* if $0 \neq ab \in I(R)$ for $a, b \in R$ implies $ab = ba$, where $I(R)$ is the set of all idempotents in R. Pseudo-reversible is seated between reversible and quasi-reversible. It is proved that the reversibility, pseudo-reversibility, and quasi-reversibility are equivalent in Dorroh extensions and direct products. Dorroh extensions are also used to construct several sorts of rings which are necessary in the process.

Throughout every ring is an associative ring with identity unless otherwise stated. Let R be a ring. Use $I(R)$, $N^*(R)$, $N_*(R)$, $W(R)$, $N(R)$, and $J(R)$ to denote the set of all idempotents, the upper nilradical (i.e., the sum of all nil ideals), the lower nilradical (i.e., the intersection of all minimal prime ideals), the Wedderburn radical (i.e., the sum of all nilpotent ideals), the set of all nilpotent elements, and the Jacobson radical in R, respectively. Note $N^*(R) \subseteq N(R)$. Write $I(R)' = \{e \in I(R) \mid e \neq 0\}$. $Z(R)$ denotes the center of R. The polynomial (power series) ring with an indeterminate x over R is denoted by $R[x]$ ($R[[x]]$). $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively. Let $n \geq 2$. Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$), and $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere, and I_n denotes the identity matrix in $\mathrm{Mat}_n(R)$. Given a set S, $|S|$ means the cardinality of S. We use $\prod$ to denote the direct product of rings.

Following Cohn [2], a ring R (possibly without identity) is called *reversible* if $ab = 0$ for $a, b \in R$ implies $ba = 0$. Anderson and Camillo [1] used the term ZC_2 for the reversibility. A ring (possibly without identity) is usually said to be *reduced* if it has no nonzero nilpotent elements. Many commutative

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rings are not reduced (e.g., $\mathbb{Z}_{n^l}$ for $n, l \geq 2$), and there exist many noncommutative reduced rings (e.g., direct products of noncommutative domains). It is easily checked that the class of reversible rings contains commutative rings and reduced rings. A ring (possibly without identity) is called *Abelian* if every idempotent is central. It is simple to check that reversible rings are Abelian. A ring R is usually called *directly finite* (or *Dedekind finite*) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. Abelian rings are clearly directly finite. Due to Lambek [12], a ring R (possibly without identity) is called *symmetric* if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$. Commutative rings are clearly symmetric, and reduced rings are symmetric by [14, Lemma 1.1]. It is well-known that the converses need not hold. By [12, Proposition 1], a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $r_i \in R$ and n is any positive integer. Symmetric rings are clearly reversible, but the converse need not hold by [1, Example I.5] or Marks [13, Examples 5 and 7]. We will use freely the symmetric ring property of reduced rings.

1. Reversibility and pseudo-reversibility

In this section we study the structure of reversible and pseudo-reversible rings in relation with various ring properties which have roles in ring theory. We will prove that the reversibility, pseudo-reversibility, and quasi-reversibility are equivalent in direct products. We first recall the equivalent conditions to the reversibility.

Lemma 1.1. (1) *For a ring R the following conditions are equivalent*:

(1) *R is reversible*;
(2) *$ab \in I(R)$ for $a, b \in R$ implies $ba \in I(R)$*;
(3) *$ab \in I(R)$ for $a, b \in R$ implies $ab = ba$*;
(4) *$ab \in I(R)$ for $a, b \in R$ implies $bra = braab$ for all $r \in R$*;
(5) *$ab \in I(R)$ for $a, b \in R$ implies $ba = baab$.*

Proof. The equivalences of the conditions (1), (2), (3), and (4) are obtained from [7, Proposition 1.4 and Corollary 1.5] and [8, Proposition 2.5]. (4) $\Rightarrow$ (5) is obvious. Assume (5) and let $ab = 0$ for $a, b \in R$. Then $ba = baab = 0$, and so R is reversible. $\square$

We consider next the case of nonzero idempotent in Lemma 1.1(3).

Definition 1.2. A ring R (possibly without identity) is called *pseudo-reversible* provided that $I(R)'$ is empty, or else $ab \in I(R)'$ for $a, b \in R$ implies $ab = ba$.

Reversible rings are pseudo-reversible by Lemma 1.1. We will use this fact freely. The converse need not hold as we see later. Following Jung et al. [8], a ring R (possibly without identity) is called *quasi-reversible* if $ab \in I(R)'$ for $a, b \in R$ implies $ba \in I(R)$. Following Grover et al. [5], a ring is said to be *connected* if $I(R) = \{0, 1\}$. Both domains and local rings are clearly connected.

Lemma 1.3. (1) *Pseudo-reversible rings are Abelian.*

(2) *Pseudo-reversible rings are quasi-reversible.*

(3) *The class of pseudo-reversible rings is closed under subrings* (*with or without identity*).

(4) *Connected rings are pseudo-reversible.*

(5) *Let R be a ring with or without identity. R is pseudo-reversible if and only if $ab \in I(R)'$ for $a, b \in R$ implies $ba = baab$.*

Proof. (1) Let R be a pseudo-reversible ring. Assume on the contrary that there exist $e \in I(R)$ and $r \in R$ such that $er - ere \neq 0$ or $re - ere \neq 0$. Let $er - ere \neq 0$. Then $e + (er - ere) \in I(R)'$ (otherwise, $e = -er + ere$ implies $e = 0$). Since R is pseudo-reversible, $e(e + (er - ere)) = e + (er - ere) \in I(R)'$ implies $e + (er - ere) = e(e + (er - ere)) = (e + (er - ere))e = e$. This entails $er - ere = 0$, contrary to $er - ere \neq 0$. Next $re - ere \neq 0$ also induces a contradiction similarly. Thus R is Abelian.

(2) and (3) are obvious.

(4) Let R be a ring with $I(R) = \{0, 1\}$. Then R is Abelian (hence directly finite). Let $ab \in I(R)'$ for $a, b \in R$. Then $ab = 1$ since $I(R)' = \{1\}$. But R is directly finite, so $ba = 1$. Thus R is pseudo-reversible.

(5) Let R be a pseudo-reversible ring and suppose that $ab \in I(R)'$ for $a, b \in R$. Then $ab = ba$, and so $ba = ab = (ab)ab = (ba)ab$.

Let R satisfy the necessity. We first claim that R is Abelian. Assume on the contrary that there exist $e \in I(R)$ and $r \in R$ such that $er - ere \neq 0$ or $re - ere \neq 0$. Let $er - ere \neq 0$. Then $e + (er - ere) \in I(R)'$ (otherwise, $e = -er + ere$ implies $e = 0$). Set $a = e$ and $b = e + (er - ere)$. Then $ab = b \in I(R)'$. So we get $ba = baab$ by the necessity. But $ba = e \neq b = baab$, a contradiction. Letting $re - ere \neq 0$, we also get to a contradiction similarly. Thus R is Abelian.

Next let $ab \in I(R)'$ for $a, b \in R$. Then $ba = baab$ by the necessity. Since ab is central, we have $ba = ba(ab) = b(ab)a = (ba)^2$, entailing $ba \in I(R)$. Then ba is also central, and hence $ba = (ba)ab = a(ba)b = (ab)^2 = ab$. Thus R is pseudo-reversible. $\square$

Lemma 1.3(5) can be compared with Lemma 1.1(5).

Given any reversible ring with a kind of minimal prime ideal, we can construct a pseudo-reversible ring that is not reversible, via the factorization by such an ideal. Let R be a reversible ring. Then it is easily checked that $N(R) = N^*(R) = N_*(R) = W(R)$. So every minimal prime ideal P of R is completely prime (i.e., R/P is a domain) by [14, Proposition 1.11].

Given a ring A and an ideal I of A, define a ring

$$F_3(A) = \left\{ \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ 0 & a & \alpha_3 \\ 0 & 0 & a \end{pmatrix} \mid a \in A \text{ and } \alpha_i \in A/I \text{ for all } i \right\},$$

where the ordinary additions and multiplications of matrices are available by using the fact that A/I is an (A,A)-bimodule.

Proposition 1.4. *Let A be a reversible ring and P be a minimal prime ideal of A satisfying $P \cap I(A) = 0$. Then $F_3(A)$ is a pseudo-reversible ring but not reversible.*

Proof. Write $R = F_3(A)$. Since A is reversible, A is Abelian. We use this fact freely. Write $\bar{A} = A/P$ and $\bar{r} = r + P$ for $r \in A$. Note that $\bar{A}$ is an (A,A)-bimodule with the operation $a\bar{r}b = \bar{a}\bar{r}\bar{b}$ for $r, a, b \in A$.

We first show that R is Abelian. Let $E = \left(\begin{smallmatrix} e & \bar{b} & \bar{c} \\ 0 & e & \bar{d} \\ 0 & 0 & e \end{smallmatrix}\right) \in I(R)$. Then $e^2 = e$, $\bar{e}\bar{b} + \bar{b}\bar{e} = \bar{b}$, $\bar{e}\bar{c} + \bar{b}\bar{d} + \bar{c}\bar{e} = \bar{c}$, $\bar{e}\bar{d} + \bar{d}\bar{e} = \bar{d}$. Multiplying $\bar{e}\bar{b} + \bar{b}\bar{e} = \bar{b}$ by $\bar{e}$, we get $2\bar{e}\bar{b} = \bar{e}\bar{b}$ and $\bar{e}\bar{b} = 0$ follows, entailing $\bar{b} = 0$, because $\bar{A}$ is a domain and $P \cap I(A) = 0$. Similarly $\bar{d} = 0$. Consequently $E = \left(\begin{smallmatrix} e & 0 & \bar{c} \\ 0 & e & 0 \\ 0 & 0 & e \end{smallmatrix}\right)$ and $\bar{e}\bar{c} + \bar{c}\bar{e} = \bar{c}$. Similarly $\bar{c} = 0$. Summarizing, $E = eI_3$. This implies that R is Abelian.

Next we claim that R is pseudo-reversible. Let $BC \in I(R)'$ for $B = \left(\begin{smallmatrix} a & \bar{b} & \bar{c} \\ 0 & a & \bar{d} \\ 0 & 0 & a \end{smallmatrix}\right)$, $C = \begin{pmatrix} a_1 & \bar{b}_1 & \bar{c}_1 \\ 0 & a_1 & \bar{d}_1 \\ 0 & 0 & a_1 \end{pmatrix} \in R$. Then $BC = eI_3$ with $e \in I(A)'$ by the preceding argument; hence we have that $aa_1 = e \in I(A)'$, and $\bar{a}\bar{b}_1 + \bar{b}\bar{a}_1 = 0$, $\bar{a}\bar{c}_1 + \bar{b}\bar{d}_1 + \bar{c}\bar{a}_1 = 0$, $\bar{a}\bar{d}_1 + \bar{d}\bar{a}_1 = 0$. Since A is reversible, $aa_1 = a_1a$ by Lemma 1.1. But $P \cap I(A) = 0$ by hypothesis, and this implies $\bar{a}\bar{a}_1 = \bar{1}$ because $\bar{R}$ is a domain by [14, Proposition 1.11]. Multiply $\bar{a}\bar{b}_1 + \bar{b}\bar{a}_1 = 0$ by $\bar{a}_1$ on the left, we get

$$0 = \bar{a}_1\bar{a}\bar{b}_1 + \bar{a}_1\bar{b}\bar{a}_1 = \bar{b}_1\bar{a}\bar{a}_1 + \bar{a}_1\bar{b}\bar{a}_1 = (\bar{b}_1\bar{a} + \bar{a}_1\bar{b})\bar{a}_1$$

and $0 = (\bar{b}_1\bar{a} + \bar{a}_1\bar{b})\bar{a}_1\bar{a} = \bar{b}_1\bar{a} + \bar{a}_1\bar{b}$ follows. Similarly $\bar{a}_1\bar{d} + \bar{d}_1\bar{a} = 0$. Next multiplying $\bar{a}\bar{c}_1 + \bar{b}\bar{d}_1 + \bar{c}\bar{a}_1 = 0$ by $\bar{a}_1$ on the left, we get

$$\begin{aligned} 0 &= \bar{a}_1\bar{a}\bar{c}_1 + (\bar{a}_1\bar{b})\bar{d}_1 + \bar{a}_1\bar{c}\bar{a}_1 = \bar{c}_1\bar{a}\bar{a}_1 + (-\bar{b}_1\bar{a})\bar{d}_1 + \bar{a}_1\bar{c}\bar{a}_1 \\ &= \bar{c}_1\bar{a}\bar{a}_1 + \bar{b}_1(-\bar{a}\bar{d}_1) + \bar{a}_1\bar{c}\bar{a}_1 = \bar{c}_1\bar{a}\bar{a}_1 + \bar{b}_1(\bar{d}\bar{a}_1) + \bar{a}_1\bar{c}\bar{a}_1 \\ &= (\bar{c}_1\bar{a} + \bar{b}_1\bar{d} + \bar{a}_1\bar{c})\bar{a}_1 \end{aligned}$$

and $0 = (\bar{c}_1\bar{a} + \bar{b}_1\bar{d} + \bar{a}_1\bar{c})\bar{a}_1\bar{a} = \bar{c}_1\bar{a} + \bar{b}_1\bar{d} + \bar{a}_1\bar{c}$ follows. Summarizing, $BC = eI_3 = CB$, and therefore R is pseudo-reversible. But R is not reversible by the computation of $(\bar{1}E_{12})(\bar{1}E_{23}) \neq (\bar{1}E_{23})(\bar{1}E_{12})$. $\square$

Every commutative local ring with nonzero nilpotent Jacobson radical satisfies the condition in Proposition 1.4. Let R be such a ring. Then $D_2(R)$ (resp., $D_2(R)[x]$) also satisfies the condition, in fact, $J(D_2(R)) = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a \in J(R), b \in R\}$ (resp., $J(D_2(R))[x]$) is the unique minimal prime ideal of R (resp., $R[x]$). Note that $D_2(R)/J(D_2(R))$ is a division ring.

In the following we see a relation between reversibility and pseudo-reversibility via direct products.

Theorem 1.5. *Let $R = \prod_{i \in I} R_i$ be the direct product of rings R_i for $i \in I$. Suppose $|I| \geq 2$. Then the following conditions are equivalent*:

(1) *R is pseudo-reversible*;
(2) *R is quasi-reversible*;
(3) *R_i is reversible for all $i \in I$*;
(4) *R is reversible.*

Proof. (2) $\Rightarrow$ (3). Let R be quasi-reversible and j be arbitrary in I. Assume $ab = 0$ for $a, b \in R_j$. Let $\alpha = (a_i)$ and $\beta = (b_i)$ be such that $a_j = a$, $b_j = b$, and $a_i = b_i = 1$ for all $i \in I$ with $i \neq j$. Then $\alpha\beta = (c_i) \in I(R)'$ such that $c_j = 0$ and $c_i = 1$ for all $i \neq j$. Since R is quasi-reversible, $\beta\alpha \in I(R)$. Note that $\beta\alpha = (d_i)$ such that $d_j = ba$ and $d_i = 1$ for all $i \neq j$. So $(d_i) \in I(R)$ implies

$$ba = d_j = d_j^2 = (ba)^2 = baba = 0,$$

concluding that R_j is reversible.

(4) $\Rightarrow$ (1) is proved by Lemma 1.1.

(1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious. $\square$

Recall that a ring R is *semiperfect* if $R/J(R)$ is semisimple (i.e., $R/J(R)$ is Artinian) and $J(R)$ is idempotent-lifting. Local rings are clearly semiperfect. Following [11], an idempotent e of a given ring R is called *local* if eRe is a local ring.

Theorem 1.6. (1) *Let R be a semiperfect ring. If R is Abelian, then R is either a local ring or a finite direct product of two or more local rings. Especially, R is pseudo-reversible in the former case.*

(2) *Let R be a semiperfect ring. If R is pseudo-reversible, then R is either a local ring or a finite direct product of two or more reversible local rings. Especially, R is reversible in the latter case.*

Proof. (1) Let R be Abelian. Since R is semiperfect, R contains a finite orthogonal set of local idempotents whose sum is 1 by [11, Corollary 3.7.2]. Say $\{e_1, \ldots, e_n\}$. Then $R = \prod_{i=1}^{n} e_i R$, where every $e_i R e_i$ is a local ring. But since R is Abelian, $e_i R = e_i R e_i$ for all i. Hence R is a direct product of local rings. If $n = 1$, then R is a local ring. In this case, R is connected and hence pseudo-reversible.

(2) Let R be pseudo-reversible. Then R is Abelian by Lemma 1.3(1). By (1), $R = \prod_{i=1}^{n} e_i R$, where every $e_i R e_i$ is a local ring and $e_i R = e_i R e_i$ for all i. If $n = 1$, then R is a local ring. Assume $n \geq 2$. Then R is a direct product of two or more pseudo-reversible rings. This implies, by Theorem 1.5, that every R_i is reversible. Therefore R is reversible. $\square$

Considering Theorem 1.6, one may ask whether semiperfect Abelian (resp., semiperfect pseudo-reversible) rings are pseudo-reversible (resp., reversible). However the answers are negative as follows.

Example 1.7. (1) There exists a semiperfect Abelian ring that is not pseudo-reversible. Let A be a division ring and $R_0 = A \times A$. Next set $R = D_n(R_0)$ for $n \geq 3$. Then $R/J(R) \cong R_0$ and $J(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\}$ is nil; hence R is semiperfect. Moreover R is Abelian by [6, Lemma 2]. But $R \cong D_n(A) \times D_n(A)$, and $D_n(A)$ is not reversible as can be seen by $E_{12}E_{23} = E_{13} \neq 0 = E_{23}E_{12}$. Therefore R is not pseudo-reversible by Theorem 1.5.

(2) There exists a semiperfect pseudo-reversible ring that is not reversible. Consider $R = D_n(A)$ over a division ring for $n \geq 3$. Then R is a semiperfect pseudo-reversible ring. But R is not reversible as above.

Now we will elaborate Lemma 1.3, asserting that the pseudo-reversibility is seated between the reversibility and the quasi-reversibility. To do it we need the following. Given a ring R and $k \geq 2$, write $N_k(R) = \{a \in R \mid a^k = 0\}$. Note $N(R) = \cup_{k=1}^{\infty} N_k(R)$. It is easily checked that a ring R is reduced if and only if $N_2(R) = 0$.

Proposition 1.8. *Let R be a ring. If $D_2(R)$ is reversible, then $N_2(R) \subseteq Z(R)$.*

Proof. Suppose that $D_2(R)$ is reversible. Then R is reversible clearly. Let $a \in N_2(R)$ and $b \in R$. Consider two matrices

$$A = \begin{pmatrix} ba & b \\ 0 & ba \end{pmatrix} \text{ and } B = \begin{pmatrix} a & -1 \\ 0 & a \end{pmatrix}$$

in $D_2(R)$. Then $AB = 0$ because $a^2 = 0$. Since $D_2(R)$ is reversible, we get $BA = \left(\begin{smallmatrix} 0 & ab-ba \\ 0 & 0 \end{smallmatrix}\right) = 0$ and $ab - ba = 0$ follows, noting that $ba^2 = 0$ implies $aba = 0$ by the reversibility of R. Thus $ab = ba$ and $a \in Z(R)$. □

This result is not valid to the case of $D_2(R)$ being pseudo-reversible. In fact, considering $R = D_n(A)$ over a domain A for $n \geq 3$, $D_2(R)$ is pseudo-reversible by Lemma 1.3(4) and The argument in [10, Example 1.7] can be simplified by Proposition 1.8 as we see in the following.

Example 1.9. (1) There exist pseudo-reversible rings but not reversible.

(i) We follow the construction in [10, Example 2.1]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra generated by noncommuting indeterminates a_0, a_1, a_2, b_0, b_1, b_2, c over $\mathbb{Z}_2$; and set $B = \{f \in A \mid \text{the constant term of } f \text{ is zero}\}$. Next let I be the ideal of A generated by a_0b_0, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, $a_1b_2 + a_2b_1$, a_2b_2, a_0rb_0, a_2rb_2, b_0a_0, $b_0a_1 + b_1a_0$, $b_0a_2 + b_1a_1 + b_2a_0$, $b_1a_2 + b_2a_1$, b_2a_2, b_0ra_0, b_2ra_2, $(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$, $(b_0 + b_1 + b_2)r(a_0 + a_1 + a_2)$, and $r_1r_2r_3r_4$, where $r, r_1, r_2, r_3, r_4 \in B$. Set $R = A/I$. Then R is reversible but $R[x]$ is not reversible by [10, Example 2.1]. By [9, Lemma 8], $I(R) = I(R[x])$ because R is Abelian. Moreover $I(R) = \{0, 1\}$ by [8, Example 2.1]. Thus $R[x]$ is pseudo-reversible by Lemma 1.3(4).

(ii) Let $R = D_n(A)$ for $n \geq 3$ over a reduced ring A with $I(A) = \{0, 1\}$. Then $I(R) = \{0, I_n\}$ by [6, Lemma 2]; hence R is pseudo-reversible by Lemma 1.3(4). But R is not reversible by [10, Example 1.5].

(iii) We use the construction and argument in [10, Example 1.7]. Let $\mathbb{H}$ be the Hamilton quaternions over the real number field and $R_0 = D_2(\mathbb{H})$.

Then R_0 is an Abelian ring with $I(R_0) = \{0, 1\}$ by [6, Lemma 2]. Moreover $I(D_2(R_0)) = \{0, 1\}$ also by [6, Lemma 2]. So $D_2(R_0)$ is pseudo-reversible by Lemma 1.3(4). However $D_2(R_0)$ is not reversible by Proposition 1.8 because $N_2(R_0) = \left(\begin{smallmatrix} 0 & \mathbb{H} \\ 0 & 0 \end{smallmatrix}\right)$ is not contained in $Z(R_0)$ as can be seen by $(iE_{12})(jI_2) = kE_{12} \neq (-k)E_{12} = (jI_2)(iE_{12})$.

(2) There exists an Abelian ring that is not pseudo-reversible. Let R_1 be a pseudo-reversible ring that is not reversible (e.g., $R[x]$ in (1)) and R_2 be a reversible ring. Then $R = R_1 \times R_2$ is Abelian, but not pseudo-reversible by Theorem 1.5.

(3) There exist quasi-reversible rings which are not pseudo-reversible.

(i) $\mathrm{Mat}_2(\mathbb{Z}_2)$ is quasi-reversible by [8, Theorem 1.8], but this ring is clearly not Abelian (hence not pseudo-reversible by Lemma 1.3(1)).

(ii) Consider $T_2(R)$ over a reversible ring R with $I(R) = \{0, 1\}$. Then $T_2(R)$ is quasi-reversible by [8, Theorem 1.4]. But $T_2(R)$ is clearly not Abelian, and so $T_2(R)$ is not pseudo-reversible by Lemma 1.3(2).

If R is a reduced ring, then $D_2(R)$ is reversible; but the converse need not hold. In fact, $D_2(\mathbb{Z}_{m^n})$ is commutative but $\mathbb{Z}_{m^n}$ is not reduced, where $m, n \geq 2$. One may conjecture that $D_2(R)$ is pseudo-reversible over a (pseudo-)reversible ring R. However the answer is negative as we see in the following.

Example 1.10. Let R be an Abelian ring and suppose that $AB \in I(D_2(R))'$ for $A = \left(\begin{smallmatrix} a & a_1 \\ 0 & a \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b & b_1 \\ 0 & b \end{smallmatrix}\right)$ in $D_2(R)$. Then $ab \in I(R)'$ and $ab_1 + a_1b = 0$ by [6, Lemma 2]. Here, assuming that $D_2(R)$ is pseudo-reversible, R is also pseudo-reversible by Lemma 1.3(3). So $ab = ba$, and moreover R is Abelian by Lemma 1.3(1). Then every nonzero idempotent of $D_2(R)$ is of the form $\left(\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix}\right)$ with $f \in I(R)'$ by [6, Lemma 2].

Suppose $ba_1 + b_1a \neq 0$. Then $BA = \left(\begin{smallmatrix} ab & ba_1+b_1a \\ 0 & ab \end{smallmatrix}\right) \notin I(D_2(R))$, and hence $D_2(R)$ is not pseudo-reversible. We will show the existence of such example. We apply the construction and argument in [10, Example 1.7]. Let $\mathbb{H}$ be the Hamilton quaternions over the real number field and $R_0 = \mathbb{H} \times \mathbb{H}$. Next set $R = D_2(R_0)$. Then R is reversible by [10, Proposition 1.6] because R_0 is reduced. Now consider $D_2(R)$. Let

$$A = \begin{pmatrix} \begin{pmatrix} (1,0) & (0,i) \\ (0,0) & (1,0) \end{pmatrix} & \begin{pmatrix} (0,j) & (0,0) \\ (0,0) & (0,j) \end{pmatrix} \\ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} & \begin{pmatrix} (1,0) & (0,i) \\ (0,0) & (1,0) \end{pmatrix} \end{pmatrix} \text{ and } B = \begin{pmatrix} \begin{pmatrix} (1,0) & (0,1) \\ (0,0) & (1,0) \end{pmatrix} & \begin{pmatrix} (0,k) & (0,0) \\ (0,0) & (0,k) \end{pmatrix} \\ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} & \begin{pmatrix} (1,0) & (0,1) \\ (0,0) & (1,0) \end{pmatrix} \end{pmatrix}$$

in $D_2(R)$. Then

$$AB = \begin{pmatrix} \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix} & \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} \\ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} & \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix} \end{pmatrix} \in I(D_2(R))'.$$

But

$$BA = \begin{pmatrix} \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix} & \begin{pmatrix} (0,0) & (0,2j) \\ (0,0) & (0,0) \end{pmatrix} \\ \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (0,0) \end{pmatrix} & \begin{pmatrix} (1,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix} \end{pmatrix} \notin I(D_2(R))$$

because $2j \neq 0$. Therefore $D_2(R)$ is not pseudo-reversible.

Example 1.10 also says that $I(R) = \{0, 1\}$ in Lemma 1.3(4) is a kind of neat condition for R to be pseudo-reversible, when R is a noncommutative ring.

Following [4], a ring R is said to be *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Every von Neumann regular ring is clearly semiprimitive.

Proposition 1.11. *Let R be a von Neumann regular ring. Then the following conditions are equivalent*: (1) *R is reduced*; (2) *R is reversible*; (3) *R is pseudo-reversible*; (4) *R is Abelian.*

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are shown by Lemma 1.1 and Lemma 1.3(1), respectively. (4) $\Rightarrow$ (1) is shown by [4, Theorem 3.2]. $\square$

But the quasi-reversibility need not be a condition in Proposition 1.11. In fact, considering $R = \mathrm{Mat}_2(\mathbb{Z}_2)$, R is quasi-reversible by [8, Theorem 1.8] and clearly von Neumann regular; but R is not Abelian.

The pseudo-reversibility does not go up to polynomial rings by the following.

Example 1.12. There exists a reversible ring over which the polynomial ring is not reversible as we see in Example 1.9(1). Let R_1 be the reversible ring R in Example 1.9(1), and R_2 be any reversible ring. Next set $R = R_1 \times R_2$. Then R is clearly reversible (hence pseudo-reversible). Consider $R[x]$ and note $R[x] \cong R_1[x] \times R_2[x]$. Since $R_1[x]$ is not reversible, $R[x]$ is not quasi-reversible by Theorem 1.5.

In fact, let

$$f(x) = (a_0, 1) + (a_1, 0)x + (a_2, 0)x^2 \text{ and } g(x) = (b_0c, 1) + (b_1c, 0)x + (b_2c, 0)x^2$$

in $R[x]$, where $a_0, a_1, a_2, b_0, b_1, b_2, c$ are identified with the images in R_1 for simplicity. Then $f(x)g(x) = (0, 1) \in I(R[x])'$. But

$$\begin{aligned} g(x)f(x) &= ((b_0c, 1) + (b_1c, 0)x + (b_2c, 0)x^2)((a_0, 1) + (a_1, 0)x + (a_2, 0)x^2) \\ &= (0, 1) + (b_0ca_1 + b_1ca_0, 0)x + \cdots \notin I(R[x]) \end{aligned}$$

because $b_0ca_1 + b_1ca_0 \neq 0$, and R being Abelian implies $I(R) = I(R[x])$ by [9, Lemma 8]. Therefore $R[x]$ is not pseudo-reversible.

We will find conditions under which the pseudo-reversibility can go up to polynomial rings.

Proposition 1.13. (1) *If R is a connected ring, then $R[x]$ is connected* (*hence pseudo-reversible*).

(2) *If R is a connected ring, then $R[[x]]$ is connected* (*hence pseudo-reversible*).

Proof. (1) If $I(R) = \{0, 1\}$, then R is Abelian and we also have $I(R[x]) = \{0, 1\}$ by [9, Lemma 8]. Then $R[x]$ is pseudo-reversible by Lemma 1.3(4).

The proof of (2) is almost same as one of (1). □

Next we construct a kind of pseudo-reversible ring from connected rings which have ideals I satisfying $I^2 = 0$.

Proposition 1.14. *Let R be a ring, $n \geq 2$, and I be an ideal of R satisfying $I^2 = 0$. Define*

$$E_n(R) = \{(a_{ij}) \in \mathrm{Mat}_n(R) \mid a_{ij} \in I \text{ for all } i, j \text{ with } i \neq j, \text{ and } a_{11} = \cdots = a_{nn}\}.$$

(1) *Let R be an Abelian ring. Then every idempotent in $E_n(R)$ is of the form eI_n with $e \in I(R)$, and $E_n(R)$ is an Abelian ring.*

(2) *If R is a connected ring, then $E_n(R)$ is a connected ring. Especially $E_n(R)$ is a pseudo-reversible ring.*

Proof. (1) Let R be Abelian. We first observe the form of idempotents in $E_n(R)$. We will proceed by induction on n. In the procedure we use the condition of R being Abelian freely.

Let $A = \left(\begin{smallmatrix} e & a \\ b & e \end{smallmatrix}\right) \in I(E_2(R))'$. Then clearly $e \neq 0$, and $\begin{pmatrix} e^2 & ea+ae \\ be+eb & e^2 \end{pmatrix} = \left(\begin{smallmatrix} e & a \\ b & e \end{smallmatrix}\right)$, entailing $e^2 = e$, $ea + ae = a$, and $be + eb = b$. Multiplying $ea + ae = a$ by e, we obtain $ae = 0$ and $a = 0$ follows. Similarly $b = 0$. Hence $A = eI_2$ with $e \in I(R)'$.

Let $A = \left(\begin{smallmatrix} e & a_{12} & a_{13} \\ a_{21} & e & a_{23} \\ a_{31} & a_{32} & e \end{smallmatrix}\right) \in I(E_3(R))'$. Clearly $e \neq 0$. Write $A_1 = \left(\begin{smallmatrix} e & a_{12} \\ a_{21} & e \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} e & a_{23} \\ a_{32} & e \end{smallmatrix}\right)$. Then $A_1, A_2 \in I(E_2(R))$ because $A^2 = A$. So $A_1 = A_2 = eI_2$ by the result in the case of $n = 2$. Consequently $A = \left(\begin{smallmatrix} e & 0 & a_{13} \\ 0 & e & 0 \\ a_{31} & 0 & e \end{smallmatrix}\right)$, and $ea_{13} + a_{13}e = a_{13}$, $ea_{31} + a_{31}e = a_{31}$. Multiplying this equality by e, we get $ea_{13} = 0$ and $a_{13} = 0$ follows. Similarly $a_{31} = 0$. Thus $A = eI_3$ with $e \in I(R)'$.

Consider the case of $n \geq 4$. Let $A = (a_{ij}) \in I(E_n(R))'$ such that $e = a_{ii}$ for all i. Write $A_1 = (b_{lm}), A_2 = (c_{st}) \in E_{n-1}(R)$ where $b_{lm} = a_{ij}$ with $l = i$, $m = j$ for $1 \leq i, j \leq n-1$, and $c_{st} = a_{ij}$ with $s = i-1$, $t = j-1$ for $2 \leq i, j \leq n$. From $A^2 = A$, we get $A_1^2 = A_1$ and $A_2^2 = A_2$. So, by the induction hypothesis, we get $A_1 = eI_{n-1} = A_2$; hence $A = \left(\begin{smallmatrix} A_1 & B \\ C & e \end{smallmatrix}\right)$ with $B = \begin{pmatrix} a_{1n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathrm{Mat}_{(n-1)\times 1}(R)$, $C = \left(\begin{smallmatrix} a_{n1} & 0 & \cdots & 0 \end{smallmatrix}\right) \in \mathrm{Mat}_{1\times(n-1)}(R)$. It then follows that $ea_{1n} + a_{1n}e = a_{1n}$ and $ea_{n1} + a_{n1}e = a_{n1}$, from $A^2 = A$. Similarly we get $a_{1n} = 0 = a_{n1}$. Thus $A = eI_n$ with $e \in I(R)'$.

This result implies that $E_n(R)$ is also Abelian.

(2) is an immediate consequence of (1). □

The condition in Proposition 1.14 can be found in many rings. For example, every local ring with nonzero nilpotent Jacobson radical (e.g., $\mathbb{Z}_{p^n}$, p is a prime and $n \geq 2$) satisfies the condition.

2. Dorroh extensions

In this section we study the structure of Dorroh extensions in relation with pseudo-reversible and reversible rings. In the procedure we also deal with the case of without identity for our purpose. In fact we will prove that the reversibility, pseudo-reversibility, and quasi-reversibility are equivalent in Dorroh extensions when given rings have identities.

We first obtain the following basic facts which can be compared with Lemma 1.3.

Lemma 2.1. (1) *Let R be a ring without identity. If R is reversible, then R is pseudo-reversible.*

(2) *Let R be a ring without identity. If R is pseudo-reversible, then R is Abelian.*

Proof. (1) Let R be a reversible ring with $I(R)' \neq \emptyset$. We first show that R is Abelian. Assume on the contrary that there exist $e \in I(R)$ and $r \in R$ such that $er - ere \neq 0$ or $re - ere \neq 0$. But $(er - ere)e = 0$ and $e(re - ere) = 0$ implies $er - ere = e(er - ere) = 0$ and $re - ere = (re - ere)e = 0$, contrary to the assumption. Thus R is Abelian.

Suppose $ab \in I(R)'$. Then $ab = abab$ yields $a(b - bab) = 0$. Since R is reversible, $(b - bab)a = 0$ and $ba = (ba)^2$ follows. Since R is Abelian, we have $ba = baba = abba = abab = ab$. Thus R is pseudo-reversible.

(2) The proof of Lemma 1.3(1) is applicable to this case. □

By Lemma 2.1 or the proof of Lemma 2.1(1), reversible rings without identity are also Abelian. A pseudo-reversible ring without identity is also quasi-reversible by definition. When given a ring does not have an identity, we also have the following that is almost same as Lemma 1.1.

Proposition 2.2. (1) *For a ring R without identity, the following conditions are equivalent*:

(1) *R is reversible*;

(2) *$ab \in I(R)$ for $a, b \in R$ implies $ba \in I(R)$*;

(3) *$ab \in I(R)$ for $a, b \in R$ implies $ab = ba$*;

(4) *$ab \in I(R)$ for $a, b \in R$ implies $ba = baab$.*

Proof. (1) $\Rightarrow$ (3). The proof is almost same as one of Lemma 2.1(1). (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (1). Let $ab = 0$. Then $ba = baba = 0$ by the condition (2), so R is reversible. The proof of (4) $\Rightarrow$ (1) is similar to one of (2) $\Rightarrow$ (1). □

Considering Lemma 1.1, one may conjecture that a ring R is reversible if and only if $ab \in I(R)$ for $a, b \in R$ implies $bra = braab$ for all $r \in R$, even though R does not have an identity. But the following erases the possibility.

Example 2.3. We first claim that if R is a reversible ring without identity, then $ab \in I(R)$ for $a, b \in R$ implies $bra = braab$ for all $r \in R$. Assume the sufficiency. Let $ab \in I(R)$ for $a, b \in R$. Then $a(b - bab)r = 0$ for all $r \in R$, and so $(b - bab)ra = 0$; hence $bra = babra$. But R is Abelian by Lemma 2.1, so $bra = babra = braab$.

However the converse need not hold. Let $A = \mathbb{Z}_8$ and $B = 2\mathbb{Z}_8$. Consider $R = \left(\begin{smallmatrix} B & B \\ 0 & 0 \end{smallmatrix}\right)$, a subring of $T_2(A)$. Then R is not reversible as can be seen by $(2E_{11})(2E_{12}) = 4E_{12} \neq 0 = (2E_{12})(2E_{11})$. Let $ab \in I(R)$ for $a, b \in R$. Then $ab = 0$ since $R^3 = 0$ (hence $R = N(R)$). Moreover $R^3 = 0$ implies $bra = 0 = braab$ for all $r \in R$. Therefore R satisfies the necessity.

Next we observe the structure of Abelian rings without identity in relation with a kind of ring extensions. Let A be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [3], the *Dorroh extension* of A by S is the Abelian group $A \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. We use $A \times_{dor} S$ to denote this extension. The characteristic of a given ring R is denoted by $ch(R)$.

Proposition 2.4. (1) *Let A be an Abelian ring with or without identity. Then $A \times_{dor} \mathbb{Z}$ is an Abelian ring.*

(2) *Let A be an Abelian ring with or without identity with $ch(A) = 2$. Then $A \times_{dor} \mathbb{Z}_2$ is an Abelian ring.*

Proof. Let $R = A \times_{dor} \mathbb{Z}$ and suppose that $(a, n)^2 = (a, n)$ for $(a, n) \in R$. Then $a^2 + 2na = a$ and $n^2 = n$. From $n^2 = n$, we get $n = 0$ or $n = 1$.

Let $n = 0$. Then $a^2 = a$. Since A is Abelian, $a \in Z(A)$ and hence

$$(a, 0)(r, m) = (ar + ma, 0) = (ra + ma, 0) = (r, m)(a, 0)$$

for all $(r, m) \in R$.

Let $n = 1$. Then $a^2 + 2a = a$ and $a^2 = -a$. This implies $(-a)^2 = a^2 = -a$, i.e., $-a \in I(A)$. Since A is Abelian, $-a \in Z(A)$ and $a \in Z(A)$ follows. This yields

$$(a, 1)(r, m) = (ar + ma + r, m) = (ra + ma + r, m) = (r, m)(a, 1)$$

for all $(r, m) \in R$.

Therefore R is Abelian.

(2) Let $R = A \times_{dor} \mathbb{Z}_2$ and suppose that $(a, n)^2 = (a, n)$ for $(a, n) \in R$. Then $a = a^2 + 2na = a^2$ (i.e., $a \in I(A)$) because $ch(A) = 2$. Since A is Abelian, $a \in Z(A)$ and hence

$$(a, n)(r, m) = (ar + ma + nr, nm) = (ra + ma + nr, mn) = (r, m)(a, n)$$

for all $(r, m) \in R$. Thus R is Abelian. □

Pseudo-reversible rings are Abelian by Lemma 2.1. So one may conjecture that the Dorroh extension of a pseudo-reversible ring without identity by $\mathbb{Z}$ is pseudo-reversible. But the following provides us with counterexamples.

Example 2.5. There exists a reversible ring without identity over which the Dorroh extension by $\mathbb{Z}$ is not quasi-reversible (hence not pseudo-reversible). To see this, we apply [13, Example 4]. Let $A = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over $\mathbb{Z}_4$. Let I be the ideal of A generated by $x^2, xy + 2x, 2yx, y^2x$. Set $A_1 = A/I$. We identify x, y with their images in A_1 for simplicity. Next set $A_2 = A_1x + A_1y$. Then A_2 is a reversible ring without identity by the argument in [13, Example 4].

Now let $R_0 = A_2 \times \mathbb{Z}_2$. Then R_0 is also a reversible ring without identity. Next consider $R = R_0 \times_{dor} \mathbb{Z}$. Take

$$\alpha = ((x, 1), 0) \text{ and } \beta = ((y, 1), 2)$$

in R. Then

$$\alpha\beta = ((xy, 1) + (2x, 0), 0) = ((xy + 2x, 1), 0) = ((0, 1), 0) \in I(R)'.$$

But

$$\beta\alpha = ((y, 1), 2)((x, 1), 0) = ((yx, 1) + (2x, 0), 0) = ((yx + 2x, 1), 0) \notin I(R)'$$

because $yx+2x \neq 0$ and $(yx+2x)^2 = yxyx+2yx^2+2xyx+4x^2 = y(2x)x = 0$. Therefore R is not quasi-reversible.

However we obtain an affirmative result when given rings have identities by help of [13, Proposition 3]. Recall that the reversibility and pseudo-reversibility are equivalent in direct products by Theorem 1.5. In the following we see another kind of extension in which the equivalence of reversibility and pseudo-reversibility is possible.

Theorem 2.6. *Let A be a ring with identity. Then the following conditions are equivalent*:
(1) $A \times_{dor} \mathbb{Z}$ *is pseudo-reversible*;
(2) $A \times_{dor} \mathbb{Z}$ *is quasi-reversible*;
(3) A *is reversible*;
(4) $A \times_{dor} \mathbb{Z}$ *is reversible.*

Proof. Let $R = A \times_{dor} \mathbb{Z}$. (2) $\Rightarrow$ (3). Suppose that $ab = 0$ for $a, b \in A$. Set $a_0 = a - 1$ and $b_0 = b - 1$. Then, for $(a_0, 1), (b_0, 1) \in R$, we have

$$(a_0, 1)(b_0, 1) = (a_0b_0 + a_0 + b_0, 1) = (ab - a - b + 1 + a + b - 2, 1) = (-1, 1).$$

But $(-1, 1) \in I(R)'$. Since R is quasi-reversible, $(b_0a_0+a_0+b_0, 1) = (b_0, 1)(a_0, 1) \in I(R)$. Then we get $(b_0a_0 + a_0 + b_0)^2 = -(b_0a_0 + a_0 + b_0)$ by the proof of Proposition 2.4(1). Furthermore we have

$$\begin{aligned}
&(b_0a_0 + a_0 + b_0 + 1)^2 \\
={}& b_0a_0b_0a_0 + b_0a_0^2 + b_0a_0b_0 + b_0a_0 + a_0b_0a_0 + a_0^2 + a_0b_0 + a_0 \\
&+ b_0^2a_0 + b_0a_0 + b_0^2 + b_0 + b_0a_0 + a_0 + b_0 + 1 \\
={}& b_0(a_0b_0 + a_0 + b_0 + 1)a_0 + b_0(a_0b_0 + a_0 + b_0 + 1) \\
&+ (a_0b_0 + a_0 + b_0 + 1)a_0 + (a_0b_0 + a_0 + b_0 + 1) = 0,
\end{aligned}$$

because $a_0b_0 + a_0 + b_0 = -1$ as above. This yields

$$\begin{aligned}0 &= (b_0a_0 + a_0 + b_0 + 1)^2\\ &= (b_0a_0 + a_0 + b_0)^2 + 2(b_0a_0 + a_0 + b_0) + 1\\ &= -(b_0a_0 + a_0 + b_0) + 2(b_0a_0 + a_0 + b_0) + 1\\ &= (b_0a_0 + a_0 + b_0) + 1,\end{aligned}$$

entailing $b_0a_0 + a_0 + b_0 = -1 = a_0b_0 + a_0 + b_0$. Then $a_0b_0 = b_0a_0$. This result gives us $(a - 1)(b - 1) = (b - 1)(a - 1)$ and $ba = ab = 0$ follows. Thus A is reversible.

(3) ⇒ (4) is proved by [13, Proposition 3], and (4) ⇒ (1) is obtained from Lemma 1.1. (1) ⇒ (2) is obvious. □

Note that $I(\mathbb{Z}_n) = \{0, 1\}$ when n is prime, but $I(\mathbb{Z}_n) \supsetneq \{0, 1\}$ may arise when n is not prime (e.g., $I(\mathbb{Z}_6) = \{0, 1, 3, 4\}$). So we take $I(\mathbb{Z}_n) = \{0, 1\}$ as a condition in the following to apply the proof of Theorem 2.6.

Proposition 2.7. *Let A be a ring with identity and $n \geq 2$. Suppose that $ch(A) = n$ and $I(\mathbb{Z}_n) = \{0, 1\}$. Then the following conditions are equivalent*:

(1) $A \times_{dor} \mathbb{Z}_n$ *is pseudo-reversible*;
(2) $A \times_{dor} \mathbb{Z}_n$ *is quasi-reversible*;
(3) A *is reversible*;
(4) $A \times_{dor} \mathbb{Z}_n$ *is reversible.*

Proof. Let $R = A \times_{dor} \mathbb{Z}_n$ and $(f, l) \in I(R)$ for $(f, l) \in R$. Since $I(\mathbb{Z}_n) = \{0, 1\}$, l is either 0 or 1. Then the proof of Theorem 2.6 is applicable. □

In the following we see concrete shapes of idempotents of $A \times_{dor} \mathbb{Z}$ in Theorem 2.6.

Remark 2.8. Let $R = A \times_{dor} \mathbb{Z}$ and suppose that $(a, m)(b, n) \in I(R)'$ for $(a, m), (b, n) \in R$. Then $(ab + na + mb, mn) \in I(R)'$, and mn is zero or 1. Notice that $\mathbb{Z} \subseteq A$ because $1 \in A$. R is Abelian by Proposition 2.4 because A is Abelian by Lemma 1.3(1). If $ab + na + mb = 0$, then $mn = 1$ because $(a, m)(b, n) \in I(R)'$; hence $(a, m)(b, n) = (0, 1) = 1$ implies $(b, n)(a, m) = 1$ because R is directly finite, entailing $(a, m)(b, n) = (b, n)(a, m)$. Thus we will deal with the case of $ab + na + mb \neq 0$.

Consider the case of $mn = 0$ (i.e., $m = 0$ or $n = 0$). Here we must have $ab + na + mb \in I(A)'$ because $(ab + na + mb, 0) \in I(R)'$. Let $m = 0$. Then $a(b + n) \in I(A)'$, and since A is pseudo-reversible, we have $a(b + n) = (b + n)a$. This yields $ab = ba$, and so $(a, m)(b, n) = (b, n)(a, m)$. The computation for the case of $n = 0$ is similar.

Consider the case of $mn = 1$ (i.e., $m = n = 1$ or $m = n = -1$). Let $m = n = 1$. Then $(ab + a + b, 1) \in I(R)'$, and this yields

$$(ab + a + b)^2 = -(ab + a + b).$$

From this equality, we get

$$\begin{aligned}((ab+a+b)+1)^2 &= (ab+a+b)^2+2(ab+a+b)+1\\ &= -(ab+a+b)+2(ab+a+b)+1\\ &= (ab+a+b)+1,\end{aligned}$$

entailing $(ab+a+b)+1 \in I(A)$. Note $ab+a+b+1=(a+1)(b+1)$. Since A is reversible, we get $(a+1)(b+1)=(b+1)(a+1)$ by Lemma 1.1. This yields $ab=ba$, and $(a,m)(b,n)=(b,n)(a,m)$ follows.

Let $m=n=-1$. Then $(ab-a-b,1) \in I(R)'$, and this yields

$$(ab-a-b)^2=-(ab-a-b).$$

From this equality, we get

$$\begin{aligned}((ab-a-b)+1)^2 &= (ab-a-b)^2+2(ab-a-b)+1\\ &= -(ab-a-b)+2(ab-a-b)+1\\ &= (ab-a-b)+1,\end{aligned}$$

entailing $(ab-a-b)+1 \in I(A)$. Note $ab-a-b+1=(a-1)(b-1)$. Since A is reversible, we get $(a-1)(b-1)=(b-1)(a-1)$ by Lemma 1.1. This yields $ab=ba$, and $(a,m)(b,n)=(b,n)(a,m)$ follows.

If A is a commutative ring with or without identity, then clearly $A \times_{dor} \mathbb{Z}$ is commutative. Considering Theorem 2.6, one may ask whether $1 \in A$ is a necessary condition for $A \times_{dor} \mathbb{Z}$ to be reversible. But the answer is negative as we see in the following. Theorem 2.6 can be applied in constructing a reversible ring $A \times_{dor} \mathbb{Z}$ with $1 \notin A$.

Example 2.9. Let A_0 be a reversible ring with identity and $A_1 = A_0 \times_{dor} \mathbb{Z}$. Then A_1 is a reversible ring with identity by Theorem 2.6. Next consider $A_2 = A_1 \times_{dor} \mathbb{Z}$, and set $A_{k+1} = A_k \times_{dor} \mathbb{Z}$ for $k \geq 1$. Then, for all $n \geq 1$, A_n is a reversible ring with identity inductively, by help of Theorem 2.6. Define a map $\sigma : A_k \to A_{k+1}$, defined by $(a_k, m_k) \mapsto ((a_k, m_k), 0)$, for all $k \geq 1$. Then σ is clearly a monomorphism and so A_k is considered as a subring of A_{k+1} via σ. Now let $A = \bigcup_{i=1}^{\infty} A_i$ and $R = A \times_{dor} \mathbb{Z}$. We will show that R is a reversible ring.

Let $(a,m)(b,n)=0$ for $(a,m),(b,n) \in R$. Then there exists $k \geq 1$ such that $a, b \in A_k$. Note $A_k = A_{k-1} \times_{dor} \mathbb{Z}$ and that A_k is reversible.

From $(a,m)(b,n)=0$, we get $ab+na+mb=0$ and $mn=0$ (hence $m=0$ or $n=0$).

Assume $m=0$. Then $ab+na=0$ and so $a(b+n)=0$. But since A_k is reversible, we get $(b+n)a$=0 and $(b,n)(a,m)=0$. The computation for the case of $n=0$ is similar.

Therefore R is a reversible ring. However A does not have an identity because every element a of A is an element of A_k for some $k \geq 1$ and a cannot be the identity of A_{k+1}, noting $a=\sigma(a)=(a,0)$.

Using the Dorroh extension, we can always construct a reduced ring with identity, which is not a domain, over any nonzero domain of nonzero characteristic with or without identity.

Remark 2.10. Let A be a domain of nonzero characteristic p with or without identity, where $|A| \geq 2$. Then p is clearly prime. Consider $R = A \times_{dor} \mathbb{Z}$. Take elements $u = (a, 0)$ and $v = (0, p)$ in R, where $a \neq 0$. Then $u, v \in R \backslash \{0\}$ clearly. But $uv = (pa, 0) = 0$, concluding that R is not a domain. Next assume $(b, m)^k = 0$ for some $(b, m) \in R$ and $k \geq 2$. Then $m = 0$ and $b^k = 0$ follows. But since A is a domain, we get $b = 0$; hence $(b, m) = 0$. Therefore R is a reduced ring with identity.

Let R be the reduced ring $A \times_{dor} \mathbb{Z}$ in Remark 2.10. Then $B = D_2(R)$ is a reversible ring with identity by Proposition 1.8. Next consider $E = B \times_{dor} \mathbb{Z}$. Then E is also a (pseudo-)reversible ring by Theorem 2.6. Clearly E is not reduced. Therefore, over any nonzero domain of nonzero characteristic with or without identity, we can construct a non-reduced reversible ring with identity, via Dorroh extensions.

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