

ON THE GEOMETRY OF VECTOR BUNDLES WITH FLAT CONNECTIONS

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ABSTRACT. Let $E \rightarrow M$ be an arbitrary vector bundle of rank k over a Riemannian manifold M equipped with a fiber metric and a compatible connection D^E . R. Albuquerque constructed a general class of (two-weights) spherically symmetric metrics on E . In this paper, we give a characterization of locally symmetric spherically symmetric metrics on E in the case when D^E is flat. We study also the Einstein property on E proving, among other results, that if $k \geq 2$ and the base manifold is Einstein with positive constant scalar curvature, then there is a 1-parameter family of Einstein spherically symmetric metrics on E , which are not Ricci-flat.

Introduction and main results

In the framework of Riemannian geometry, many special kinds of vector bundles have been considered and extensively studied, such as the cotangent bundle or the tangent bundle the literature of whose is very rich. Indeed, a wide range of interesting works have been published on the geometry of tangent bundles endowed with special types of metrics (Sasaki, Cheeger-Gromoll, . . .) or more generally with g -natural metrics (cf. [1–3], [7]). For the general case of an arbitrary vector bundle, to the best of our knowledge, the situation becomes substantially different (cf. [5], [6]).

Let (E, π, M) be a vector bundle equipped with a fiber metric h and a connection D^E compatible with h . Classically, the total space E , as a Riemannian manifold, have been “naturally” equipped with the metric $\pi^*g \oplus \pi^*h$. More recently, in [4], R. Albuquerque considered a more general class of two-weights metrics with the weight functions depending on the fibre norm of E , i.e., metrics of the form

$$\tilde{g} = e^{2\varphi_1} \pi^*g \oplus e^{2\varphi_2} \pi^*h,$$

where φ_1, φ_2 are smooth scalar functions on E depending only of the norm $r = h(e, e)$ for $e \in E$, and smooth at $r = 0$ on the right. He called such metrics

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spherically symmetric metrics. In this work we shall deal with this class of metrics, and we will study some problems of heredity and rigidity with respect to the base metric. Firstly, we shall prove that every spherically symmetric metric \tilde{g} on E has the following hereditary properties:

Theorem 1. *Let \tilde{g} be a spherically symmetric metric on a vector bundle (E, π, M) . If the manifold (E, \tilde{g}) is flat, or of constant sectional curvature, or of constant scalar curvature, or an Einstein manifold, respectively, then the manifold (M, g) possesses the same property.*

In the rest of the paper, we shall assume that $\dim M \geq 2$ and that the connection D^E is flat. At first, we shall give a characterization of the local symmetry of (E, \tilde{g}) . More precisely, we have:

Theorem 2. *Let \tilde{g} be a spherically symmetric metric on a vector bundle (E, π, M) of rank $k \geq 2$, equipped with a fiber metric h and a flat connection D^E compatible with h . Then (E, \tilde{g}) is locally symmetric if and only if the following conditions hold*

- i) φ_1 is constant;
- ii) either φ_2 is constant or $\varphi_2(r) = -\ln(r + c)$ for all $r \in \mathbb{R}_+$, for a constant $c > 0$;
- iii) (M, g) is locally symmetric.

In particular, (E, \tilde{g}) is of constant sectional curvature if and only if φ_1 and φ_2 are constant and (M, g) is of constant sectional curvature.

As concerns the constant sectional curvature property, we have the following rigidity result:

Theorem 3. *Let \tilde{g} be a spherically symmetric metric on a vector bundle (E, π, M) of rank $k \geq 2$, equipped with a fiber metric h and a flat connection D^E compatible with h . Then (E, \tilde{g}) is of constant sectional curvature if and only if φ_1 and φ_2 are constant and (M, g) is flat.*

In this case, (E, \tilde{g}) is also flat.

Then, we will prove that the locally symmetric metrics on E obtained in Theorem 2 are also Einstein metrics, provided that (M, g) is a locally symmetric Einstein manifold. More precisely, we have:

Theorem 4. *Let (M, g) be a locally symmetric Einstein manifold of positive constant scalar curvature and (E, π, M) be a vector bundle of rank $k \geq 2$, equipped with a fiber metric h and a flat connection D^E compatible with h . Then there is a 1-parameter family of locally symmetric Einstein metrics, which are not Ricci-flat.*

If we assume that $k \geq 3$ and we consider spherically symmetric metrics such that φ_1 is constant (in particular if we require π to be a Riemannian submersion), then we shall prove that the only Einstein spherically symmetric metrics on E are those of the family of Theorem 4 (Proposition 10).

Finally, concerning the sectional curvatures, we shall prove that the property of having a bounded sectional curvature is hereditary and that this not as rigid as the case of constant sectional curvature. Indeed we shall give an example where φ_1 and φ_2 are not necessarily constant, and where the sectional curvature of (E, \tilde{g}) is bounded if and only if that of (M, g) is bounded.

1. Preliminaries

Unless otherwise stated, the Einstein summation convention is used throughout the paper. Let (M, g) be a Riemannian manifold, ∇ its Levi-Civita connection, R its curvature tensor, and Ric its Ricci curvature tensor. Let (E, π, M) be a vector bundle equipped with a connection D^E . Assume h is a fiber metric on E and that D^E is compatible with h , i.e., $D^E h = 0$.

The connection D^E allows the splitting of the tangent bundle

$$TE = \mathcal{H} \oplus \mathcal{V},$$

where \mathcal{V} is the vertical sub-bundle defined as $\mathcal{V} = \cup_{e \in E} V_e E$, with $V_e E = \ker(d\pi)_e$ is the vertical subspace, which is canonically identified with the pull-back vector bundle $\pi^* E \rightarrow E$, and \mathcal{H} is the horizontal sub-bundle with respect to the connection D^E which is also naturally isomorphic to the pullback vector bundle $\pi^* TM \rightarrow E$. Then

$$\mathcal{H} \oplus \mathcal{V} = TE \simeq \pi^* TM \oplus \pi^* E.$$

The latter splitting gives rise to a splitting of vector fields of E : If $Z \in \mathfrak{X}(E)$, then $Z = Z^H + Z^V$, Z^H being the horizontal part and Z^V the vertical part.

Remark 1. Hereafter, we shall use the identification of the vertical (resp. the horizontal) sub-bundle $\mathcal{V} \rightarrow E$ (resp. $\mathcal{H} \rightarrow E$) with the pull-back bundle $\pi^* E \rightarrow E$ (resp. $\pi^* TM \rightarrow E$).

We consider the following tautological section of the vertical sub-bundle defined by $\xi_e = e \in \pi^* E$. It is easy to see that ξ satisfies the following result.

Lemma 1. *For all $X \in \mathfrak{X}(E)$ we have*

$$(\pi^* D^E)_X \xi = X^V.$$

We consider the scalar function $r = \|\xi\|_E^2 = h(\xi, \xi)$. So $dr = 2\xi^b$ where the ‘b’ is taken with respect to h . As a consequence, we have:

Lemma 2. *Let f be a smooth real scalar function. Then, for any horizontal (resp. vertical) vector X^H (resp. Y^V) on E , we have*

- i) $X^H(f(r)) = 0$;
- ii) $Y^V(f(r)) = 2f'(r)\xi^b(Y^V)$.

Clearly, E inherits “naturally”, from the metrics g and h , the Riemannian metric $\pi^* g \oplus \pi^* h$, which we denote for the sake of simplicity by \langle , \rangle . Throughout this paper, we will use the following other notations:

Notations 1. (1) The norm with respect to the metric \langle , \rangle is denoted by $\| \cdot \|$.

(2) The ‘b’ is taken with respect to \langle , \rangle , in such a way that, for $X = X^H + X^V \in TE$, we have $\xi^b(X) = \xi^b(X^V)$. If we restrict ourselves to the vertical distribution on E , the ‘b’ in ξ^b is no other than that with respect to the fiber-metric h .

(3) For $X, Y \in T_e E, e \in E$ (resp. $X, Y \in \mathfrak{X}(E)$), we denote by $(X \wedge Y)$ the endomorphism on $T_e E$ (resp. the $(1, 1)$ -tensor field on E), given by

$$(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$$

for all $Z \in T_e E$ (resp. $Z \in \mathfrak{X}(E)$).

(4) For all $X, Y, Z, T \in T_e E$ (resp. $\mathfrak{X}(E)$), we denote

$$\langle X \wedge Y, Z \wedge T \rangle = \langle X, Z \rangle \langle Y, T \rangle - \langle X, T \rangle \langle Y, Z \rangle$$

so that $\|X \wedge Y\|^2 := \langle X \wedge Y, X \wedge Y \rangle = \|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2$, which is exactly the squared area of the parallelogram in $T_e E$ constituted by the vectors X and Y .

Now, we endow the manifold E with a class of Riemannian metrics (see [4]) given by

$$(1.1) \quad \tilde{g} = e^{2\varphi_1} \pi^* g \oplus e^{2\varphi_2} \pi^* h,$$

where φ_1, φ_2 are smooth scalar functions on E depending only of r and smooth at $r = 0$ on the right, i.e., $\varphi_i, \varphi'_i = \frac{\partial \varphi_i}{\partial r}, \varphi''_i, \dots$ exist and are continuous at $r = 0$. Explicitly, for all $e \in E$, we have

$$\begin{cases} \tilde{g}_e(X_1^H, X_2^H) = e^{2\varphi_1} g(d\pi_e(X_1^H), d\pi_e(X_2^H)), & X_1^H, X_2^H \in \mathcal{H}_e, \\ \tilde{g}_e(X^H, Y^V) = 0, & X^H \in \mathcal{H}_e, Y^V \in \mathcal{V}_e, \\ \tilde{g}_e(Y_1^V, Y_2^V) = e^{2\varphi_2} \pi^* h(Y_1^V, Y_2^V), & Y_1^V, Y_2^V \in \mathcal{V}_e. \end{cases}$$

We have a connection $D^{**} = \pi^* \nabla \oplus \pi^* D^E$ on E which is clearly metric with respect to \tilde{g} for $\varphi_1 = \varphi_2 = 0$. Moreover its torsion is given, in the general setting, by:

Proposition 1 (See [4]). *Let $X, Y \in \mathfrak{X}(E)$. Then*

- (1) $d\pi(T^{D^{**}}(X, Y)) = 0$.
- (2) $(T^{D^{**}}(X, Y))^V = \pi^* R^E(X, Y)\xi$, where R^E is the curvature of D^E .

When the connection D^E is flat, D^{**} is torsion-free. But it's not metric with respect to \tilde{g} . As it was pointed out by R. Albuquerque in [4], we can derive from a torsion-free connection D^{**} which is metric with respect to \tilde{g} as follows:

We consider $\tilde{D} = D^{**} + C$ with $C \in \Omega^0(S^2(T^*E) \otimes TE)$ such that

$$\begin{aligned} C_X Y &= a(\xi^b(X)Y^H + \xi^b(Y)X^H) + b(\xi^b(X)Y^V + \xi^b(Y)X^V) \\ &\quad + [c_1 \langle X^H, Y^H \rangle + c_2 \langle X^V, Y^V \rangle] \xi \end{aligned}$$

for all $X = X^H + X^V, Y = Y^H + Y^V \in \mathfrak{X}(E)$. Here, $S^2(T^*E)$ denotes the set of symmetric $(0, 2)$ -tensor fields on E .

Proposition 2 (See [4]). *The linear connection \tilde{D} on the Riemannian manifold E is a metric connection if and only if*

$$\begin{cases} a = 2\varphi'_1, & c_1 = -2\varphi'_1 e^{2(\varphi_1 - \varphi_2)}, \\ b = 2\varphi'_2, & c_2 = -2\varphi'_2. \end{cases}$$

So, by choosing the coefficients a, b, c_1, c_2 of Proposition 2, we get a metric connection with respect to \tilde{g} which is torsion-free. Thus the Levi-Civita connection of the metric \tilde{g} , when D^E is flat, is given by

$$\tilde{\nabla}_X Y = D_X^{**} Y + C_X Y$$

for all vector fields X, Y on E .

Remarks 1. From now on, we will suppose the following:

- (1) E is endowed with a metric \tilde{g} of the form (1.1);
- (2) Unless otherwise stated, the functions φ_1, φ_2 and their successive derivatives are evaluated at $r = \|\xi\|_E^2$.

Thus, it is easy to see that:

Proposition 3. *Assume that D^E is flat. Let X^H and Y^H (resp. Z^V and T^V) be two horizontal (resp. vertical) vector fields on E . Then*

- (1) $\tilde{\nabla}_{X^H} Y^H = (\pi^* \nabla)_{X^H} Y^H - 2\varphi'_1 e^{2(\varphi_1 - \varphi_2)} \langle X^H, Y^H \rangle \xi$;
- (2) $\tilde{\nabla}_{X^H} Z^V = (\pi^* D^E)_{X^H} Z^V + 2\varphi'_1 \xi^b(Z^V) X^H$;
- (3) $\tilde{\nabla}_{Z^V} X^H = 2\varphi'_1 \xi^b(Z^V) X^H$;
- (4) $\tilde{\nabla}_{X^H} \xi = 2r\varphi'_1 X^H$;
- (5) $\tilde{\nabla}_{T^V} Z^V = \pi^* D_{T^V}^E Z^V + 2\varphi'_2 (\xi^b(T^V) Z^V + \xi^b(Z^V) T^V - \langle T^V, Z^V \rangle \xi)$;
- (6) $\tilde{\nabla}_{T^V} \xi = (1 + 2r\varphi'_2) T^V$.

The curvature of the manifold (E, \tilde{g}) is computed in [4]:

Proposition 4. *Assume that D^E is flat, and let $X, Y, Z \in \mathfrak{X}(E)$. Then the curvature tensor of (E, \tilde{g}) is given by:*

1. $\tilde{R}(X^H, Y^H)Z^H = \pi^* R(X^H, Y^H)Z^H + 4r(\varphi'_1)^2 e^{2(\varphi_1 - \varphi_2)} (X^H \wedge Y^H)Z^H$;
2. $\tilde{R}(X^H, Y^H)Z^V = 0$;
3. $\tilde{R}(X^H, Y^V)Z^H = e^{2(\varphi_1 - \varphi_2)} \langle X^H, Z^H \rangle [4(\varphi''_1 + (\varphi'_1)^2 - 2\varphi'_1 \varphi'_2) \xi^b(Y^V) \xi + 2(2r\varphi'_1 \varphi'_2 + \varphi'_1) Y^V]$;
4. $\tilde{R}(X^H, Y^V)Z^V = [4(2\varphi'_1 \varphi'_2 - (\varphi'_1)^2 - \varphi''_1) \xi^b(Y^V) \xi^b(Z^V) - 2(2r\varphi'_1 \varphi'_2 + \varphi'_1) \langle Y^V, Z^V \rangle] X^H$;
5. $\tilde{R}(X^V, Y^V)Z^H = 0$;
6. $\tilde{R}(X^V, Y^V)Z^V = 4(\varphi''_2 - (\varphi'_2)^2) [\xi^b(Z^V) (X^V \wedge Y^V) \xi - \langle X^V \wedge Y^V, \xi \wedge Z^V \rangle \xi] + 4(\varphi'_2 + r(\varphi'_2)^2) (X^V \wedge Y^V) Z^V$.

2. On the heredity of some geometric properties on the vector bundles

Our purpose, in this section, is to prove Theorem 1. It was proven in [4] that the curvature tensor, the Ricci tensor and the scalar curvature, restricted to the zero section of the vector bundle (E, π, M) , are given by:

Proposition 5 ([4]). *Let us denote O_M the zero section of (E, π, M) , and let $x \in M$ and $o \in O_M$ such that $\pi(o) = x$. Then, for all $X^H, Y^H, Z^H, W^H \in \mathcal{H}_o$, we have*

$$\begin{aligned} (1) \quad & \tilde{R}_o(X^H, Y^H, Z^H, W^H) = e^{2\varphi_1(0)} \langle \pi^* R_x(X^H, Y^H) Z^H, W^H \rangle_M, \\ (2) \quad & \tilde{\text{Ric}}_o(X^H, Y^H) = \text{Ric}_x(d\pi(X^H), d\pi(Y^H)) \\ & \quad - 2k\varphi_1'(0)e^{2(\varphi_1-\varphi_2)(0)} \langle X^H, Y^H \rangle_M, \end{aligned}$$

$$(3) \quad \tilde{S}_o = e^{-2\varphi_1(0)} S_x + 4ke^{-2\varphi_2(0)}((1-k)\varphi_2'(0) - n\varphi_1'(0)),$$

where $\tilde{\text{Ric}}$ and \tilde{S} (resp. Ric and S) denote the Ricci tensor and the scalar curvature of (E, \tilde{g}) (resp. (M, g)).

Furthermore, we have the following result concerning the sectional curvatures.

Proposition 6. *For all $\{X^H, Y^H\} \subseteq H_o$ an orthonormal system, we have*

$$\tilde{K}_o(X^H, Y^H) = e^{-2\varphi_1(0)} K_x(d\pi(X^H), d\pi(Y^H)),$$

where \tilde{K} (resp. K) denote the sectional curvature of (E, \tilde{g}) (resp. (M, g)).

Proof. By the first equation of Proposition 5, we have

$$\begin{aligned} \tilde{K}_o(X^H, Y^H) &= \tilde{R}_o(X^H, Y^H, Z^H, W^H) \\ &= e^{2\varphi_1(0)} \langle \pi^* R_x(X^H, Y^H) Z^H, W^H \rangle_M \\ &= e^{-2\varphi_1(0)} K_x(d\pi(X^H), d\pi(Y^H)) \end{aligned}$$

because

$$\|X^H \wedge Y^H\|_{\tilde{g}}^2 = e^{4\varphi_1} \|X^H \wedge Y^H\|^2.$$

□

Proof of Theorem 1. Follows from Propositions 5 and 6. □

3. Vector bundles with flat connections

In this section, we shall deal with vector bundles with flat connections, endowed with a spherically symmetric metric of the form (1.1). As a consequence of Theorem 1, the properties of being flat, of constant sectional curvature, of constant scalar curvature or Einstein are hereditary on any vector bundle endowed with a spherically symmetric metric. We shall investigate these properties and other geometrical properties to treat the converse problem, i.e., to study sufficient conditions to have such properties on a vector bundle (E, \tilde{g}) with flat connection. Our purpose would be to classify (even partially) metrics

of the form (1.1) on (E, \tilde{g}) having a prescribed geometric property. In many cases, we will obtain some rigidity results.

3.1. Locally symmetric spherically symmetric metrics on vector bundles

Our purpose is to prove Theorem 2. For this, we need some technical formulas and some intermediate results. Long but routine calculation, using Lemma 2 and Propositions 3 and 4, gives the following.

Proposition 7. *Let $e \in E, X^H, Y^H, Z^H, T^H \in \mathcal{H}_e$ and $X^V, Y^V, Z^V, T^V \in \mathcal{V}_e$. Then*

$$\begin{aligned}
 \text{i)} \quad & \left(\tilde{\nabla}_{T^H} \tilde{R} \right) (X^H, Y^H) Z^H \\
 &= (\pi^* \nabla_{T^H} \pi^* R)(X^H, Y^H) Z^H - 2\varphi'_1 e^{2(\varphi_1 - \varphi_2)} \left[\pi^* R(X^H, Y^H, Z^H, T^H) \right. \\
 &\quad \left. - 2e^{2(\varphi_1 - \varphi_2)} \left(2r(\varphi''_1 - \varphi'_1 \varphi'_2) + \varphi'_1 \right) \langle X^H \wedge Y^H, Z^H \wedge T^H \rangle \right] \xi; \\
 \text{ii)} \quad & \left(\tilde{\nabla}_{T^H} \tilde{R} \right) (X^H, Y^V) Z^V \\
 &= 4\varphi'_1 e^{2(\varphi_1 - \varphi_2)} [2r(\varphi''_2 - \varphi'_1 \varphi'_2) + \varphi'_2 - \varphi'_1] \langle X^H, T^H \rangle (\xi \wedge Y^V) Z^V; \\
 \text{iii)} \quad & \left(\tilde{\nabla}_{T^H} \tilde{R} \right) (X^V, Y^V) Z^V \\
 &= 4\varphi'_1 [-2r(\varphi''_2 - (\varphi'_2)^2) + \varphi'_1(1 + 2r\varphi'_2)] \langle X^V \wedge Y^V, \xi \wedge Z^V \rangle T^H; \\
 \text{iv)} \quad & \left(\tilde{\nabla}_{T^V} \tilde{R} \right) (X^V, Y^V) Z^V \\
 &= -4(\varphi''_2 - (\varphi'_2)^2)(1 + 2r\varphi'_2) \left[\langle X^V \wedge Y^V, T^V \wedge Z^V \rangle \xi \right. \\
 &\quad \left. + \langle X^V \wedge Y^V, \xi \wedge Z^V \rangle T^V - 2\xi^b(T^V)(X^V \wedge Y^V) Z^V \right. \\
 &\quad \left. - \xi^b(Z^V)(X^V \wedge Y^V) T^V - \langle T^V, Z^V \rangle (X^V \wedge Y^V) \xi \right] \\
 &\quad + 8 \left[\varphi_2^{(3)} - 2\varphi'_2 \varphi''_2 - 4(\varphi''_2 - (\varphi'_2)^2) \varphi'_2 \right] \xi^b(T^V) \left[\xi^b(Z^V)(X^V \wedge Y^V) \xi \right. \\
 &\quad \left. - \langle X^V \wedge Y^V, \xi \wedge Z^V \rangle \xi \right].
 \end{aligned}$$

Proof of Theorem 2. Suppose that (E, \tilde{g}) is locally symmetric. Let $e \in E$ such that $r = \|e\|_h^2 \neq 0$. Taking at first $X^V = Z^V = T^V = \xi_e$ and $0 \neq Y^V \perp \xi_e$, in the fourth formula of Proposition 7, we obtain

$$0 = r^2 \left\{ 2(\varphi''_2 - (\varphi'_2)^2)(1 + 2r\varphi'_2) + r \left[\varphi_2^{(3)} - 2\varphi'_2 \varphi''_2 - 4(\varphi''_2 - (\varphi'_2)^2) \varphi'_2 \right] \right\} Y^V,$$

which gives, by virtue of $Y^V \neq 0$,

$$(3.1) \quad r(\varphi_2^{(3)} - 2\varphi'_2 \varphi''_2) + 2(\varphi''_2 - (\varphi'_2)^2) = 0$$

on \mathbb{R}_+^* and by continuity on \mathbb{R}_+ . Putting $\psi = \varphi''_2 - (\varphi'_2)^2$, equation (3.1) can be written in the form $r\psi' + 2\psi = 0$, whose general solution is $\psi = \frac{A}{r^2}$, where $A \in \mathbb{R}$. Since φ_2 is C^∞ on \mathbb{R}_+ , so is ψ , and consequently $A = 0$, i.e., $\psi = 0$. It

follows that $\varphi_2'' - (\varphi_2')^2 = 0$. Then, by regularity considerations, it is easy to see that either φ_2' vanishes identically on \mathbb{R}_+ or $\varphi_2'(r) = -\frac{1}{c+r}$ for all $r \in \mathbb{R}^+$, where $c \in \mathbb{R}_+^*$ is a constant.

Case 1: $\varphi_2' = 0$ identically. In this case, the second equation of Proposition 7 becomes

$$(3.2) \quad (\tilde{\nabla}_{T^H} \tilde{R})(X^H, Y^V)Z^V = -4(\varphi_1')^2 e^{2(\varphi_1 - \varphi_2)} \langle X^H, T^H \rangle (\xi \wedge Y^V) Z^V.$$

Putting in (3.2) $0 \neq T^H = X^H$ and $0 \neq Y^V = Z^V \perp \xi_e$, we have

$$4(\varphi_1')^2 e^{2(\varphi_1 - \varphi_2)} \|X^H\|^2 \|Y^V\|^2 \xi_2 = 0,$$

and consequently $\varphi_1' = 0$.

Case 2: $\varphi_2'(r) = -\frac{1}{c+r}$ for all $r \in \mathbb{R}_+$, where $c \in \mathbb{R}_+$ is a constant. In this case, the third equation of Proposition 7 becomes

$$(3.3) \quad (\tilde{\nabla}_{T^H} \tilde{R})(X^V, Y^V)Z^V = 4(\varphi_1')^2 \frac{c-r}{c+r} \langle X^V \wedge Y^V, \xi \wedge Z^V \rangle T^H.$$

Putting in (3.3) $0 \neq X^V = Z^V \perp Y^V = \xi_e$, and supposing that $T^H \neq 0$, we have by local symmetry

$$0 = r(\varphi_1')^2 \frac{c-r}{c+r}.$$

We deduce that φ_1' vanishes identically on $\mathbb{R}_+ \setminus \{0, c\}$, and by continuity on \mathbb{R}_+ .

In both cases, using the first equation of Proposition 7 and the fact that $\varphi_1' = 0$, we deduce from $\tilde{\nabla}_{T^H} \tilde{R}(X^H, Y^H)Z^H = 0$ that (M, g) is also locally symmetric.

Conversely suppose we have i), ii) and iii) of Theorem 2. Then $\varphi_1' = 0$ and $\varphi_2'' - (\varphi_2')^2 = 0$. Substituting into i), iii) and iv) of Proposition 7, we find respectively $(\tilde{\nabla}_{T^H} \tilde{R})(X^H, Y^H)Z^H = (\pi^* \nabla_{T^H} \pi^* R)(X^H, Y^H)Z^H = 0$, by local symmetry of (M, g) , and $(\tilde{\nabla}_{T^H} \tilde{R})(X^V, Y^V)Z^V = (\tilde{\nabla}_{T^V} \tilde{R})(X^V, Y^V)Z^V = 0$. On the other hand, substituting from $\varphi_1' = 0$ into Proposition 4, we find

$$\tilde{R}(X^H, Y^H)Z^V = \tilde{R}(X^H, Y^V)Z^H = \tilde{R}(X^H, Y^V)Z^V = \tilde{R}(X^V, Y^V)Z^H = 0.$$

We deduce that $\tilde{\nabla} \tilde{R} = 0$, which completes our proof. □

Remark 2. It is worth mentioning that, if we restrict ourselves to functions defined only on \mathbb{R}_+^* instead of \mathbb{R}_+ , then the solution $\psi = \frac{A}{r^2}$ of the differential equation $r\psi' + 2\psi = 0$ found in the previous proof, with $A \neq 0$, gives solutions of equation (3.1). Indeed, if $0 \neq A \leq \frac{1}{4}$, the functions of the form

$$(3.4) \quad \varphi_2(r) = \ln(ar^\alpha) \quad \text{for all } r \in \mathbb{R}_+^*,$$

where $\alpha = \frac{-1 \pm \sqrt{1-4A}}{2}$ and $a > 0$ are solutions of (3.1).

If we consider the “slit” vector bundle $E^* := E \setminus O_M$ of E , where O_M is the zero section of E , and if we extend the notion of spherically symmetric metrics to E^* (considering the functions φ_1 and φ_2 defined and smooth only on \mathbb{R}_+^*), then spherically symmetric metrics on E^* , whose weight functions are given

by a constant φ_1 and a φ_2 of the form (3.4), could give interesting geometric situations, although such functions don't annihilates $(\widetilde{\nabla}_{T^V} \widetilde{R})(X^V, Y^V)Z^V$ identically, so that E^* is not locally symmetric as well.

3.2. Ricci curvature

We shall be interested in the Ricci curvature of (E, \widetilde{g}) and the existence of Einstein metrics. For this, let $e \in E$ and $x = \pi(e)$. We consider an orthonormal basis $\{b_i; i = 1, \dots, n\}$ of $(T_x M, g_x)$ and we denote by b_i^h the horizontal lift of b_i to E at e , with respect to the flat connection D^E , $i = 1, \dots, n$. Let $\{e_p; p = 1, \dots, k\}$ be an orthonormal basis of (E_x, h_x) and denote $e_p^v = \pi^* e_p$, $p = 1, \dots, k$. If we put

$$(3.5) \quad E_i = e^{-\varphi_1} b_i^h \quad \text{and} \quad E_{n+p} = e^{-\varphi_2} e_p^v; \quad i = 1, \dots, n, \quad p = 1, \dots, k,$$

then $\{E_I; I = 1, \dots, n+p\}$ is an orthonormal basis of $(T_e E, \widetilde{g}_e)$. Using this basis, we can compute the Ricci tensor of (E, \widetilde{g}) , as follows: for any $X, Y \in T_e E$, we have

$$\widetilde{\text{Ric}}(X, Y) = \sum_{I=1}^{n+p} \widetilde{g}(\widetilde{R}(X, E_I)E_I, Y).$$

Simple computations, using Proposition 4, give:

Proposition 8. *The Ricci curvature of (E, \widetilde{g}) at an arbitrary point $e \in E$ is totally characterized by*

1. $\widetilde{\text{Ric}}(X^H, Y^H) = \text{Ric}(X, Y) - 4e^{2(\varphi_1 - \varphi_2)} \left[\frac{k}{2} \varphi_1' + r[\varphi_1'' + n(\varphi_1')^2 + (k-2)\varphi_1' \varphi_2'] \right] g(X, Y),$
2. $\widetilde{\text{Ric}}(X^H, Z^V) = \widetilde{\text{Ric}}(Z^V, X^H) = 0,$
3. $\widetilde{\text{Ric}}(Z^V, T^V) = - \left[2n(2r\varphi_1' \varphi_2' + \varphi_1') + 4(k-1)(\varphi_2' + r(\varphi_2')^2) + 4r(\varphi_2'' - (\varphi_2')^2) \right] \langle Z^V, T^V \rangle - \left[4n(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1' \varphi_2') + 4(k-2)(\varphi_2'' - (\varphi_2')^2) \right] \xi^b(Z^V) \xi^b(T^V)$

for all $X^H, Y^H \in \mathcal{H}_e$ and $Z^V, T^V \in \mathcal{V}_e$, where Ric is the Ricci tensor of (M, g) , $X = (d\pi)_e X^H$ and $Y = (d\pi)_e Y^H$.

Now, we shall analyse the formulas of the previous proposition and examine the existence of Einstein spherically symmetric metrics.

Theorem 5. *(E, \widetilde{g}) is an Einstein manifold with constant μ if and only if*

- (a) (M, g) is also an Einstein manifold with constant λ ,
- (b) $2e^{-2\varphi_2} \left[n(2r\varphi_1' \varphi_2' + \varphi_1') + 2(k-1)(\varphi_2' + r(\varphi_2')^2) + 2r(\varphi_2'' - (\varphi_2')^2) \right] = -\mu,$
- (c) $n(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1' \varphi_2') + (k-2)(\varphi_2'' - (\varphi_2')^2) = 0,$
- (d) $e^{2(\varphi_1 - \varphi_2)} \left[4r[\varphi_1'' + n(\varphi_1')^2] + \varphi_1'[2k + (k-2)r\varphi_2'] + \mu e^{2\varphi_2} \right] = \lambda.$

Proof. Suppose that (E, \tilde{g}) is an Einstein manifold with constant μ , i.e., $\widetilde{\text{Ric}} = \mu\tilde{g}$. Then by 1. of Proposition 8, we have $\text{Ric}(X, Y) = \lambda g(X, Y)$ for all $x \in M$ and $X, Y \in T_x M$, where λ is the function defined, on E , by

$$(3.6) \quad \lambda = e^{2(\varphi_1 - \varphi_2)} \left[4r[\varphi_1'' + n(\varphi_1')^2] + \varphi_1'[2k + (k - 2)r\varphi_2'] + \mu e^{2\varphi_2} \right].$$

Since Ric and g depend only on x , λ will depend only on x , i.e., λ is constant on each fiber of E . But it is obvious, from (3.6), that λ depends only on r . Therefore λ is constant, so (M, g) is an Einstein manifold, and (3.6) is exactly condition (d) of the theorem. On the other hand, taking $0 \neq Z^V = T^V \perp \xi_e$ in 3. of Proposition 8, we obtain by virtue of $\widetilde{\text{Ric}}(Z^V, T^V) = \mu\tilde{g}(Z^V, T^V) = \mu e^{2\varphi_2} \langle Z^V, T^V \rangle$,

$$-2 \left[n(2r\varphi_1'\varphi_2' + \varphi_1') + 2(k - 1)(\varphi_2' + r(\varphi_2')^2) + 2r(\varphi_2'' - (\varphi_2')^2) \right] = \mu e^{2\varphi_2},$$

which gives condition (b) of the theorem. Now, 3. of Proposition 8 becomes, by virtue of (b) of the theorem

$$(3.7) \quad \left[4n(\varphi_1'' + (\varphi_1')^2 - 2\varphi_1'\varphi_2') + 4(k - 2)(\varphi_2'' - (\varphi_2')^2) \right] \xi^b(Z^V) \xi^b(T^V) = 0$$

for all $Z^V, T^V \in \mathcal{V}_e$. Taking, in (3.7), $Z^V = T^V = \xi_e$, we obtain the equality (c) of the theorem on \mathbb{R}_+^* , and by continuity on \mathbb{R}_+ .

The converse follows immediately from Proposition 8. □

Generally speaking, it is not easy to solve the system of differential equations under conditions (b), (c) and (d) of Theorem 5. We shall then investigate the special situations when φ_1 or φ_2 are constant.

Case 1: φ_2 is constant. In this case we have:

Proposition 9. *If φ_2 is constant, then (E, \tilde{g}) is Einstein if and only if (E, \tilde{g}) and (M, g) are Ricci-flat, and φ_1 is constant.*

Proof. Since $\varphi_2' = 0$, then we get, from (b) of Theorem 5, $\varphi_1' = \frac{-\mu e^{2\varphi_2}}{2n}$. Substituting into (c) of Theorem 5, we obtain $\mu = 0$, and consequently $\varphi_1' = 0$. Hence (d) of Theorem 5 becomes $\lambda = \mu e^{2\varphi_1} = 0$. The converse follows immediately from Theorem 5. □

Case 2: φ_1 is constant.

Subcase 2.1: for $k \neq 2$, (c) of Theorem 5 is equivalent to $\varphi_2'' = (\varphi_2')^2$, whose solutions are given by φ_2 is constant and

$$(3.8) \quad \varphi_2(r) = -\ln(r + c) \text{ for all } r \in \mathbb{R}_+, \text{ where } c > 0 \text{ is a constant.}$$

On the other hand, (b) of Theorem 5 is equivalent to

$$(3.9) \quad 4(1 - k)\varphi_2'(1 + r\varphi_2') = \mu e^{2\varphi_2}.$$

If $k = 1$, then by the preceding equation we have $\mu = 0$, i.e., (E, \tilde{g}) is Ricci-flat, and we deduce from (d) of Theorem 5 that $\lambda = 0$.

If $k \neq 1$, then we have two eventualities:

- For φ_2 constant, we obtain $\mu = \lambda = 0$.
- For φ_2 given by (3.8), equation (3.9) gives $c = \frac{\mu}{4(k-1)}$. In particular $\mu > 0$.

Subcase 2.1: for $k = 2$, it is easy to check that the function φ_2 given by (3.8) satisfy conditions (b), (c) and (d) of Theorem 5.

Summarizing, we have just proved the following.

Proposition 10. *Let (M, g) be an Einstein manifold with constant $\lambda \geq 0$. Let (E, π, M) be a vector bundle of rank k , equipped with a fiber metric h and a flat connection D^E compatible with h and consider a spherically symmetric metric \tilde{g} on E , such that φ_1 is constant.*

- (a) *If $k = 1$, then (E, \tilde{g}) is Einstein if and only if it is Ricci-flat if and only if (M, g) is Ricci-flat and either φ_2 is constant, or $\varphi_2(r) = -\ln(r + c)$ for all $r \in \mathbb{R}_+$, where $c > 0$ is a constant.*
- (b) *If $k \geq 3$, then (E, \tilde{g}) is Einstein with constant μ if and only if either φ_2 is constant and (M, g) and (E, \tilde{g}) are Ricci-flat, or $\mu = \lambda e^{2\varphi_1} > 0$ and $\varphi_2(r) = -\ln(r + \frac{\mu}{4(k-1)})$ for all $r \in \mathbb{R}_+$.*
- (c) *If $k \neq 1$ and $\varphi_2(r) = -\ln(r + c)$ for all $r \in \mathbb{R}_+$, with $c = \frac{\lambda}{4(k-1)}e^{-2\varphi_1}$, then (E, \tilde{g}) is an Einstein manifold, which is not Ricci flat unless (M, g) is Ricci flat.*

Proof of Theorem 4. Follows from Proposition 10 and Theorem 2, taking the family of metrics $\tilde{g}_a := a\pi^*g + \frac{16(k-1)^2 a^2}{(4(k-1)ar + \lambda)^2} \pi^*h$, parameterized by $a > 0$. The Ricci tensor of \tilde{g}_a is given by $\text{Ric} = \frac{\lambda}{a} \tilde{g}_a$, and its scalar curvature is constant equal to $\frac{(n+k)\lambda}{a} > 0$. □

Notice that the metric \tilde{g}_a depends on the rank k of E . But if we impose the solutions of the system in Theorem 5 not to depend on the dimension n of M and the rank k of E , then the only solutions are φ_1 and φ_2 are constant.

3.3. Scalar curvature

Fixing $e \in E$ and taking the orthonormal basis $\{E_I, I = 1, \dots, n + k\}$ of $T_e E$, defined by (3.5), the scalar curvature \tilde{S} of (E, \tilde{g}) at e is given by

$$\tilde{S}_e = \sum_{I=1}^{n+k} \widetilde{\text{Ric}}(E_I, E_I).$$

A straightforward computation using Proposition 8 yields:

Proposition 11. *Let S denote the scalar curvature of (M, g) . Then the scalar curvature of (E, \tilde{g}) is given by*

$$\begin{aligned} \tilde{S} = e^{-2\varphi_1} S + e^{-2\varphi_2} \left\{ -4n \left[r(2\varphi_1'' + (n+1)(\varphi_1')^2 + 2(k-2)\varphi_1'\varphi_2') + k\varphi_1' \right] \right. \\ \left. + 4(1-k) \left[r(2\varphi_2'' + (k-2)(\varphi_2')^2) + k\varphi_2' \right] \right\}. \end{aligned}$$

From Theorem 1, if (E, \tilde{g}) has constant scalar curvature \tilde{S}_0 , then the scalar curvature S of (M, g) is constant. In some special cases we have the converse.

Proposition 12. *Assume that φ_1 and φ_2 are constant. Then $\tilde{S} = e^{-2\varphi_1}S$, thus \tilde{S} is constant on each fiber of E .*

Proof. Proposition 11, for $\varphi'_1 = \varphi'_2 = 0$, gives $\tilde{S} = e^{-2\varphi_1}S$. □

Concerning the sign of the scalar curvature of (E, \tilde{g}) , we can state the following result.

Proposition 13. *Assume that φ_1 (resp. φ_2) is constant and φ_2 and φ'_2 (resp. φ_1 and φ'_1) are increasing functions. If S is negative, then \tilde{S} is negative.*

Proof. If φ_1 (resp. φ_2) is constant, then Proposition 11 gives the following equation

$$\begin{aligned} \tilde{S} &= e^{-2\varphi_1}S - 4(k-1) \left[r(2\varphi''_2 + (k-2)(\varphi'_2)^2) + k\varphi'_2 \right] \\ (\text{resp. } \tilde{S} &= e^{-2\varphi_1}S - 4n \left[r(2\varphi''_1 + (n+1)(\varphi'_1)^2) + k\varphi'_1 \right]) \end{aligned}$$

which implies that, if $\varphi'_2, \varphi''_2 \geq 0$ (resp. $\varphi'_1, \varphi''_1 \geq 0$) and S is negative, then \tilde{S} is negative. □

3.4. Sectional curvatures

Using the formulas of curvature, an easy computation gives:

Proposition 14. *Let $X, Y \in T_eE$ be such that $\{X, Y\}$ is an orthonormal system of vectors, and set $X = X^H + X^V, Y = Y^H + Y^V$. Then*

$$\begin{aligned} \tilde{K}(X, Y) &= e^{2\varphi_1} \left[K((d\pi)_eX, (d\pi)_eY) - 4r(\varphi'_1)^2 e^{2\varphi_1 - 2\varphi_2} \right] \|X^H \wedge Y^H\|^2 \\ &\quad + 4(\varphi''_1 + (\varphi'_1)^2 - 2\varphi'_1\varphi'_2) e^{2\varphi_1} \left[2\xi^b(X^V)\xi^b(Y^V)\langle X^H, Y^H \rangle \right. \\ &\quad \left. - (\xi^b(Y^V))^2 \|X^H\|^2 - (\xi^b(X^V))^2 \|Y^H\|^2 \right] \\ &\quad + 2\varphi'_1(1 + r\varphi'_2) e^{2\varphi_1} \left[2\langle X^V, Y^V \rangle \langle X^H, Y^H \rangle \right. \\ &\quad \left. - \|Y^V\|^2 \|X^H\|^2 - \|X^V\|^2 \|Y^H\|^2 \right] \\ &\quad + 4(\varphi''_2 - (\varphi'_2)^2) e^{2\varphi_2} \left[2\xi^b(X^V)\xi^b(Y^V)\langle X^V, Y^V \rangle \xi(X^V) \right. \\ &\quad \left. - \xi^b(X^V)^2 \|Y^V\|^2 - \xi^b(Y^V)^2 \|X^V\|^2 \right] \\ &\quad - 4\varphi'_2(1 + r\varphi'_2) e^{2\varphi_2} \|X^V \wedge Y^V\|^2, \end{aligned}$$

where K is the sectional curvature of (M, g) .

Note that the orthonormality of $\{X, Y\}$ is equivalent to the following system:

$$(3.10) \quad \begin{cases} e^{2\varphi_1} \langle X^H, Y^H \rangle = -e^{2\varphi_2} \langle X^V, Y^V \rangle, \\ e^{2\varphi_1} \|X^H\|^2 + e^{2\varphi_2} \|X^V\|^2 = 1, \\ e^{2\varphi_1} \|Y^H\|^2 + e^{2\varphi_2} \|Y^V\|^2 = 1. \end{cases}$$

Proof of Theorem 3. Suppose that (E, \tilde{g}) is of constant sectional curvature \tilde{K} . Then it is locally symmetric. We claim that φ_2 is constant. If not, by Theorem 2, φ_2 is constant or $\varphi_2(r) = -\ln(r + c)$ for all $r \in \mathbb{R}_+$, for a constant $c > 0$. We have also φ_1 is constant. Using Proposition 14, we have

$$(3.11) \quad \tilde{K}(X, Y) = e^{2\varphi_1} K((d\pi)_e X, (d\pi)_e Y) \|X^H \wedge Y^H\|^2 + \frac{4c}{(r+c)^4} \|X^V \wedge Y^V\|^2$$

for any orthonormal system of vectors $\{X, Y\}$ in $T_e E$. If we suppose X horizontal and Y vertical in (3.11), in such a way that $X^V = Y^H = 0$, then we have $\tilde{K} = 0$. On the other hand, taking both X and Y vertical in (3.11), in such a way that (by orthonormality) $\|X^V \wedge Y^V\|^2 = e^{-4\varphi_2} = (r + c)^4$, then we have $c = 0$, which is a contradiction. We conclude that φ_2 is constant, so by (3.11) we have

$$(3.12) \quad \tilde{K}(X, Y) = e^{2\varphi_1} K((d\pi)_e X, (d\pi)_e Y) \|X^H \wedge Y^H\|^2.$$

Using the same arguments as before, we obtain $\tilde{K} = 0$ and then the sectional curvature of (M, g) is also constant equal to 0.

The converse in the Theorem is trivial. □

In the remaining of this section, we will be interested in the property for the sectional curvature of (E, \tilde{g}) to be bounded. We start with the following theorem, whose proof follows by simple computations from Proposition 4.

Theorem 6. *If the sectional curvature of (E, \tilde{g}) is bounded, then that of (M, g) is also bounded.*

Proof. From Proposition 14, for any orthonormal system $\{X^H, Y^H\}$ in \mathcal{H}_e , we have

$$(3.13) \quad \tilde{K}(X^H, Y^H) = e^{-2\varphi_1} K((d\pi)_e(X^H), (d\pi)_e(Y^H)) - 4r(\varphi'_1)^2 e^{-2\varphi_2},$$

which, restricted to the zero section, yields

$$\tilde{K}(X^H, Y^H) = e^{-2\varphi_1(0)} K((d\pi)_{0_x}(X^H), (d\pi)_{0_x}(Y^H)),$$

where 0_x is the zero vector of E_x , $x \in M$. Using the isomorphism $(d\pi)_{0_x} : \mathcal{H}_{0_x} \rightarrow M_x$ and the fact that \tilde{K} is bounded, we deduce that K is bounded. □

The converse of Theorem 6 is not true in the general setting, but we shall prove that it remains true in some situations.

Proposition 15. *Assume $\varphi_2 \geq 0$ is constant and take $\varphi_1(r) = \arctan(r)$. If the sectional curvature of (M, g) is bounded, then the sectional curvature of (E, \tilde{g}) is also bounded.*

Proof. We assume that K is bounded and $|K| \leq K_0$. Let $X, Y \in T_e E$ be such that $\{X, Y\}$ is an orthonormal system of vectors, and set $X = X^H + X^V$, $Y = Y^H + Y^V$. Under the condition $\varphi'_2 = 0$, we find that

$$\tilde{K}(X, Y) = e^{2\varphi_1} [K((d\pi)_e X, (d\pi)_e Y) - 4r(\varphi'_1)^2 e^{2\varphi_1 - 2\varphi_2}] \|X^H \wedge Y^H\|^2$$

$$\begin{aligned}
& + 4(\varphi_1'' + (\varphi_1')^2)e^{2\varphi_1} \left[2\xi^b(X^V)\xi^b(Y^V)\langle X^H, Y^H \rangle \right. \\
& - (\xi^b(Y^V))^2\|X^H\|^2 - (\xi^b(X^V))^2\|Y^H\|^2 \left. \right] \\
& + 2\varphi_1'e^{2\varphi_1} \left[2\langle X^V, Y^V \rangle \langle X^H, Y^H \rangle \right. \\
& - \|Y^V\|^2\|X^H\|^2 - \|X^V\|^2\|Y^H\|^2 \left. \right].
\end{aligned}$$

Using the orthonormality conditions (3.10), we have

$$(3.14) \quad \|X^H \wedge Y^H\|^2 = e^{-4\varphi_1}(\tilde{g}(X^H, X^H)\tilde{g}(Y^H, Y^H) - \tilde{g}(X^H, Y^H)^2).$$

By virtue of (1.1), using equalities 1., 2., 3., the Cauchy-Schwartz inequality with respect to \tilde{g} and the fact that we have $0 \leq \varphi_1 \leq \frac{\pi}{2}$, since $\varphi_1(r) = \arctan(r)$, we conclude that

$$(3.15) \quad e^{2\varphi_1}\|X^H \wedge Y^H\|^2 \leq 2e^{-2\varphi_1} \leq 2.$$

Using the Cauchy-Schwartz inequality with respect to h and \tilde{g} and the fact that $\varphi_1, \varphi_2 \geq 0$, we prove easily that

$$(3.16) \quad |2\xi^b(X^V)\xi^b(Y^V)\langle X^H, Y^H \rangle - (\xi^b(Y^V))^2\|X^H\|^2 - (\xi^b(X^V))^2\|Y^H\|^2| \leq 4r^2,$$

$$(3.17) \quad |2\langle X^V, Y^V \rangle \langle X^H, Y^H \rangle - \|Y^V\|^2\|X^H\|^2 - \|X^V\|^2\|Y^H\|^2| \leq 4,$$

and

$$(3.18) \quad e^{2\varphi_1 - 2\varphi_2} \leq e^\pi.$$

So, by virtue of (3.15)–(3.18), we have:

$$(3.19) \quad |\tilde{K}(X, Y)| \leq 2(K_0 + 4r|\varphi_1'|^2e^\pi) + 16(|\varphi_1'' + (\varphi_1')^2|)e^\pi r^2 + 8|\varphi_1'|e^\pi.$$

Since $\varphi_1(r) = \arctan(r)$, we have $0 \leq \varphi_1 \leq \frac{\pi}{2}$, $\varphi_1'(r) = \frac{1}{1+r^2}$ and $\varphi_1''(r) = \frac{-2r}{(1+r^2)^2}$. Thus, $\varphi_1' \leq 1$, $r\varphi_1' \leq \frac{1}{2}$ and $|(\varphi_1')^2 + \varphi_1''| \leq 2$. Furthermore, $\varphi_1'' + (\varphi_1')^2 = \frac{-2r}{(1+r^2)^2} + \frac{1}{(1+r^2)^2} = \frac{1-2r}{(1+r^2)^2}$, thus $r^2(|\varphi_1'' + (\varphi_1')^2|) \leq \frac{r^2(1+2r)}{(1+r^2)^2}$ which is a bounded function since $\frac{r^2(1+2r)}{(1+r^2)^2} \rightarrow 0$ when $r \rightarrow +\infty$. Denoting by K_1 the bound of the function $r \mapsto r^2(|\varphi_1'' + (\varphi_1')^2|)$, we conclude that

$$|\tilde{K}| \leq 2K_0 + 4e^\pi(3 + 4K_1),$$

which means that \tilde{K} is bounded. \square

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