Bull. Korean Math. Soc. **56** (2019), No. 5, pp. 1235–1255
https://doi.org/10.4134/BKMS.b180995
pISSN: 1015-8634 / eISSN: 2234-3016

SR-ADDITIVE CODES

Saadoun Mahmoudi and Karim Samei

Abstract. In this paper, we introduce SR-additive codes as a generalization of the classes of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ and $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, where S is an R-algebra and an SR-additive code is an R-submodule of $S^\alpha \times R^\beta$. In particular, the definitions of bilinear forms, weight functions and Gray maps on the classes of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ and $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes are generalized to SR-additive codes. Also the singleton bound for SR-additive codes and some results on one weight SR-additive codes are given. Among other important results, we obtain the structure of SR-additive cyclic codes. As some results of the theory, the structure of cyclic $\mathbb{Z}_2\mathbb{Z}_4$, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$, $\mathbb{Z}_2\mathbb{Z}_2[u]$, $(\mathbb{Z}_2)(\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2+u\mathbb{Z}_2)(\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2)(\mathbb{Z}_2+u\mathbb{Z}_2+v\mathbb{Z}_2)$ and $(\mathbb{Z}_2+u\mathbb{Z}_2)(\mathbb{Z}_2+u\mathbb{Z}_2+v\mathbb{Z}_2)$-additive codes are presented.

1. Introduction

An important class of additive codes is $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. A subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, where α and β are positive integers, is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. A comprehensive study on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes has been introduced in [9] by Borges et al. The studies on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and their algebraic structures have attracted many researchers; see [2, 6–9, 13, 15–17].

$\mathbb{Z}_2\mathbb{Z}_4$-additive codes were generalized to $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes [4, 21]. Also $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes is another generalization of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes which has been introduced by Aydogdu et al. [3].

Recently, Aydogdu and Siap generalized $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes to $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes [5]. Also, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes have been studied in [10]. Also additive codes were studied over direct product of chain rings in [11].

Note that in $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes, $\mathbb{Z}_2$ is a $\mathbb{Z}_4$-algebra and $\mathbb{Z}_{2^s}$-algebra; respectively. Also in $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes, $\mathbb{Z}_2$ is considered as a $\mathbb{Z}_2[u]$-algebra and $\mathbb{Z}_{p^r}$ is a $\mathbb{Z}_{p^s}$-algebra in $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. Also in additive codes over product of two chain rings, one of the rings is an algebra over another ring.

Received October 18, 2018; Accepted December 18, 2018.
2010 *Mathematics Subject Classification.* 94B15.
Key words and phrases. additive code, chain ring, Galois ring.

In this paper, we generalize above codes to SR-additive codes, where S is an R-algebra. In this generalization, a subset C of $S^\alpha \times R^\beta$ is called an SR-additive code if C is an R-submodule of $S^\alpha \times R^\beta$. We present the structure of SR-additive cyclic codes. Also we give the structure of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes, $(\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2)$-additive cyclic codes and cyclic codes over direct product of chain rings as results of this theory, which the structure of these codes are the main parts of [2], [10], [22] and [11]; respectively.

Also, we obtain the structure of $(\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2)$ and $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2)$-additive cyclic codes as other results of this theory.

In Section 4, we define an inner product over SR-additive codes which is a generalization of the inner products over $\mathbb{Z}_2\mathbb{Z}_4$, $\mathbb{Z}_2\mathbb{Z}_{2^s}$, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$, $\mathbb{Z}_2\mathbb{Z}_2[u]$ additive codes. We show that the dual code of any SR-additive cyclic code is also an SR-additive cyclic code.

In Section 5, we find the Singleton bound for SR-additive codes. As examples, the Singleton bound for $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes and $\mathbb{Z}_2[u]\mathbb{Z}_2$-additive codes are given. In Section 6, we investigate one weight SR-additive codes. In particular, one weight $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes are determined.

Throughout this paper R and S are finite commutative rings such that S is an R-algebra.

2. Preliminaries

In this section, we remind some facts of R-additive codes which are applied throughout this paper. Also the structure of cyclic codes over some rings are given.

Definition 2.1. Let S be an R-algebra with a ring homomorphism $f : R \to S$. A nonempty subset C of S^n is called R-additive code if C is an R-submodule of S^n, where the scalar multiplication is defined as follows: for $r \in R$ and $(a_0, a_1, \ldots, a_{n-1}) \in C$, we have

$$r.(a_0, a_1, \ldots, a_{n-1}) = (f(r)a_0, f(r)a_1, \ldots, f(r)a_{n-1}).$$

Example 2.2 (Linear codes)**.** Let R be a commutative ring with identity. A subset C of R^n is called a linear code if C is an R-submodule of R^n. Now consider R as R-algebra with identity homomorphism. Clearly, the subset C of R^n is a linear code if and only if C is an R-additive code.

Above example shows that R-additive codes is a generalization of linear codes. The following example give some special cases which R-additive codes and linear codes are the same.

Example 2.3. (1) Let $f : R \to S$ be a ring isomorphism. In this case, R-additive codes over S are exactly linear codes over S.

(2) Let $S = R/I$, where I is an ideal of R and $f : R \to R/I$ is the natural homomorphism. For any nonempty subset C of S^n, we have $I.C = 0$. Hence

R-additive codes over S are exactly linear codes over S. Moreover, if $f : R \to S$ is a surjective ring homomorphism, then R-additive codes over S are exactly linear codes.

Example 2.4 (Additive codes)**.** Let S be a local ring of characteristic p^r. A subset C of S^n is called an additive code if C is a subgroup of S^n under addition. But we have the injective ring homomorphism $f : \mathbb{Z}_{p^r} \to S$, $x \mapsto x.1_S$. It is easy to see that additive codes are exactly $\mathbb{Z}_{p^r}$-submodules of S^n. In other words, additive codes over S are exactly $\mathbb{Z}_{p^r}$-additive codes over S.

Example 2.5 ($\mathbb{F}_q$-linear codes over $\mathbb{F}_{q^t}$)**.** A subset C of $(\mathbb{F}_{q^t})^n$ is called an $\mathbb{F}_q$-linear code over $\mathbb{F}_{q^t}$ of length n, if C is an $\mathbb{F}_q$-submodule of $(\mathbb{F}_{q^t})^n$. Clearly these codes are R-additive codes, where $R = \mathbb{F}_q$ and $S = \mathbb{F}_{q^t}$.

For a positive integer n, let $R_n = R[x]/\langle x^n - 1\rangle$ and $S_n = S[x]/\langle x^n - 1\rangle$. Consider the following correspondence map.

$$\begin{aligned} \pi : S^n &\longrightarrow S_n, \\ (a_0, a_1, \ldots, a_{n-1}) &\longmapsto a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n - 1\rangle. \end{aligned} \tag{1}$$

Clearly π is an R-module isomorphism. We will identify S^n with S_n under π and for simplicity, we write the polynomial $a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ for the residue class $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n - 1\rangle$. The following proposition gives the structure of cyclic R-additive codes.

Proposition 2.6 ([19, Proposition 3.1])**.** *Let π be the correspondence map defined in* (1)*. Then a nonempty subset C of S^n is a cyclic R-additive code if and only if $\pi(C)$ is an R_n-submodule of S_n.*

Let ω be a weight function over S. If $A_S = \mathrm{Max}\{\omega(x) : x \in S\}$, then we have the following bound for minimum weight of R-additive codes.

Theorem 2.7 ([20, Theorem 3.5])**.** *Let R be a finite chain ring and S be a free R-algebra of $\dim_R(S) = m$. If there exists a nondegenerate bilinear form $\beta : S \times S \to R$, then $\lfloor \frac{d_\omega(C)-1}{A_S} \rfloor \leq n - \lceil \frac{\mathrm{rank}(C)}{m} \rceil$.*

Now we remind the structure of cyclic codes over a chain ring R of length n coprime to $\mathrm{Char}(R)$. Also the structure of cyclic codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$, $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$ for an arbitrary length are given.

Theorem 2.8. *Let R be a chain ring with the maximal ideal $\mathfrak{m} = \langle \gamma \rangle$ of nilpotency index s and C be a cyclic code of length n over R, where $(n, \mathrm{Char}(R)) = 1$. Then*

(1) *There is a unique set of pairwise co-prime monic polynomials $g_0, \ldots, g_s$ over R (possibly, some of them are equal to 1) such that $g_0g_1 \cdots g_s = x^n - 1$ in $R[x]$ and $C = \langle \widehat{g_1}, \gamma\widehat{g_2}, \ldots, \gamma^{s-1}\widehat{g_s}\rangle$, where $\widehat{g_i} = \prod_{j \neq i} g_j$. Moreover, $|C| = |R/\mathfrak{m}|^{\sum_{i=0}^{s-1}(s-i)\deg g_{i+1}}$.*

(2) *If $h_i = g_0g_{i+2} \cdots g_s$ for $i = 0, 1, \ldots, s-2$ and $h_{s-1} = g_0$. Then $h_{s-1}|h_{s-2}| \cdots |h_0|(x^n - 1)$, and $C = \langle h_0 + \gamma h_1 + \cdots + \gamma^{s-1}h_{s-1}\rangle$.*

Proof. Part (1) follows from Theorem 3.4 in [12]. We have part (2) by Theorem 3.5 in [12] and Theorem 2.4 in [11]. □

The following corollary is a result of Proposition 2.8.

Corollary 2.9. *Let C be a cyclic code of length n over $R = \mathbb{Z}_{p^s}$, where $(n,p) = 1$. Then there exists a set of polynomials $h_0, h_1, \dots, h_{s-1}$ in $R[x]$ such that $h_0|(x^n-1)$, $h_i|h_{i-1}$ for $i = 1,\dots,s-1$ and $C = \langle h_0 + ph_1 + \cdots + p^{s-1}h_{s-1}\rangle$. Moreover if $\widehat{h_i} = \frac{h_{i-1}}{h_i}$ for $i \geq 1$ and $\widehat{h_0} = \frac{x^n-1}{h_0}$, then $|C| = p^d$, where $d = \sum_{i=0}^{s-1}(s-i)\deg \widehat{h_i}$. In special case, if n is odd and C is a cyclic code of length n over $R = \mathbb{Z}_4$, then $C = \langle g(x) + 2a(x)\rangle$, where $a(x)|g(x)|(x^n-1)$ in $\mathbb{Z}_4[x]$. In this case, $|C| = 2^{2t_1+t_2}$, where $t_1 = \deg \frac{x^n-1}{g(x)}$ and $t_2 = \deg \frac{g(x)}{a(x)}$.*

Theorem 2.10 ([1, Theorem 1])**.** *Let C be a cyclic code over $\mathbb{Z}_2 + u\mathbb{Z}_2$ of length n. Then*

(1) *If n is odd, then $(\mathbb{Z}_2 + u\mathbb{Z}_2)_n$ is principal ideal ring and $C = \langle g(x) + ua(x)\rangle$, where $g(x)$ and $a(x)$ are polynomials in $\mathbb{Z}_2[x]$ such that $a(x)|g(x)|(x^n-1) \mod 2$.*

(2) *If n is not odd, then*

   (a) *$C = \langle g(x) + up(x)\rangle$ such that $g(x)|(x^n-1) \mod 2$, $(g(x)+up(x))|(x^n-1)$ in $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $g(x)|p(x)(\frac{x^n-1}{g(x)})$. Or*

   (b) *$C = \langle g(x) + up(x), ua(x)\rangle$ such that $g(x)$, $a(x)$ and $p(x)$ are polynomials in $\mathbb{Z}_2[x]$. And $a(x)|g(x)|(x^n-1) \mod 2$, $a(x)|p(x)(\frac{x^n-1}{g(x)})$ and $\deg a(x) > \deg p(x)$.*

Theorem 2.11 ([1, Theorem 2])**.** *Let C be a cyclic code over $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ of length n. Then*

(1) *If n is odd, then $(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)_n$ is principal ideal ring. $C = \langle g(x) + ua_1(x) + u^2a_2(x)\rangle$, where $a_1(x)$, $a_2(x)$ and $g(x)$ are polynomials in $\mathbb{Z}_2[x]$ such that $a_2(x)|a_1(x)|g(x)|(x^n-1) \mod 2$.*

(2) *If n is not odd, then*

   (a) *$C = \langle g + up_1 + u^2p_2\rangle$, where $p_2|p_1|g|(x^n-1) \mod 2$, $(g+up_1)|(x^n-1)$ in $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $(g + up_1 + u^2p_2)|(x^n-1)$ in $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ and $\deg p_2 < \deg p_1$.*

   (b) *$C = \langle g + up_1 + u^2p_2, u^2a_2\rangle$, where $a_2|g|(x^n-1) \mod 2$, $(g + up_1)|(x^n-1)$ in $\mathbb{Z}_2 + u\mathbb{Z}_2$, $g(x)|p_1(\frac{x^n-1}{g(x)})$ and a_2 divides $p_1(\frac{x^n-1}{g(x)})$ and $p_2(\frac{x^n-1}{g(x)})(\frac{x^n-1}{g(x)})$ and $\deg p_2 < \deg a_2$. Or*

   (c) *$C = \langle g + up_1 + u^2p_2, ua_1 + u^2q_1, u^2a_2\rangle$, where $a_2|a_1|g|(x^n-1) \mod 2$, $a_1|p_1(\frac{x^n-1}{g(x)})$ and a_2 divides $q_1(\frac{x^n-1}{a_1(x)})$ and $p_2(\frac{x^n-1}{g(x)})(\frac{x^n-1}{a_1(x)})$. Moreover, $\deg p_2 < \deg a_2$, $\deg q_1 < \deg a_2$ and $\deg p_1 < \deg a_1$.*

The following theorem gives the structure of cyclic codes over the non Frobenius ring $\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2 = \{0, 1, u, v, 1+u, 1+v, u+v, 1+u+v\}$.

Theorem 2.12. *Let C be a cyclic code over $R = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2$ of length n. Then C has a unique representation as follows*:

$$C = \langle g + up_1 + vp_2, ua_1 + vq_1, va_2\rangle,$$

where

(1) $a_2|a_1|g|(x^n - 1)$ *and* $a_1|p_1(\frac{x^n-1}{g})$,

(2) $a_2|q_1(\frac{x^n-1}{a_1})$ *and* $a_2|p_2(\frac{x^n-1}{g})(\frac{x^n-1}{a_1})$,

(3) $\deg p_2, \deg q_1 < \deg a_2$.

Moreover if n is odd, then $C = \langle g + ua_1, va_2\rangle$, where $a_2|a_1|g|(x^n - 1)$.

Proof. See Theorems 1 and 2, Lemmas 3 and 4 and Corollary 1 in [18]. □

3. *SR*-additive cyclic codes

The structure of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes investigated in [2]. As generalizations of these codes, recently $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ and $\mathbb{Z}_2\mathbb{Z}_2[u]$ additive codes have been introduced in [3] and [5]. Also the generator polynomials of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes were given in [10]. Moreover, additive codes studied over direct product of chain rings with the same residue fields in [11]. In this section, we define and extend these codes to SR-additive codes, where R is a finite commutative ring and S is a finite commutative R-algebra. A theory to find the generators of SR-additive cyclic codes is given. As results, we obtain the generators of $\mathbb{Z}_2\mathbb{Z}_4$, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$, $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive cyclic codes. Also the results in [11] on the structure of cyclic codes over direct product of chain rings with the same residue fields are given as a result of the theory. Moreover the structure of $(\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2)$, $(\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2)$ and $(\mathbb{Z}_2 + u\mathbb{Z}_2)(\mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2)$-additive cyclic codes as new examples of SR-additive cyclic codes are given, which we can not obtain their structures by previous works.

Definition 3.1. Let α and β be two positive integers. A nonempty subset C of $S^\alpha \times R^\beta$ is called an SR-additive code if C is an R-submodule with the following scalar multiplication: for $r \in R$ and $(s_\alpha, r_\beta) = (s_0, s_1, \ldots, s_{\alpha-1}, r_0, r_1, \ldots, r_{\beta-1}) \in C$,

$$r.(s_\alpha, r_\beta) = (f(r)s_\alpha, rr_\beta) = (f(r)s_0, f(r)s_1, \ldots, f(r)s_{\alpha-1}, rr_0, rr_1, \ldots, rr_{\beta-1}).$$

We say that an SR-additive code C is cyclic if $(s_{\alpha-1}, s_0, \ldots, s_{\alpha-2}, r_{\beta-1}, r_0, \ldots, r_{\beta-2}) \in C$ whenever $(s_0, s_1, \ldots, s_{\alpha-1}, r_0, r_1, \ldots, r_{\beta-1}) \in C$.

Consider the map $\pi' : S^\alpha \times R^\beta \to S_\alpha \times R_\beta, (s_0, s_1, \ldots, s_{\alpha-1}, r_0, r_1, \ldots, r_{\beta-1}) \mapsto (s_0 + s_1x + \cdots + s_{\alpha-1}x^{\alpha-1} + \langle x^\alpha - 1\rangle, r_{0+}r_1x + \cdots + r_{\beta-1}x^{\beta-1} + \langle x^\beta - 1\rangle)$. Clearly π' is an R-module isomorphism. We will identify $S^\alpha \times R^\beta$ with $S_\alpha \times R_\beta$ under π' and for simplicity we write $(s_0 + s_1x + \cdots + s_{\alpha-1}x^{\alpha-1}, r_{0+}r_1x + \cdots + r_{\beta-1}x^{\beta-1})$ for above residue class.

Lemma 3.2. *A subset C of $S^\alpha \times R^\beta$ is an SR-additive cyclic code if and only if $\pi'(C)$ is an $R[x]$-submodule of $S_\alpha \times R_\beta$.*

Proof. Clearly $S_\alpha \times R_\beta$ is an $R[x]$-module. Since π' is an R-module isomorphism, C is an R-submodule if and only if $\pi'(C)$ is an R-submodule. Now for an element $(s_\alpha, r_\beta) = (s_0, s_1, \dots, s_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in C$, the cyclic shift $\sigma(s_\alpha, r_\beta) = (s_{\alpha-1}, s_0, \dots, s_{\alpha-2}, r_{\beta-1}, r_0, \dots, r_{\beta-2}) \in C$ if and only if $x\pi'(s_\alpha, r_\beta) = \pi'(\sigma(s_\alpha, r_\beta)) \in \pi'(C)$. This completes the proof. □

We identify C with $\pi'(C)$. Now we find the generator polynomials of C.

Theorem 3.3. *A subset C of $S_\alpha \times R_\beta$ is an SR-additive cyclic code if and only if $C = \langle (g_1, 0), \dots, (g_s, 0), (h_1, f_1), \dots, (h_r, f_r)\rangle_{R[x]}$ such that*
(1) *$C_1 = \langle f_1, \dots, f_r\rangle_{R[x]}$ is a cyclic linear code over R of length β,*
(2) *$C_2 = \langle g_1, \dots, g_s\rangle_{R[x]}$ is a cyclic R-additive code over S of length α,*
(3) *$h_1, \dots, h_r$ are elements of S_α,*
(4) *$|C| = |C_1||C_2|$.*

Proof. Let $C \subseteq S_\alpha \times R_\beta$ be an SR-additive cyclic code. Clearly the projection map $\phi : C \to R_\beta$ is an $R[x]$-homomorphism. Hence $Im(\phi)$ is an $R[x]$-submodule of R_β. As $\langle x^\beta - 1\rangle.Im(\phi) \subseteq \langle x^\beta - 1\rangle.R_\beta = 0$, $Im(\phi)$ is an ideal of R_β. In other words $Im(\phi)$ is a linear cyclic code over R of length β, say C_1. Let $C_1 = \langle f_1, \dots, f_r\rangle_{R[x]} = \langle \phi(h_1, f_1), \dots, \phi(h_r, f_r)\rangle_{R[x]}$. Now, $\ker\phi$ is an $R[x]$-submodule of C. Let $C_2 = \{g \in S_\alpha : (g, 0) \in \ker\phi\}$, then clearly C_2 is an $R[x]$-submodule of S_α. Since $\langle x^\alpha - 1\rangle.C_2 \subseteq \langle x^\alpha - 1\rangle.S_\alpha = 0$, C_2 is an R_α-module. In other words C_2 is a cyclic R-additive code of length α over S. If $C_2 = \langle g_1, \dots, g_s\rangle_{R_\alpha}$, then $\ker\phi = \langle (g_1, 0), \dots, (g_s, 0)\rangle_{R[x]}$. Therefore $C = \langle (g_1, 0), \dots, (g_s, 0), (h_1, f_1), \dots, (h_r, f_r)\rangle_{R[x]}$. Since ϕ is an $R[x]$-homomorphism, $\frac{C}{\ker\phi} \cong C_1$, hence $|C| = |\ker\phi||C_1| = |C_2||C_1|$. □

Proposition 3.4. *With the above assumptions, let $f : R \to S$ be a surjective ring homomorphism and $C = \langle (g_1, 0), \dots, (g_s, 0), (h_1, f_1), \dots, (h_r, f_r)\rangle_{R[x]}$ be an SR-additive cyclic code. Also let $\{g_{i_1}, \dots, g_{i_t}\}$ be a subset of $\{g_1, \dots, g_s\}$ such that g_{i_j} is monic for all j; $j = 1, \dots, t$. Then we can assume that* $\deg h_i < \min\{\deg g_{i_j} : 1 \le j \le t\}$ *for all i; $1 \le i \le r$.*

Proof. Since f is surjective, every R-additive code over S is linear. In particular, C_2 is a cyclic linear code over S. Let g_j be monic and $\deg h_i \ge \deg g_j$ for some i. Let $\deg h_i - \deg g_j = \ell$ and $a \in S$ be the leading coefficient of h_i. Then $(h_i, f_i) = (h_i - ax^\ell g_j, f_i) + ax^\ell(g_j, 0)$. Thus $\langle (h_i, f_i), (g_j, 0)\rangle = \langle (h_i - ax^\ell g_j, f_i), (g_j, 0)\rangle$. Hence we can use $h_i - ax^\ell g_j$ instead of h_i. By this method we can reduce $\deg h_i$. □

Proposition 3.5. *Let $C = \langle (g_1, 0), \dots, (g_s, 0), (h_1, f_1), \dots, (h_r, f_r)\rangle_{R[x]}$ be an SR-additive cyclic code as in Theorem 3.3. Then*

$$(x^\beta - 1)h_i \in C_2 = \langle g_1, \dots, g_s\rangle_{R[x]}.$$

Proof. Clearly $(x^\beta - 1)(h_i, f_i) = ((x^\beta - 1)h_i, 0) \in \ker\phi$. Hence $(x^\beta - 1)h_i \in C_2 = \langle g_1, \dots, g_s\rangle_{R[x]}$. □

Corollary 3.6 ($(R/\mathfrak{m})R$-additive cyclic codes)**.** *Let R be a finite local ring with the unique maximal ideal $\mathfrak{m}$ and $C \subseteq (R/\mathfrak{m})^{\alpha} \times R^{\beta}$ be an $(R/\mathfrak{m})R$-additive cyclic code. Then $C = \langle (g, 0), (h_1, f_1), \ldots, (h_r, f_r)\rangle$ with the following conditions*:

(a) $g | x^{\alpha} - 1$ *over* $(R/\mathfrak{m})$,

(b) $h_i \in (R/\mathfrak{m})_{\alpha}$,

(c) $C_1 = \langle f_1, \ldots, f_r\rangle$ *is a linear cyclic code over* R.

Proof. $R/\mathfrak{m}$ is an R-algebra with the natural ring homomorphism $f : R \to R/\mathfrak{m}$. Since f is surjective, R-additive codes over $R/\mathfrak{m}$ are linear over $R/\mathfrak{m}$. Now, we have the results by Theorem 3.3. □

$\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes is an example of $(R/\mathfrak{m})R$-additive cyclic codes. This class of codes is discussed in [2]. We obtain the structure of these codes as a result of above discussion.

Corollary 3.7 ($\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes)**.** *Let $C \subseteq (\mathbb{Z}_2)_{\alpha} \times (\mathbb{Z}_4)_{\beta}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. If β is an odd integer, then*

(1) $C = \langle (h(x), 0), (\ell(x), g(x) + 2a(x))\rangle$, *where*

(a) $h(x)$ *is a monic polynomial over* $\mathbb{Z}_2$ *such that* $h(x)|(x^{\alpha} - 1)$,

(b) $a(x)|g(x)|(x^{\beta} - 1)$ *in* $\mathbb{Z}_4[x]$,

(c) $\ell(x) \in (\mathbb{Z}_2)_{\alpha}$ *and* $\deg \ell(x) < \deg h(x)$.

(2) *If* $t_1 = \deg \frac{x^{\beta}-1}{g(x)}$, $t_2 = \deg \frac{g(x)}{a(x)}$ *and* $t = \deg h(x)$, *then* $|C| = 2^{2t_1+t_2+\alpha-t}$.

Proof. By above corollary, $C = \langle (h(x), 0), (\ell_1, f_1), \ldots, (\ell_r, f_r)\rangle$, where $h(x)$ is a monic polynomial over $\mathbb{Z}_2$ such that $h(x)|(x^{\alpha} - 1)$. Also $C_1 = \langle f_1, \ldots, f_r\rangle$ is a linear cyclic code over $\mathbb{Z}_4$. By Corollary 2.9, there exist polynomials $g(x)$ and $a(x)$ over $\mathbb{Z}_4$ such that $C_1 = \langle g(x) + 2a(x)\rangle$, where $a(x)|g(x)|(x^{\beta} - 1)$ in $\mathbb{Z}_4[x]$. Hence $C = \langle (h(x), 0), (\ell(x), g(x) + 2a(x))\rangle$, where $\ell(x) \in (\mathbb{Z}_2)_{\alpha}$ and $\deg \ell(x) < \deg h(x)$. By Corollary 2.9, $|C_1| = 2^{2t_1+t_2}$, where $t_1 = \deg \frac{x^{\beta}-1}{g(x)}$ and $t_2 = \deg \frac{g(x)}{a(x)}$. Also $|C_2| = |\langle h(x)\rangle| = 2^{\alpha-t}$, where $t = \deg h(x)$. Therefore by Theorem 3.3, $|C| = |C_1||C_2| = 2^{2t_1+t_2+\alpha-t}$. □

Another example of SR-additive codes is the class of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes (see [10]). We give the structure of these codes as another result of above discussion.

Corollary 3.8 ($\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes)**.** *Let $1 \le r < s$ and $C \subseteq (\mathbb{Z}_{p^r})_{\alpha} \times (\mathbb{Z}_{p^s})_{\beta}$ be a $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic code. If $(p, \beta) = 1$ and $(p, \alpha) = 1$, then*

(1) $C = \langle (h'_0 + ph'_1 + \cdots + p^{r-1}h'_{r-1}, 0), (\ell(x), h_0 + ph_1 + \cdots + p^{s-1}h_{s-1})\rangle$, *where*

(a) $h_0, h_1, \ldots, h_{s-1}$ *are polynomials in* $\mathbb{Z}_{p^s}[x]$ *such that* $h_0|(x^{\beta} - 1)$ *and* $h_i|h_{i-1}$ *for* $i = 1, \ldots, s-1$,

(b) $h'_0, h'_1, \ldots, h'_{r-1}$ *are polynomials in* $Z_{p^r}[x]$ *such that* $h'_0|(x^{\alpha} - 1)$ *and* $h'_i|h'_{i-1}$ *for* $i = 1, \ldots, r-1$.

(2) $|C| = p^{d_1+d_2}$, *where* $d_1 = \sum_{i=0}^{s-1}(s-i)\deg \widehat{h}_i$ *and* $d_2 = \sum_{i=0}^{r-1}(r-i)\deg \widehat{h}'_i$.

Proof. Since $f : \mathbb{Z}_{p^s} \to \mathbb{Z}_{p^r}$ is surjective, by the same argument of Corollary 3.7, $C = \langle (h(x), 0), (\ell(x), g(x))\rangle$, where $g(x) \in (\mathbb{Z}_{p^s})_\beta$ is a generator of a cyclic code over $\mathbb{Z}_{p^s}$ of length β, $h(x) \in (\mathbb{Z}_{p^r})_\alpha$ is a generator of a cyclic code over $\mathbb{Z}_{p^r}$ of length α and $\ell(x) \in (\mathbb{Z}_{p^r})_\alpha$ is a polynomial. By Corollary 2.9, there exists a set of polynomials $h_0, h_1, \ldots, h_{s-1}$ in $\mathbb{Z}_{p^s}[x]$ such that $h_0|(x^\beta - 1)$ and $h_i|h_{i-1}$ for $i = 1, \ldots, s-1$ and $g(x) = h_0 + ph_1 + \cdots + p^{s-1}h_{s-1}$. Similarly, there exists a set of polynomials $h'_0, h'_1, \ldots, h'_{r-1}$ in $\mathbb{Z}_{p^r}[x]$ such that $h'_0|(x^\alpha - 1)$ and $h'_i|h'_{i-1}$ for $i = 1, \ldots, r-1$ and $h(x) = h'_0 + ph'_1 + \cdots + p^{r-1}h'_{r-1}$. In this case, $|C| = p^{d_1+d_2}$, where $d_1 = \sum_{i=0}^{s-1}(s-i)\deg \widehat{h}_i$ and $d_2 = \sum_{i=0}^{r-1}(r-i)\deg \widehat{h}'_i$. □

Recently, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes generalized to codes over direct product of two finite chain rings in some special case [11]. More precisely, let R_1 and R_2 be two chain rings with the maximal ideals $\mathfrak{m}_1 = \langle \gamma_1 \rangle$ and $\mathfrak{m}_2 = \langle \gamma_2 \rangle$ of the nilpotency indexes e_1 and e_2; respectively. Let $e_1 \le e_2$, and R_1 and R_2 have the same residue field $R_1/\mathfrak{m}_1 = R_2/\mathfrak{m}_2 = \mathbb{F}$. If $a_1 \in R_1$ and $a_2 \in R_2$, then a_1 and a_2 can be uniquely written as follows:

$$a_1 = a_{1,0} + a_{1,1}\gamma_1 + \cdots + a_{1,e_1-1}\gamma_1^{e_1-1}, \quad a_2 = a_{2,0} + a_{2,1}\gamma_2 + \cdots + a_{2,e_2-1}\gamma_2^{e_2-1},$$

where the $a_{1,i}$s and $a_{2,i}$s can be viewed as elements in $\mathbb{F}$ (see [14, Lemma 2]). Now define $\psi : R_2 \to R_1$ by $\psi(\sum_{i=0}^{e_2-1} a_i\gamma_2^i) = \sum_{i=0}^{e_1-1} a_i\gamma_1^i$. It is easy to see that ψ is a ring homomorphism. Hence R_1 is an R_2-algebra. For positive integers α and β, an R_2-submodule $C \subseteq R_1^\alpha \times R_2^\beta$ is called an R_1R_2-additive code. When α and β are coprime integers with $\mathrm{Char}(R_i/\mathfrak{m})$, the structure of these codes have been given (see [11, Theorem 4.3]). Now we obtain the structure of these codes as a result of the structure of SR-additive codes.

Corollary 3.9 (Additive cyclic codes over direct product of finite chain rings)**.** *With above assumptions, let $C \subseteq (R_1)_\alpha \times (R_2)_\beta$ be an R_1R_2-additive cyclic code. If α and β are coprime integers with* $\mathrm{Char}(R_i/\mathfrak{m})$*, Then*

(1) $C = \langle (h'_0 + \gamma_1 h'_1 + \cdots + \gamma_1^{e_1-1}h'_{e_1-1}, 0), (\ell(x), h_0 + \gamma_2 h_1 + \cdots + \gamma_2^{e_2-1}h_{e_2-1})\rangle$, *where*
  - (a) $h_0, h_1, \ldots, h_{e_2-1}$ *are polynomials in* $R_2[x]$ *such that* $h_0|(x^\beta - 1)$ *and* $h_i|h_{i-1}$ *for* $i = 1, \ldots, e_2 - 1$,
  - (b) $h'_0, h'_1, \ldots, h'_{e_1-1}$ *are polynomials in* $R_1[x]$ *such that* $h'_0|(x^\alpha - 1)$ *and* $h'_i|h'_{i-1}$ *for* $i = 1, \ldots, e_1 - 1$.

(2) $|C| = p^{d_1+d_2}$, *where* $d_1 = \sum_{i=0}^{e_2-1}(e_2-i)\deg \widehat{h}_i$ *and* $d_2 = \sum_{i=0}^{e_1-1}(e_1-i)\deg \widehat{h}'_i$.

Proof. By the same argument as Corollary 3.8, it follows from Theorem 3.3 and Theorem 2.8. □

Now we give new examples of SR-additive codes. First we give some examples of additive codes over direct products of chain rings that we can not obtain their structures by [11]; see Corollaries 3.10, 3.11 and 3.12. Note that in [11], they considered an additive code $C \subseteq R_1^\alpha \times R_2^\beta$ over the chain rings R_1 and R_2 in a case that α and β are coprime integers with $\mathrm{Char}(R_i/\mathfrak{m})$. But in the structure of SR-additive codes we haven't any restriction on α and β.

Let $R_1 = \mathbb{Z}_2$, $R_2 = \mathbb{Z}_2 + u\mathbb{Z}_2 = \{0, 1, u, 1+u\}$ such that $u^2 = 0$ and $R_3 = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2 = \{0, 1, u, 1+u, u^2, 1+u^2, 1+u+u^2, u+u^2\}$ such that $u^3 = 0$. By the following maps, R_i is an R_j-algebra for $1 \le i < j \le 3$.

$$\begin{aligned} f_{2,1} : R_2 &\longrightarrow R_1; &\quad \lambda_0 + \lambda_1 u &\longmapsto \lambda_0, \\ f_{3,1} : R_3 &\longrightarrow R_1; &\quad \lambda_0 + \lambda_1 u + \lambda_2 u^2 &\longmapsto \lambda_0, \\ f_{3,2} : R_3 &\longrightarrow R_2; &\quad \lambda_0 + \lambda_1 u + \lambda_2 u^2 &\longmapsto \lambda_0 + \lambda_1 u. \end{aligned}$$

We want to describe R_iR_j-additive cyclic codes for $1 \le i < j \le 3$. First we find the generators of R_1R_2-additive cyclic codes which are known as $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes and studied in [3, 22].

Corollary 3.10 (R_1R_2-additive cyclic codes)**.** *Let $C \subseteq (R_1)_\alpha \times (R_2)_\beta$ be an R_1R_2-additive cyclic code.*

(1) *If β is odd, then $C = \langle (h(x), 0), (\ell(x), g(x) + ua(x)) \rangle$ such that $h(x) | (x^\alpha - 1) \mod 2$, $\ell(x) \in (\mathbb{Z}_2)_\alpha$ and $g(x) + ua(x) \in (R_2)_\beta$ with the same condition as the part* (1) *of Theorem* 2.10*.*

(2) *If β is not odd, then*
  (a) *$C = \langle (h(x), 0), (\ell(x), g(x) + up(x)) \rangle$, where $h(x)$ and $\ell(x)$ are such as* (1)*. $g(x)$ and $p(x)$ have the same conditions as Theorem* 2.10 *part* 2(a)*. Or*
  (b) *$C = \langle (h(x), 0), (\ell_1(x), g(x) + up(x)), (\ell_2(x), ua(x)) \rangle$, where $h(x)$ and $\ell_i(x)$ are such as* (1)*. $g(x)$, $p(x)$ and $a(x)$ have the same conditions as Theorem* 2.10 *part* 2(b)*.*

Proof. By Corollary 3.6, $C = \langle (h(x), 0), (\ell_1, f_1), \ldots, (\ell_r, f_r) \rangle$, where $h(x)$ is a monic polynomial over R_1 such that $h(x) | (x^\alpha - 1)$. Also $C_1 = \langle f_1, \ldots, f_r \rangle$ is a linear cyclic code over R_2. Now we have the result by Theorem 2.10. □

Corollary 3.11 (R_1R_3-additive cyclic codes)**.** *Let $C \subseteq (R_1)_\alpha \times (R_3)_\beta$ be an R_1R_3-additive cyclic code.*

(1) *If β is odd, then $C = \langle (h(x), 0), (\ell(x), g(x) + ua_1(x) + u^2a_2(x)) \rangle$, where $h(x)$, $\ell(x)$ are elements of $\mathbb{Z}_2[x]$, $h(x) | (x^\alpha - 1)$ in $\mathbb{Z}_2[x]$ and g, a_1, a_2 have the same conditions as Theorem* 2.11 *part* (1)*.*

(2) *If β is not odd, then*
  (a) *$C = \langle (h(x), 0), (\ell(x), g(x) + up_1(x) + u^2p_2(x)) \rangle$, where ℓ, h are such as* (1) *and g, p_1, p_2 have the same conditions as Theorem* 2.11 *part* 2(a)*.*
  (b) *$C = \langle (h(x), 0), (\ell_1(x), g(x) + up_1(x) + u^2p_2(x)), (\ell_2(x), u^2a_2(x)) \rangle$, where ℓ_i and h are such as* (1) *and g, p_1, p_2, a_2 have the same conditions as Theorem* 2.11 *part* 2(b)*.*

(c) $C = \langle (h(x), 0), (\ell_1(x), g(x) + up_1(x) + u^2p_2(x)), (\ell_2(x), ua_1(x) + u^2q_1(x)), (\ell_3, u^2a_2(x))\rangle$, *where* ℓ_i *and* h *are such as* (1) *and* g, p_1, p_2, a_1, q_1, a_2 *have the same conditions as Theorem* 2.11 *part* 2(c).

Proof. By the same argument as Corollary 3.10, it follows from Corollary 3.6 and Theorem 2.11. □

Corollary 3.12 (R_2R_3-additive cyclic codes)**.** *Let* $C \subseteq (R_2)_\alpha \times (R_3)_\beta$ *be an* R_2R_3*-additive cyclic code.*

(1) *If* β *and* α *are odd, then* $C = \langle (h(x), 0), (\ell(x), g(x)+ua_1(x)+u^2a_2(x))\rangle$, *where* $h(x)$, $\ell(x)$ *are elements of* $(R_2)_\alpha$. $h(x)$ *is a generator of a code such as Theorem* 2.10 *part* (1) *and* g, a_1, a_2 *have the same conditions as Theorem* 2.11 *part* (1).

(2) *If* β *is odd and* α *is not odd, then*
 (a) $C = \langle (g + up, 0), (\ell, f)\rangle$, *where* g, p *have the same conditions as Theorem* 2.10 *part* 2(*a*). $\ell \in (R_2)_\alpha$ *and* $f \in (R_3)_\beta$ *is a generator of a code such as Theorem* 2.11 *part* (1). *Or*
 (b) $\langle (g + up, 0), (ua, 0), (\ell, f)\rangle$, *where* g, p, a *are polynomials with the same conditions as Theorem* 2.10 *part* 2(*b*). $\ell \in (R_2)_\alpha$ *and* $f \in (R_3)_\beta$ *is a generator of a code such as Theorem* 2.11 *part* (1).

(3) *If* α *is odd and* β *is not odd, then*
 (a) $\langle (f, 0), (\ell, g + ua_1 + u^2a_2)\rangle$, *where* $\ell \in (R_2)_\alpha$, f *is a generator of a code such as Theorem* 2.10 *part* (1) *and* g, a_1, a_2 *are such as Theorem* 2.11 *part* 2(a). *Or*
 (b) $C = \langle (f, 0), (\ell_1, g + up_1 + u^2p_2), (\ell_2, u^2a_2)\rangle$, *where* f *and* ℓ_i *are such as* (a) *and* g, p_1, p_2, a_2 *have the same conditions as Theorem* 2.11 *part* 2(b). *Or*
 (c) $C = \langle (f, 0), (\ell_1, g + up_1 + u^2p_2), (\ell_2, ua_1 + u^2q_1)(\ell_3, u^2a_2)\rangle$, *where* f *and* ℓ_i *are such as* (a) *and* $g, p_1, p_2, a_1, a_2, q_1$ *have the same conditions as Theorem* 2.11 *part* 2(c).

(4) *If* α *and* β *are not odd, then we have one of the following states.*
 (a) $C = \langle (g_1, 0), (\ell_1, f_1)\rangle$, *where* g_1 *is a generator of a code in Theorem* 2.10 *part* 2(a), f_1 *is a generator of a code in Theorem* 2.11 *part* 2(a) *and* ℓ_1 *is an elements of* $(R_2)_\alpha$.
 (b) $C = \langle (g_1, 0), (\ell_1, f_1), (\ell_2, f_2)\rangle$, *where* g_1 *is a generator of a code in Theorem* 2.10 *part* 2(a), f_i *are generators of a code in Theorem* 2.11 *part* 2(b) *and* ℓ_i *are elements of* $(R_2)_\alpha$.
 (c) $C = \langle (g_1, 0), (\ell_1, f_1), (\ell_2, f_2), (\ell_3, f_3)\rangle$, *where* g_1 *is a generator of a code in Theorem* 2.10 *part* 2(a), f_i *are generators of a code in Theorem* 2.11 *part* 2(c) *and* ℓ_i *are elements of* $(R_2)_\alpha$.
 (d) $C = \langle (g_1, 0), (g_2, 0), (\ell_1, f_1)\rangle$, *where* g_i *are generators of a code in Theorem* 2.10 *part* 2(b), f_1 *is a generator of a code in Theorem* 2.11 *part* 2(a) *and* ℓ_1 *is an element of* $(R_2)_\alpha$.

(e) $C = \langle (g_1, 0), (g_2, 0), (\ell_1, f_1), (\ell_2, f_2)\rangle$, *where* g_i *are generators of a code in Theorem* 2.10 *part* 2(b), f_i *are generators of a code in Theorem* 2.11 *part* 2(b) *and* ℓ_i *is an element of* $(R_2)_\alpha$.

(f) $C = \langle (g_1, 0), (g_2, 0), (\ell_1, f_1), (\ell_2, f_2), (\ell_3, f_3)\rangle$, *where* g_i *are generators of a code in Theorem* 2.10 *part* 2(b). f_i *are generators of a code in Theorem* 2.11 *part* 2(c) *and* ℓ_i *are elements of* $(R_2)_\alpha$.

Proof. By Theorem 3.3, $C = \langle (g_1, 0), \dots, (g_s, 0), (h_1, f_1), \dots, (h_r, f_r)\rangle_{R[x]}$ such that $C_1 = \langle f_1, \dots, f_r\rangle_{R_3[x]}$ is a cyclic linear code over R_3 of length β and $C_2 = \langle g_1, \dots, g_s\rangle_{R_3[x]}$ is a cyclic R_3-additive code over R_2 of length α. Since $f_{3,2} : R_3 \to R_2$ is a surjective map, C_2 is a linear code over R_2. Now the result follows from Theorems 2.10 and 2.11. □

Now we give some examples that the ring R in SR-additive codes is not a chain ring (moreover this ring is not a Frobenius ring). Let $R_4 = \mathbb{Z}_2 + u\mathbb{Z}_2 + v\mathbb{Z}_2 = \{0, 1, u, v, 1+u, 1+v, u+v, 1+u+v\}$ such that $u^2 = v^2 = uv = 0$. This ring is not a chain ring. Moreover R_4 is a non Frobenius ring. Consider the rings $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_2 + u\mathbb{Z}_2$ in above corollaries. It is easy to see that the following maps are ring homomorphisms:

$$f_{4,1} : R_4 \longrightarrow R_1; \qquad \lambda_1 + \lambda_2 u + \lambda_3 v \longmapsto \lambda_1,$$
$$f_{4,2} : R_4 \longrightarrow R_2; \qquad \lambda_1 + \lambda_2 u + \lambda_3 v \longmapsto \lambda_1 + \lambda_2 u.$$

Hence R_4 is an R_i-algebra for $i = 1, 2$. Now we want to describe R_1R_4 and R_2R_4-additive cyclic codes.

Corollary 3.13 (R_1R_4-additive cyclic codes)**.** *Let* $C \subseteq (R_1)_\alpha \times (R_4)_\beta$ *be an* R_1R_4*-additive cyclic code. Then* $C = \langle (f, 0), (h_1, g + up_1 + vp_2), (h_2, ua_1 + vq_1), (h_3, va_2)\rangle$, *where* $f|(x^\alpha - 1)$, $h_i \in (R_1)_\alpha$ *and* p_1, p_2, q_1, a_1, a_2 *have the same conditions as Theorem 2.12. Moreover if* β *is odd, then* $C = \langle (f, 0), (h_1, g + ua_1), (h_2, va_2)\rangle$, *where* $a_2|a_1|g|(x^n - 1)$.

Proof. It follows from Corollary 3.6 and Theorem 2.12. □

Corollary 3.14 (R_2R_4-additive cyclic codes)**.** *Let* $C \subseteq (R_2)_\alpha \times (R_4)_\beta$ *be an* R_2R_4*-additive cyclic code. Then*

(1) *If* α *is odd, then* $C = \langle (g + ua, 0), (h_1, g_1 + up_1 + vp_2), (h_2, ua_1 + vq_1), (h_3, va_2)\rangle$, *where* g *and* a *are polynomials in* $\mathbb{Z}_2[x]$ *such that* $a|g|(x^\alpha - 1) \mod 2$, $h_i \in (R_2)_\alpha$ *and* $p_1, p_2, q_1, g_1, a_1, a_2$ *have the same conditions as Theorem 2.12.*

(2) *If* α *is not odd, then*

(a) $C = \langle (g + up, 0), (h_1, g_1 + up_1 + vp_2), (h_2, ua_1 + vq_1), (h_3, va_2)\rangle$ *such that* $g|(x^\alpha - 1) \mod 2$, $(g + up)|(x^\alpha - 1)$ *in* $\mathbb{Z}_2 + u\mathbb{Z}_2$ *and* $g|p(\frac{x^\alpha - 1}{g})$. *Or*

(b) $C = \langle (ua, 0), (g + up, 0), (h_1, g_1 + up_1 + vp_2), (h_2, ua_1 + vq_1), (h_3, va_2)\rangle$ *such that* g, a *and* p *are polynomials in* $\mathbb{Z}_2[x]$. $a|g|(x^\alpha - 1) \mod 2$, $a|p(\frac{x^\alpha - 1}{g})$ *and* $\deg a > \deg p$.

Where $h_i \in (R_2)_\alpha$ and $p_1, p_2, q_1, g_1, a_1, a_2$ have the same conditions as Theorem 2.12.

Proof. It follows from Theorem 3.3 and Theorem 2.12. $\square$

In the above examples the ring homomorphisms between R_i and R_j are surjective, hence cyclic R_iR_j-additive codes are constructed by linear cyclic codes over R_i and R_j. But when f is not surjective to construct cyclic SR-additive codes we need the structure of R-additive codes over S. See the following examples.

Example 3.15. Let $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_2 + u\mathbb{Z}_2$ be the rings in above corollaries. Then R_2 is an R_1-algebra with the including map. Let $C \subseteq (R_2)_\alpha \times (R_1)_\beta$ be an R_2R_1-additive cyclic code. Then $C = \langle (g_1, 0), \dots, (g_s, 0), (h, f)\rangle$, where $f|(x^\beta - 1)$, $h \in (R_2)_\alpha$, and $C_1 = \langle g_1, \dots, g_s\rangle$ is a cyclic R_1-additive code over R_2 (C_1 is an additive cyclic code over R_2).

Example 3.16. Let $R = GR(p^s, m)$ and $S = R[\xi] = GR(p^s, m\ell)$ be the Galois extension of R. Then S is an R-algebra with the including map. Let $C \subseteq S_\alpha \times R_\beta$ be an SR-additive cyclic code. If $\gcd(\beta, p) = 1$ and $\gcd(\alpha, p) = 1$, then $C = \langle (g_1, 0), \dots, (g_\ell, 0), (h, f)\rangle$, where $C_2 = \langle f\rangle$ is a cyclic code over R, $C_1 = \langle g_1, \dots, g_\ell\rangle$ is a cyclic R-additive code over S of length α and $h \in S_\alpha$ is a polynomial.

4. Duality of *SR*-additive codes

In this section, we define a bilinear form on SR-additive codes which is a generalization of the bilinear forms over $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes in [2], $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes in [3] and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive cyclic codes in [10].

Definition 4.1. Let $\tau : S \to R$ be an R-module homomorphism, then

$$\begin{array}{rcl} \beta' : (S^\alpha \times R^\beta) \times (S^\alpha \times R^\beta) & \longrightarrow & R \\ ((x_1, y_1), (x_2, y_2)) & \longmapsto & \tau(x_1.x_2) + (y_1.y_2) \end{array}$$

is an R-bilinear form where $x_1.x_2$ and $y_1.y_2$ are standard inner products. For an SR-additive code C, $C^\perp$ is the dual of C with respect to β'.

Proposition 4.2. *Let R be a chain ring with maximal ideal $\mathfrak{m} = \langle\gamma\rangle$ of nilpotency index e. If β' is a bilinear form on $(R/\mathfrak{m})R$-additive codes defined by an R-module homomorphism $\tau : R/\mathfrak{m} \to R$, then there is a unit element $a \in R$ such that*

$$\begin{array}{rcl} \beta' : ((R/\mathfrak{m})^\alpha \times R^\beta) \times ((R/\mathfrak{m})^\alpha \times R^\beta) & \longrightarrow & R \\ ((\overline{x}_1, y_1), (\overline{x}_2, y_2)) & \longmapsto & a\gamma^{e-1}(x_1.x_2) + (y_1.y_2). \end{array}$$

Where $\overline{x}_1 = (x_{1,i} + \mathfrak{m})$, $\overline{x}_2 = (x_{2,i} + \mathfrak{m})$, and $x_1 = (x_{1,i})$ and $x_2 = (x_{2,i})$.

Proof. By the definition of β', it suffices to determine $\mathrm{Hom}_R(R/\mathfrak{m}, R)$. But we have the following R-module isomorphism

$$\begin{array}{rcl}\mathrm{Hom}_R(R/\mathfrak{m}, R) & \longrightarrow & \mathrm{Ann}_R(\mathfrak{m}) \\ \tau & \longmapsto & \tau(1+\mathfrak{m}).\end{array}$$

Since R is a chain ring and $\mathrm{Ann}_R(\mathfrak{m})$ is an ideal of R, $\mathrm{Ann}_R(\mathfrak{m}) = \langle \gamma^j \rangle$ for some $j; 1 \leq j \leq e$. Clearly $\gamma^{e-1}\mathfrak{m} = 0$. On other hand $\gamma^{e-2}\gamma \neq 0$. Hence $\mathrm{Ann}_R(\mathfrak{m}) = \langle \gamma^{e-1} \rangle$. Thus there is a unit element $a \in R \setminus \mathfrak{m}$ such that $\tau(1+\mathfrak{m}) = a\gamma^{e-1}$. Hence for $r + \mathfrak{m} \in R/\mathfrak{m}$, $\tau(r+\mathfrak{m}) = r\tau(1+\mathfrak{m}) = ra\gamma^{e-1}$. This completes the proof. □

Now we give some examples of this bilinear form over SR-additive codes, which we see some of them in [2] and [3].

Corollary 4.3 (The bilinear form of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes)**.** *The following bilinear form is the only form on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes defined by Definition 4.1.*

$$\beta' : (\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta) \times (\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta) \longrightarrow \mathbb{Z}_4,\ ((x_1, y_1), (x_2, y_2)) \longmapsto 2(x_1.x_2) + (y_1.y_2).$$

Where the elements x_1 and x_2 in the inner product $2(x_1.x_2)$ are considered as elements of $\mathbb{Z}_4^\beta$; naturally.

Proof. $\mathbb{Z}_4$ is a chain ring with maximal ideal $2\mathbb{Z}_4$ of nilpotency index 2. Also $\frac{\mathbb{Z}_4}{2\mathbb{Z}_4} \cong \mathbb{Z}_2$. Now we have the result by Proposition 4.2. □

Proposition 4.4 (The bilinear forms of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes, $r < s$)**.** *Let β' be a bilinear form on $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes defined by Definition 4.1. Then β' is defined as follows*:

$$\begin{array}{l}\beta' : (\mathbb{Z}_{p^r}^\alpha \times \mathbb{Z}_{p^s}^\beta) \times (\mathbb{Z}_{p^r}^\alpha \times \mathbb{Z}_{p^s}^\beta) \longrightarrow \mathbb{Z}_{p^s}, \\ \qquad ((x_1, y_1), (x_2, y_2)) \qquad \longmapsto ap^{s-r}(x_1.x_2) + (y_1.y_2),\end{array}$$

where $a \in \mathbb{Z}_{p^s}$ and the elements x_1 and x_2 in the inner product $ap^{s-r}(x_1.x_2)$ are considered as elements of $\mathbb{Z}_{p^s}^\beta$; naturally.

Proof. $\mathrm{Hom}_{\mathbb{Z}_{p^s}}(\mathbb{Z}_{p^r}, \mathbb{Z}_{p^s}) = \mathrm{Hom}_{\mathbb{Z}_{p^s}}(\frac{\mathbb{Z}_{p^s}}{p^r\mathbb{Z}_{p^s}}, \mathbb{Z}_{p^s}) \cong \mathrm{Ann}_{\mathbb{Z}_{p^s}}(p^r\mathbb{Z}_{p^s}) = \langle p^{s-r} \rangle$. Now by the same argument of Proposition 4.2 we have the result. □

Let R_1 and R_2 be the finite chain rings with the assumptions of Corollary 3.9. We have the isomorphism $\psi : \frac{R_2}{\gamma_2^{e_1}R_2} \to R_1$. Let $p : R_2 \to \frac{R_2}{\gamma_2^{e_1}R_2}$ be defined naturally. Hence $\iota = p^{-1}\psi^{-1} : R_1 \to R_2$ is well defined, where p^{-1} is a right inverse of p. The following proposition gives the bilinear forms over direct product of chain rings.

Proposition 4.5 (The bilinear forms of additive codes over product of chain rings)**.** *Let R_1 and R_2 be the finite chain rings with the assumptions Corollary*

3.9. If β' is a bilinear form on R_1R_2-additive codes defined by Definition 4.1, then β' is defined as follows:

$$\beta' : (R_1^\alpha \times R_2^\beta) \times (R_1^\alpha \times R_2^\beta) \longrightarrow R_2,$$
$$((x_1, y_1), (x_2, y_2)) \longmapsto a\gamma^{e_2-e_1}\iota(x_1.x_2) + (y_1.y_2),$$

where $a \in R_2$.

Proof. $\mathrm{Hom}_{R_2}(R_1, R_2) = \mathrm{Hom}_{R_2}(\frac{R_2}{\gamma_2^{e_1}R_2}, R_2) \cong \mathrm{Ann}_{R_2}(\gamma_2^{e_1}R_2) = \gamma_2^{e_2-e_1}R_2$. Now by the same argument of Proposition 4.2 we have the result. □

Proposition 4.6 (The bilinear forms of R_iR_j-additive codes, $i < j$)**.** *Let R_i and R_j be such as Corollaries 3.10, 3.11, 3.12, 3.13, 3.14. Then, we have the following bilinear forms on R_iR_j-additive codes.*

$$\beta_{1,2} : (R_1^\alpha \times R_2^\beta) \times (R_1^\alpha \times R_2^\beta) \to R_2, ((x_1, y_1), (x_2, y_2)) \mapsto u(x_1.x_2) + y_1.y_2,$$
$$\beta_{1,3} : (R_1^\alpha \times R_3^\beta) \times (R_1^\alpha \times R_3^\beta) \to R_3, ((x_1, y_1), (x_2, y_2)) \mapsto u^2(x_1.x_2) + y_1.y_2,$$
$$\beta_{2,3} : (R_2^\alpha \times R_3^\beta) \times (R_2^\alpha \times R_3^\beta) \to R_3, ((x_1, y_1), (x_2, y_2)) \mapsto au(x_1.x_2) + y_1.y_2,$$
$$\beta_{1,4} : (R_1^\alpha \times R_4^\beta) \times (R_1^\alpha \times R_4^\beta) \to R_4, ((x_1, y_1), (x_2, y_2)) \mapsto h(x_1.x_2) + y_1.y_2,$$
$$\beta_{2,4} : (R_2^\alpha \times R_4^\beta) \times (R_2^\alpha \times R_4^\beta) \to R_4, ((x_1, y_1), (x_2, y_2)) \mapsto h(x_1.x_2) + y_1.y_2,$$

where $a \in R_3$, $h \in R_4u + R_4v$.

Proof. R_2 and R_3 are chain rings with maximal ideals $R_2\langle u\rangle$ and $R_3\langle u\rangle$ of nilpotency indices 2 and 3; respectively. Also $\frac{R_2}{\langle u\rangle} \cong \frac{R_3}{\langle u\rangle} \cong R_1$. Hence we have the bilinear forms $\beta_{1,2}$ and $\beta_{1,3}$ by Proposition 4.2. To obtain $\beta_{2,3}$, $\beta_{1,4}$ and $\beta_{2,4}$ note that

$$\mathrm{Hom}_{R_3}(R_2, R_3) \cong \mathrm{Hom}_{R_3}(\frac{R_3}{\langle u^2\rangle}, R_3) \cong \mathrm{Ann}_{R_3}(\langle u^2\rangle) = \langle u\rangle,$$
$$\mathrm{Hom}_{R_4}(R_1, R_4) \cong \mathrm{Hom}_{R_4}(\frac{R_4}{\langle u, v\rangle}, R_4) \cong \mathrm{Ann}_{R_4}(\langle u, v\rangle) = R_4u + R_4v,$$
$$\mathrm{Hom}_{R_4}(R_2, R_4) \cong \mathrm{Hom}_{R_4}(\frac{R_4}{\langle v\rangle}, R_4) \cong \mathrm{Ann}_{R_4}(\langle v\rangle) = R_4u + R_4v.$$

Now by the same argument of the proof of Proposition 4.2 we have the result. □

Proposition 4.7. *Let $\tau : S \to R$ be an R-module homomorphism and $C \subseteq S^\alpha \times R^\beta$ be an SR-additive cyclic code. If $C^\perp$ is the dual of C with respect to the bilinear form defined by τ in Definition 4.1, then $C^\perp$ is an SR-additive cyclic code.*

Proof. Clearly $C^\perp$ is an R-submodule of $S^\alpha \times R^\beta$, hence $C^\perp$ is an SR-additive code. Now let

$$(x, y) = (x_0 \cdots x_{\alpha-1}, y_0 \cdots y_{\beta-1}) \in C^\perp \text{ and}$$

$$\phi(x,y) = (x_{\alpha-1}\cdots x_{\alpha-2}, y_{\beta-1}\cdots y_{\beta-2}).$$

Let $j = \mathrm{lcm}(\alpha,\beta)$ and $(v,w) \in C$. Since C is cyclic, $\phi^{j-1}(v,w) \in C$. Now

$$\begin{aligned}(v,w).\phi(x,y) &= \tau(v.\phi(x)) + w.\phi(y)\\ &= \tau(\phi^{j-1}(v).x) + \phi^{j-1}(w).y\\ &= \phi^{j-1}(v,w).(x,y) = 0.\end{aligned}$$

Therefore $\phi(x,y) \in C^{\perp}$ and hence $C^{\perp}$ is cyclic. □

5. Singleton bounds for SR-additive codes

Aydogdu and Siap obtained some bounds on the minimum distance of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes [4]. In this section, we generalize the definitions of weight functions and Gray maps on the classes of $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ and $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes to SR-additive codes. We obtain singleton bounds for SR-additive codes. As results, singleton bounds for $\mathbb{Z}_2\mathbb{Z}_2[u]$ and $\mathbb{Z}_2[u]\mathbb{Z}_2$-additive codes are given.

Definition 5.1. Let T be a commutative finite ring. For every $x = (x_1,\ldots,x_n) \in T^n$ and $t \in T$, the complete weight of x is defined by

$$n_t(x) := |\{i : x_i = t\}|.$$

For $t \in T\setminus\{0\}$, let a_t be a positive integer, and set $a_0 = 0$. The general weight function over T is defined as follows:

$$\omega_T(x) := \sum_{t\in T} a_t n_t(x).$$

Now let ω_R and ω_S be two weight functions over R and S. A weight function ω over $S^\alpha \times R^\beta$ is defined as follows: for $(x,y) \in S^\alpha \times R^\beta$, $\omega(x,y) = \omega_S(x) + \omega_R(y)$.

Definition 5.2. Let $n_s \in \mathbb{N}$ be a positive integer. A map $\phi : R \to S^{n_S}$ with the following conditions is called a gray map:
(a) ϕ is injective.
(b) for $x,y \in R$, $\omega_R(x-y) = \omega_S(\phi(x)-\phi(y))$.

A gray map ϕ is called R-linear if ϕ is an R-module homomorphism. ϕ generalize on R^β naturally; for $x = (x_1,\ldots,x_\beta) \in R^\beta$, $\phi(x) = (\phi(x_1),\ldots,\phi(x_\beta)) \in S^{n_s\beta}$. We generalize ϕ to a map Φ over $S^\alpha \times R^\beta$ as follows:

$$\begin{aligned}\Phi : S^\alpha \times R^\beta &\longrightarrow S^{\alpha+n_s\beta}\\ (x,y) &\longmapsto (x,\phi(y)).\end{aligned}$$

Clearly for any $(x,y) \in S^\alpha \times R^\beta$, $\omega(x,y) = \omega_S(\Phi(x,y))$. Moreover Φ is an injective map. Now let $C \subseteq S^\alpha \times R^\beta$ be an SR-additive code, the minimum general weight of C is

$$d_\omega(C) := \min\{\omega(x,y) : \ (x,y) \in C\setminus\{0\}\}.$$

Let $A_s = \max\{a_s : s \in S\}$. The following theorem gives singleton bounds for SR-additive codes.

Theorem 5.3. *With above notations, let R be a chain ring and S be an R-algebra with a ring homomorphism $f : R \to S$. If $C \subseteq S^\alpha \times R^\beta$ is an SR-additive code such that $\Phi(C) \subseteq S^{\alpha+n_s\beta}$ is an R-additive code, then*

(1) *If S is a principal ideal ring and f is surjective, then $\lfloor \frac{d_\omega(C)-1}{A_s} \rfloor \leq \alpha + n_s\beta - \mathrm{rank}(C)$.*
(2) *If S is a free R-algebra of dimension m, then $\lfloor \frac{d_\omega(C)-1}{A_s} \rfloor \leq \alpha + n_s\beta - \lceil \frac{\mathrm{rank}(C)}{m} \rceil$.*

Proof. (1) $\Phi(C)$ is an R-additive code. Since f is surjective, hence $\Phi(C)$ is a linear code over S. If $d_{\omega_s}(\Phi(C))$ is the minimum weight of $\Phi(C)$ with respect to the weight function ω_S, then by Theorem 3.7 of [20], we have that $\lfloor \frac{d_{\omega_s}(\Phi(C))-1}{A_s} \rfloor \leq \alpha + n_s\beta - \mathrm{rank}(\Phi(C))$. But $d_{\omega_s}(\Phi(C)) = d_\omega(C)$ and $\mathrm{rank}(\Phi(C)) = \mathrm{rank}(C)$. This completes the proof of part (1).

(2) $\Phi(C)$ is an R-additive code and S is a free R-algebra. Hence by Theorem 2.7, $\lfloor \frac{d_{\omega_s}(\Phi(C))-1}{A_s} \rfloor \leq \alpha + n_s\beta - \lceil \frac{\mathrm{rank}(\Phi(C))}{m} \rceil$. Since $d_{\omega_s}(\Phi(C)) = d_\omega(C)$ and $\mathrm{rank}(\Phi(C)) = \mathrm{rank}(C)$, we have the result. □

Corollary 5.4. *With above assumptions, let $\omega_S = \omega_H$ be the Hamming weight. Then*

(1) *If S is a free R-algebra of dimension m, then $d_\omega(C) \leq \alpha + n_s\beta - \lceil \frac{\mathrm{rank}(C)}{m} \rceil + 1$.*
(2) *If S is a principal ideal ring and f is surjective, then $d_\omega(C) \leq \alpha + n_s\beta - \mathrm{rank}(C) + 1$.*

Remark 5.5. Let R be a finite commutative ring and S be a finite R-algebra with a surjective ring homomorphism $f : R \to S$. With above assumptions, if $\omega_S = \omega_H$ is the Hamming weight, then $d_\omega(C) \leq \alpha + n_s\beta - \log_{|S|} |C| + 1$.

Proof. Since f is surjective, $\Phi(C)$ is a linear code over S. By the singleton bound for linear codes we have the result. □

Example 5.6. Consider $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes. The Lee weight over $\mathbb{Z}_2[u] = \{0, 1, u, 1+u\}$ is defined as follows:

$$\omega_L(0) = 0,\ \omega_L(1) = 1,\ \omega_L(u) = 2,\ \omega_L(1+u) = 1.$$

For any element $(x, y) = (x_0, \ldots, x_{\alpha-1}; y_0, \ldots, y_{\beta-1}) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_2[u]^\beta$, the weight function ω is defined in the following way:

$$\omega(x, y) = \sum_{i=0}^{\alpha-1} \omega_H(x_i) + \sum_{i=0}^{\beta-1} \omega_L(y_i),$$

where ω_H is the hamming weight over $\mathbb{Z}_2$ and ω_L is the Lee weight over $\mathbb{Z}_2[u]$. Now we have the following Gray map:

$$\begin{array}{rcl}\phi : \mathbb{Z}_2[u] & \longrightarrow & \mathbb{Z}_2^2 \\ a+bu & \longmapsto & (b, a+b).\end{array}$$

It is easy to see that $\omega_L(a+bu) = \omega_H(b, a+b)$ for any element $a+bu \in \mathbb{Z}_2[u]$. This map generalizes to the Gray map Φ:

$$\begin{array}{rcl}\phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_2[u]^\beta & \longrightarrow & \mathbb{Z}_2^{\alpha+2\beta} \\ (x,y) & \longmapsto & (x, \phi(y)).\end{array}$$

Clearly $\omega(x,y) = \omega_H(\phi(x,y))$. Now if $C \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_2[u]^\beta$ is a $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive code, then we have the following bounds for minimum weight $d_\omega(C)$:

$$d_\omega(C) \leq \alpha + 2\beta - \mathrm{rank}(C) + 1,$$
$$d_\omega(C) \leq \alpha + 2\beta - \log_2|C| + 1.$$

Example 5.7. Consider $\mathbb{Z}_2[u]\mathbb{Z}_2$-additive codes in Example 3.15. The subset $C \subseteq \mathbb{Z}_2[u]^\alpha \times \mathbb{Z}_2^\beta$ is a $\mathbb{Z}_2[u]\mathbb{Z}_2$-additive code if and only if C is a subgroup under addition. For any element $(x,y) \in \mathbb{Z}_2[u]^\alpha \times \mathbb{Z}_2^\beta$, the weight function ω is defined as follows:

$$\omega(x,y) = \omega_L(x) + \omega_H(y),$$

where ω_L is the Lee weight over $\mathbb{Z}_2[u]$ in above example and ω_H is the Hamming weight over $\mathbb{Z}_2$. Let $j : \mathbb{Z}_2 \to \mathbb{Z}_2[u]$ be the including map. We define a Gray map as follows:

$$\begin{array}{rcl}\Phi : \mathbb{Z}_2[u]^\alpha \times \mathbb{Z}_2^\beta & \longrightarrow & \mathbb{Z}_2[u]^{\alpha+\beta} \\ (x,y) & \longmapsto & (x, j(y)).\end{array}$$

It is easy to see that $\omega(x,y) = \omega_L(\Phi(x,y))$. Since $\mathbb{Z}_2[u]$ is a free $\mathbb{Z}_2$-algebra of dimension 2, by Theorem 5.3, we have the following bound for minimum weight:

$$\lfloor \frac{d_\omega(C)-1}{2} \rfloor \leq \alpha + \beta - \lceil \frac{\mathrm{rank}(C)}{2} \rceil.$$

6. One weight SR-additive codes

Recently, Dougherty et al. described one weight $\mathbb{Z}_2\mathbb{Z}_4$-additive codes [15]. In this section, we generalize this theory over SR-additive codes where S and R are chain rings. As applications of the theory, we obtain some results on one weight $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes (with respect to homogeneous weight) and one weight $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}$-additive codes (with respect to Lee weight). In particular, we obtain the structure of one weight $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes. First we remind the following definition of a pre-homogeneous weight in [23].

Definition 6.1. Let T be a commutative finite ring. A weight function $\omega_T : T \to \mathbb{R}$ is pre-homogeneous if $a_0 = 0$ and there exists a constant $c_T > 0$ such that for $t \neq 0$,

$$\sum_{t' \in \langle t \rangle} a_{t'} = c_T |\langle t \rangle|,$$

where $\langle t \rangle$ is the principal ideal generated by an element t of T. In this case c_T is called the average weight.

Example 6.2 ([23, Example 3.7])**.** Let $R = \mathbb{Z}_{2^s}$. Then Lee weight is pre-homogeneous with average weight $c_R = 2^{s-2}$.

Lemma 6.3. *Let R and S be two chain rings, where S is an R-algebra with a surjective ring homomorphism $f : R \to S$. Also let ω_S and ω_R be two pre-homogeneous weights with average weights c_R and c_S. If $C \subseteq S^\alpha \times R^\beta$ is an SR-additive code with no all zero columns, then*

$$\sum_{c \in C} \omega(c) = |C|(\alpha c_S + \beta c_R),$$

where ω is the weight function defined by ω_S and ω_R over $S^\alpha \times R^\beta$.

Proof. Let S be a chain ring with maximal ideal $\mathfrak{m} = \langle \gamma \rangle$ of nilpotency index υ. Write the codewords of C as rows of a matrix G. Consider the column j of G, where $1 \leq j \leq \alpha$. Let J be the ideal of S generated by all elements of the column j. Then there exists $1 \leq t \leq \upsilon$ that $J = \langle \gamma^t \rangle$. Since f is surjective and C is an R-submodule, any element of J is an element of the column j. Now we show that any two elements of J have the same repetition number in the column j. Consider two elements γ^t and γ^{t+1} of J with the repetition numbers n_t and n_{t+1}, respectively. Since $\gamma^{t+1} = \gamma\gamma^t$, hence $n_t \leq n_{t+1}$. On the other hand $\gamma^t(\gamma - 1) = \gamma^{t+1} - \gamma^t$. Since $\gamma - 1$ is invertible, $\gamma^t = (\gamma - 1)^{-1}(\gamma^{t+1} - \gamma^t)$. Hence $n_{t+1} \leq n_t$ and hence $n_t = n_{t+1}$. Thus all elements of J have the same repetition number $\frac{|C|}{|J|}$ in the column j. Therefore the sum of the weights of all elements of the column j is equal to

$$\frac{|C|}{|J|}(\sum_{s \in J} a_s) = \frac{|C|}{|J|}(c_S|J|) = |C||c_S|.$$

By the same argument, the sum of the weights of all elements of the columns of β coordinates is equal to $|C||c_R|$. Therefore

$$\sum_{c \in C} \omega(c) = |C|(\alpha c_S + \beta c_R).$$

□

Theorem 6.4. *With the assumptions of above lemma, let $C \subseteq S^\alpha \times R^\beta$ be a one weight SR-additive code with weight m such that there exists no zero columns in the generator matrix of C. Then there exists a unique positive integer λ such that $m = \lambda|C|$ and $\alpha c_S + \beta c_R = \lambda(|C| - 1)$.*

Proof. By above lemma, we have that

$$\sum_{c\in C}\omega(c) = |C|(\alpha c_S + \beta c_R).$$

On the other hand, the sum of the weights of all codewords is $(|C|-1)m$. Hence $|C|(\alpha c_S+\beta c_R) = (|C|-1)m$. But $\gcd(|C|, (|C|-1)) = 1$. Therefore there exists a positive integer λ such that $m = \lambda|C|$ and hence $\alpha c_S + \beta c_R = \lambda(|C|-1)$. $\square$

Let T be a finite chain ring with maximal ideal $\langle\gamma\rangle$, nilpotency index e, and residue field $T/\langle\gamma\rangle = \mathbb{F}_{p^k}$. A homogenous weight is defined as follows

$$\omega_{hom}(t) = \begin{cases} (p^k-1)p^{k(e-2)}, & t \in T \setminus \langle\gamma^{e-1}\rangle; \\ p^{k(e-1)}, & t \in \langle\gamma^{e-1}\rangle \setminus \langle 0\rangle; \\ 0, & t = 0. \end{cases}$$

Lemma 6.5. *With above assumptions, let T be a chain ring. Then ω_{hom} is pre-homogeneous with average weight $c_T = (p^k-1)p^{k(e-2)}$.*

Proof. Let $\langle t\rangle$ be an ideal of T. By the structure of chain rings,$\langle t\rangle = \langle\gamma^j\rangle$ for some $j; 1 \le j \le e$. Hence $|\langle\gamma^{e-1}\rangle| = |\langle\gamma^j\rangle| = p^{k(e-j)}$. Therefore

$$\begin{aligned}\sum_{t'\in\langle t\rangle} a_{t'} &= \sum_{t'\in\langle\gamma^j\rangle\setminus\langle\gamma^{e-1}\rangle} a_{t'} + \sum_{t'\in\langle\gamma^{e-1}\rangle} a_{t'} \\ &= (p^k-1)p^{k(e-2)}(|\langle\gamma^j\rangle| - |\langle\gamma^{e-1}\rangle|) + p^{k(e-1)}(|\langle\gamma^{e-1}\rangle| - 1) \\ &= (p^k-1)p^{k(e-2)}(p^{k(e-j)} - p^k) + p^{k(e-1)}(p^k-1) \\ &= (p^k-1)p^{k(e-2)}p^{k(e-j)} \\ &= c_T|\langle t\rangle|.\end{aligned}$$

This completes the proof. $\square$

Theorem 6.6. *Let ω be the weight function defined by ω_{hom} over $\mathbb{Z}_{p^r}$ and $\mathbb{Z}_{p^s}$ on $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes. If $C \subseteq \mathbb{Z}_{p^r}^{\alpha} \times \mathbb{Z}_{p^s}^{\beta}$ is a one weight $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive code with weight m such that there exists no zero columns in the generator matrix of C, then there exists a unique positive integer λ such that $m = \lambda|C|$ and $(p-1)p^{r-2}(\alpha + p^{s-r}\beta) = \lambda(|C|-1)$.*

Proof. By Lemma 6.5, $c_{\mathbb{Z}_{p^r}} = (p-1)p^{r-2}$ and $c_{\mathbb{Z}_{p^s}} = (p-1)p^{s-2}$. Now we have the result by Theorem 6.4. $\square$

By Example 6.2, the Lee weight over $\mathbb{Z}_{2^r}$ and $\mathbb{Z}_{2^s}$ is pre-homogeneous. Hence we have the following result on one weight $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}$-additive codes.

Theorem 6.7. *Let $C \subseteq \mathbb{Z}_{2^r}^{\alpha} \times \mathbb{Z}_{2^s}^{\beta}$ be a $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}$-additive code. Consider the weight ω defined by Lee weight over $\mathbb{Z}_{2^r}$ and $\mathbb{Z}_{2^s}$. If C is a one weight $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}$-additive code with weight m such that there exists no zero columns in the generator matrix of C, then there exists a unique positive integer λ such that $m = \lambda|C|$ and $2^{r-2}(\alpha + 2^{s-r}\beta) = \lambda(|C|-1)$.*

Proof. It follows from Example 6.2 and Theorem 6.4. □

The structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes is studied in [4]. If a $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive code $C \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$ is isomorphic to an abelian structure $\mathbb{Z}_2^{k_0+k_s} \times \mathbb{Z}_{2^s}^{k_1} \times \cdots \times \mathbb{Z}_4^{k_{s-1}}$, then we say that C is of type $(\alpha, \beta; k_0, k_1, k_2, \ldots, k_s)$. The following theorem gives the structure of one weight $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes which is a generalization of Theorem 3.10 in [15].

Theorem 6.8. *Let $C \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$ be a one weight $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive code of type $(\alpha, \beta; k_0, k_1, k_2, \ldots, k_s)$ with weight m. Let $k = k_0 + sk_1 + (s-1)k_2 + \cdots + k_s$. Then there exists a positive integer λ such that $m = \lambda 2^{k-1}$, where α and β satisfy $\alpha + 2^{s-1}\beta = \lambda(2^k - 1)$. Furthermore, if m is an odd integer, then α is odd and $C = \{(\mathtt{0}_\alpha, \mathtt{0}_\beta), (\mathtt{1}_\alpha, \mathtt{2}_\beta^{s-1})\}$, where $\mathtt{1}_\alpha = (1, \ldots, 1) \in \mathbb{Z}_2^\alpha$ and $\mathtt{2}_\beta^{s-1} = (2^{s-1}, \ldots, 2^{s-1}) \in \mathbb{Z}_{2^s}^\beta$.*

Proof. By Lemma 6.3, $\sum_{c \in C} \omega(c) = |C|(\frac{\alpha}{2} + 2^{s-2}\beta) = \frac{|C|}{2}(\alpha + 2^{s-1}\beta)$. On the other hand, the sum of the weights of all codewords is $(|C| - 1)m$. But $\gcd(\frac{|C|}{2}, (|C| - 1)) = \gcd(2^{k-1}, 2^k - 1) = 1$. Therefore there exists a positive integer λ such that $m = \lambda\frac{|C|}{2} = \lambda 2^{k-1}$ and hence $\alpha + 2^{s-1}\beta = \lambda(2^k - 1)$.

If m is odd, then $\lambda 2^{k-1}$ is odd. Hence λ is odd and $k = 1$. Moreover the equality $m = \lambda = \alpha + 2^{s-1}\beta$ implies that α is odd. Since $|C| = 2$ and $(\mathtt{1}_\alpha, \mathtt{2}_\beta^{s-1})$ is the only word with weight $\alpha + 2^{s-1}\beta$ and addition order 2, we have that $C = \{(\mathtt{0}_\alpha, \mathtt{0}_\beta), (\mathtt{1}_\alpha, \mathtt{2}_\beta^{s-1})\}$. □

References

[1] T. Abualrub and I. Siap, *Cyclic codes over the rings $Z_2 + uZ_2$ and $Z_2 + uZ_2 + u^2Z_2$*, Des. Codes Cryptogr. **42** (2007), no. 3, 273–287. https://doi.org/10.1007/s10623-006-9034-5

[2] T. Abualrub, I. Siap, and N. Aydin, *$\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes*, IEEE Trans. Inform. Theory **60** (2014), no. 3, 1508–1514. https://doi.org/10.1109/TIT.2014.2299791

[3] I. Aydogdu, T. Abualrub, and I. Siap, *On $\mathbb{Z}_2\mathbb{Z}_2[u]$-additive codes*, Int. J. Comput. Math. **92** (2015), no. 9, 1806–1814. https://doi.org/10.1080/00207160.2013.859854

[4] I. Aydogdu and I. Siap, *The structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$-additive codes: bounds on the minimum distance*, Appl. Math. Inf. Sci. **7** (2013), no. 6, 2271–2278. https://doi.org/10.12785/amis/070617

[5] ———, *On $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$-additive codes*, Linear Multilinear Algebra **63** (2015), no. 10, 2089–2102. https://doi.org/10.1080/03081087.2014.952728

[6] J. J. Bernal, J. Borges, C. Fernández-Córdoba, and M. Villanueva, *Permutation decoding of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes*, Des. Codes Cryptogr. **76** (2015), no. 2, 269–277. https://doi.org/10.1007/s10623-014-9946-4

[7] M. Bilal, J. Borges, S. T. Dougherty, and C. Fernández-Córdoba, *Maximum distance separable codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$*, Des. Codes Cryptogr. **61** (2011), no. 1, 31–40. https://doi.org/10.1007/s10623-010-9437-1

[8] J. Borges and C. Fernández-Córdoba, *There is exactly one $\mathbb{Z}_2\mathbb{Z}_4$-cyclic 1-perfect code*, Des. Codes Cryptogr. **85** (2017), no. 3, 557–566. https://doi.org/10.1007/s10623-016-0323-3

[9] J. Borges, C. Fernández-Córdoba, J. Pujol, J. Rifa, and M. Villanueva, $\mathbb{Z}_2\mathbb{Z}_4$-*linear codes*: *generator matrices and duality*, Des. Codes Cryptogr. **54** (2010), no. 2, 167–179. https://doi.org/10.1007/s10623-009-9316-9

[10] J. Borges, C. Fernández-Córdoba, and R. Ten-Valls, *On* $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$*-additive cyclic codes*, (2016).

[11] ———, *Linear and cyclic codes over direct product of chain rings*, Math. Meth. Appl. Sci. (2017).

[12] H. Q. Dinh and S. R. López-Permouth, *Cyclic and negacyclic codes over finite chain rings*, IEEE Trans. Inform. Theory **50** (2004), no. 8, 1728–1744. https://doi.org/10.1109/TIT.2004.831789

[13] S. T. Dougherty and C. Fernández-Córdoba, $\mathbb{Z}_2\mathbb{Z}_4$-*additive formally self-dual codes*, Des. Codes Cryptogr. **72** (2014), no. 2, 435–453. https://doi.org/10.1007/s10623-012-9773-4

[14] S. T. Dougherty, H. Liu, and Y. H. Park, *Lifted codes over finite chain rings*, Math. J. Okayama Univ. **53** (2011), 39–53.

[15] S. T. Dougherty, H. Liu, and L. Yu, *One weight* $\mathbb{Z}_2\mathbb{Z}_4$ *additive codes*, Appl. Algebra Engrg. Comm. Comput. **27** (2016), no. 2, 123–138. https://doi.org/10.1007/s00200-015-0273-4

[16] C. Fernández-Córdoba, J. Pujol, and M. Villanueva, $\mathbb{Z}_2\mathbb{Z}_4$-*linear codes*: *rank and kernel*, Des. Codes Cryptogr. **56** (2010), no. 1, 43–59. https://doi.org/10.1007/s10623-009-9340-9

[17] J. Rifà, F. I. Solov'eva, and M. Villanueva, *On the intersection of* $\mathbb{Z}_2\mathbb{Z}_4$-*additive perfect codes*, IEEE Trans. Inform. Theory **54** (2008), no. 3, 1346–1356. https://doi.org/10.1109/TIT.2007.915917

[18] K. Samei and M. R. Alimoradi, *Cyclic codes over the ring* $F_2+uF_2+vF_2$, Comput. Appl. Math. **37** (2018), no. 3, 2489–2502. https://doi.org/10.1007/s40314-017-0460-y

[19] K. Samei and S. Mahmoudi, *Cyclic R-additive codes*, Discrete Math. **340** (2017), no. 7, 1657–1668. https://doi.org/10.1016/j.disc.2016.11.007

[20] ———, *Singleton bounds for R-additive codes*, Adv. Math. Commun. **12** (2018), no. 1, 107–114. https://doi.org/10.3934/amc.2018006

[21] K. Samei and S. Sadeghi, *Maximum distance separable codes over* $\mathbb{Z}_2 \times \mathbb{Z}_{2^s}$, J. Algebra Appl. **17** (2018), no. 7, 1850136, 12 pp. https://doi.org/10.1142/S0219498818501360

[22] B. Srinivasulu and M. Bhaintwal, $\mathbb{Z}_2(\mathbb{Z}_2 + u\mathbb{Z}_2)$-*additive cyclic codes and their duals*, Discrete Math. Algorithms Appl. **8** (2016), no. 2, 1650027, 19 pp. https://doi.org/10.1142/S1793830916500270

[23] J. A. Wood, *The structure of linear codes of constant weight*, Trans. Amer. Math. Soc. **354** (2002), no. 3, 1007–1026. https://doi.org/10.1090/S0002-9947-01-02905-1

Saadoun Mahmoudi
Department of Mathematics
Bu Ali Sina University
Hamedan, Iran
Email address: mahmoudi.math89@yahoo.com

Karim Samei
Department of Mathematics
Bu Ali Sina University
Hamedan, Iran
Email address: samei@ipm.ir