

## BOUNDS FOR EXPONENTIAL MOMENTS OF BESSEL PROCESSES

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ABSTRACT. Let  $0 < \alpha < \infty$  be fixed, and let  $X = (X_t)_{t \geq 0}$  be a Bessel process with dimension  $0 < \theta \leq 1$  starting at  $x \geq 0$ . In this paper, it is proved that there are positive constants  $A$  and  $D$  depending only on  $\theta$  and  $\alpha$  such that

$$\mathbf{E}_x \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) \leq A \mathbf{E}_x \left( \exp [D\tau] \right)$$

for any stopping time  $\tau$  of  $X$ . This inequality is also shown to be sharp.

### 1. Introduction

Bounds for moments of the maximum of Bessel processes are given in [1], [2], [3], [4] and [8]. These bounds are derived in terms of the moments of stopping times of Bessel processes, and are mainly of the power-type. The purpose of this paper is to establish bounds for exponential moments of the maximum of Bessel processes. Our method of proof is essentially based on the optimal stopping theory of the maximum process for Bessel processes attributed to Dubins, Shepp and Shiryaev [3]. However, it is worth pointing out that the bounds established in the present paper are quite new.

Let  $X = (X_t)_{t \geq 0}$  be a Bessel process defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ , given by

$$(1) \quad dX_t = \frac{\theta - 1}{2X_t} dt + dB_t; \quad X(0) = x$$

with dimension  $\theta > 0$  starting at  $x \geq 0$  under  $\mathbf{P} := \mathbf{P}_x$ , where  $B = (B_t)_{t \geq 0}$  is a standard Brownian motion. It is important to note here that  $X$  is a nonnegative continuous Markov process, and moreover,  $X$  is a supermartingale when the dimension is  $\theta \leq 0$  and is a submartingale in the case  $\theta \geq 1$ . For the remaining case,  $0 < \theta < 1$ , the process  $X$  in (1) is not a semimartingale.

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A detailed exposition on various properties, and extension of Bessel processes can be found in [5], [7] and [9].

In this paper, we prove the following result.

**Theorem 1.1.** *Let  $X = (X_t)_{t \geq 0}$  be a Bessel process with dimension  $0 < \theta \leq 1$  starting at  $x \geq 0$  under  $\mathbf{P}_x$ . Suppose that there are positive constants  $M$ ,  $N$  and  $L$  such that  $MN = \frac{\alpha^2}{\theta}$  for  $0 < \theta \leq 1$  and  $0 < \alpha < \infty$ , and  $1 < L < \frac{1}{N}$  provided that  $0 < N < 1$ . Then, we have*

$$(2) \quad \mathbf{E}_x \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) \leq \frac{N}{1 - NL} \mathbf{E}_x \left( \exp[M\tau] \right)$$

for any stopping time  $\tau$  of  $X$ . Inequality (2) is sharp.

One of the distinct features of the above result is that it permits the construction of explicit forms of the constants  $M$ ,  $N$  and  $L$  in (2) for particular cases of independent interest. Here is one simple example.

**Example 1.2.** Choose  $\alpha = \sqrt{\theta}$  for  $0 < \theta \leq 1$ . Let  $L = \frac{2}{1+N}$  and  $M = \frac{1}{N}$  for  $0 < N < 1$ . Clearly, the condition  $1 < L < \frac{1}{N}$  is satisfied in this case. Hence,  $\frac{N}{1-NL} = \frac{N(1+N)}{1-N}$  for  $0 < N < 1$ . This example excludes the interesting case  $M = N$ . In what follows, we now let  $0 < \alpha < \sqrt{\theta}$  for  $0 < \theta \leq 1$ . Then,  $MN = \frac{\alpha^2}{\theta}$  implies that  $M = N = \frac{\alpha}{\sqrt{\theta}}$ . Consequently, we have

$$\frac{N(1+N)}{1-N} = \frac{\frac{\alpha}{\sqrt{\theta}} \left( 1 + \frac{\alpha}{\sqrt{\theta}} \right)}{1 - \frac{\alpha}{\sqrt{\theta}}}$$

which completes the example. The special case of a Brownian motion (case  $\theta = 1$ ) covered in the proof of our theorem is also another example.

For the proof of Theorem 1.1, a variant of the optimal stopping problem treated in [3] will play an important role. Let  $X = (X_t)_{t \geq 0}$  be a Bessel process given by (1) with dimension  $\theta > 0$  starting at  $x \geq 0$ . Throughout, we denote by  $S = (S_t)_{t \geq 0}$  the maximum process associated with  $X$ , and defined by

$$(3) \quad S_r = \left( \max_{0 \leq t \leq r} X_t \right) \vee s$$

for all  $0 \leq x \leq s$ .

We consider the optimal stopping problem:

$$(4) \quad \Phi_\theta(x, s) := \sup_{\tau} \mathbf{E}_{x,s} \left( e^{\alpha S_\tau} - \int_0^\tau c(S_r) dr \right),$$

where the supremum in (4) is taken over all stopping times  $\tau$  for  $X$  such that the integral has finite expectation,  $\mathbf{E}_{x,s}$  is the expectation with respect to the probability law  $\mathbf{P}_{x,s}$  of the Markov process  $(X, S)$  starting at  $(x, s)$  for all  $0 \leq x \leq s$ ,  $s \mapsto c(s)$  is a positive continuously differentiable cost function, and  $0 < \alpha < \infty$  is fixed here and elsewhere.

In [3], the optimal stopping problem (4) is considered with a linear terminal reward of the form  $S_\tau$  and a constant cost function  $c(s) = c > 0$ . The point here is that the construction of  $\Phi_\theta(x, s)$  and the optimal stopping time in question follows by arguing similarly as in [3]. The crucial part is to establish sharp two-sided estimates for a solution(s) of a nonlinear differential equation (6) associated with the optimal stopping boundary of the problem (4). This is established with the aid of a comparison principle.

**2. Proof of Theorem 1.1**

*Proof.* Using similar arguments as in [3], we have

$$(5) \quad \Phi_\theta(x, s) = e^{\alpha s} + \left( \frac{1}{2-\theta} g_*^2(s) - \frac{2}{\theta(2-\theta)} x^{2-\theta} g_*^\theta(s) + \frac{1}{\theta} x^2 \right) c(s)$$

if  $g_*(s) \leq x \leq s$  and

$$\Phi_\theta(x, s) = e^{\alpha s} \quad \text{if } 0 \leq x \leq g_*(s)$$

for the optimal stopping problem (4), where  $s \mapsto g_*(s)$  is the optimal stopping boundary which is a non-negative solution of the nonlinear differential equation

$$(6) \quad g'(s) = \frac{\alpha e^{\alpha s} + \left( \frac{1}{\theta} s^2 - \frac{2}{\theta(2-\theta)} s^{2-\theta} g^\theta(s) + \frac{1}{2-\theta} g^2(s) \right) c'(s)}{\frac{2}{2-\theta} \left( s^{2-\theta} g^{\theta-1}(s) - g(s) \right) c(s)}$$

such that  $g_*(s) \leq s$  for all  $s \geq 0$ , and  $\frac{g_*(s)}{s} \rightarrow 1$  as  $s \rightarrow \infty$ .

For the rest of the proof we will now assume that  $c(s) = ce^{\alpha s}$  for all  $s \geq 0$ , where  $c > 0$  is a fixed constant. Then,

$$(7) \quad g'(s) = \frac{\alpha + c\alpha \left( \frac{1}{\theta} s^2 - \frac{2}{\theta(2-\theta)} s^{2-\theta} g^\theta(s) + \frac{1}{2-\theta} g^2(s) \right)}{\frac{2c}{2-\theta} \left( s^{2-\theta} g^{\theta-1}(s) - g(s) \right)}$$

Fix  $0 < \theta \leq 1$ . Let

$$(8) \quad D_g = \left\{ (s, g) \mid g(s) < s \text{ for all } s \geq 0 \right\}$$

and let

$$(9) \quad H(s, g(s)) := \frac{\alpha + c\alpha \left( \frac{1}{\theta} s^2 - \frac{2}{\theta(2-\theta)} s^{2-\theta} g^\theta(s) + \frac{1}{2-\theta} g^2(s) \right)}{\frac{2c}{2-\theta} \left( s^{2-\theta} g^{\theta-1}(s) - g(s) \right)}$$

be a continuous function on  $D_g$ .

Given that  $H(s, g(s))$  is continuous on  $D_g$ , we will show that there is a non-negative continuous function  $U(s, g(s))$  on  $D_g$  such that

$$(10) \quad H(s, g(s)) \leq U(s, g(s)).$$

By the reverse Young inequality, we have

$$(11) \quad s^{2-\theta} g^{\theta-1}(s) \geq (2-\theta)s + (\theta-1)g(s)$$

on  $D_g$  and for  $0 < \theta \leq 1$ .

Consequently,

$$(12) \quad s^{2-\theta} g^{\theta-1}(s) - g(s) \geq (2-\theta)(s - g(s))$$

and

$$(13) \quad \frac{1}{\theta} s^2 - \frac{2}{\theta(2-\theta)} s^{2-\theta} g^\theta(s) + \frac{1}{2-\theta} g^2(s) \leq \frac{1}{\theta} (s - g(s))^2$$

on  $D_g$ .

Hence,

$$(14) \quad H(s, g(s)) \leq \frac{\alpha + \frac{c\alpha}{\theta} (s - g(s))^2}{2c(s - g(s))}$$

on  $D_g$ . This follows immediately using (12) and (13) in (9). We have established the upper estimate in (10).

Now consider (7), and a related nonlinear differential equation

$$(15) \quad h'(s) = \frac{\alpha + \frac{c\alpha}{\theta} (s - h(s))^2}{2c(s - h(s))}$$

on  $D_h$ .

We proceed to construct explicitly the minimal and maximal solutions of Eq. (15). It follows that

$$(16) \quad \widehat{h}(s) = s - \left( \frac{\theta}{\alpha} + \sqrt{\frac{\theta^2}{\alpha^2} - \frac{\theta}{c}} \right)$$

is the minimal solution, and

$$(17) \quad q(s) = s - \left( \frac{\theta}{\alpha} - \sqrt{\frac{\theta^2}{\alpha^2} - \frac{\theta}{c}} \right)$$

is the maximal solution which are defined for all  $s \geq 0$ , where  $c > \frac{\alpha^2}{\theta}$  with  $0 < \theta \leq 1$  and  $0 < \alpha < \infty$ .

Then, any non-negative solution  $s \mapsto g_*(s)$  of (7) satisfies the sharp two-sided estimate

$$(18) \quad \widehat{h}_+(s) \leq g_*(s) \leq q_+(s) \quad (s \geq 0)$$

which is implied by a comparison principle for (7) and (15), where  $m_+$  denotes the positive part of the function  $m$ . Using the estimate (18), we can easily show that  $g_*(s) \leq s$  for all  $s \geq 0$ , and  $\lim_{s \rightarrow \infty} \frac{g_*(s)}{s} = 1$ .

Assuming that  $c(s) = ce^{\alpha s}$  in (4) and (5), then

$$\begin{aligned}
 \mathbf{E}_{x,x} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) &\leq c \mathbf{E}_{x,x} \left( \int_0^\tau \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] dr \right) + \Phi_\theta(x, x) \\
 (19) \qquad \qquad \qquad &\leq c \mathbf{E}_{x,x} \left( \tau \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) + \Phi_\theta(x, x)
 \end{aligned}$$

for any stopping time  $\tau$  of the Bessel process  $X$  starting at  $x \geq 0$  under  $\mathbf{P}_x$ .

It is easy to estimate  $\Phi_\theta(x, x)$  from above using (18). For fixed  $0 < \theta \leq 1$ , and all  $x \geq 0$ , we have

$$\begin{aligned}
 \Phi_\theta(x, x) &= \left( 1 + \frac{c}{\theta} x^2 \right) e^{\alpha x} + \frac{c}{2-\theta} g_*^2(x) e^{\alpha x} - \frac{2c}{\theta(2-\theta)} x^{2-\theta} g_*^\theta(x) e^{\alpha x} \\
 &\leq \left( 1 + \frac{c}{\theta} x^2 \right) e^{\alpha x} + \frac{c}{2-\theta} \left( x - \frac{\theta}{\alpha} + \sqrt{\frac{\theta^2}{\alpha^2} - \frac{\theta}{c}} \right)_+^2 e^{\alpha x} \\
 (20) \qquad \qquad &\quad - \frac{2c}{\theta(2-\theta)} x^{2-\theta} \left( x - \frac{\theta}{\alpha} - \sqrt{\frac{\theta^2}{\alpha^2} - \frac{\theta}{c}} \right)_+^\theta,
 \end{aligned}$$

where  $n_+$  is the positive part of the function  $n$ .

Now passing to the limit in (19) as  $c \downarrow \frac{\alpha^2}{\theta}$ , taking into account (20), we get

$$\begin{aligned}
 \mathbf{E}_{x,x} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) &\leq \frac{\alpha^2}{\theta} \mathbf{E}_{x,x} \left( \tau \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) \\
 (21) \qquad \qquad \qquad &\quad + \frac{2}{2-\theta} \left( \frac{\alpha^2}{\theta^2} x^2 - \frac{\alpha^2}{\theta^2} x^{2-\theta} \left( x - \frac{\theta}{\alpha} \right)_+^\theta - \alpha x + 1 \right) e^{\alpha x}.
 \end{aligned}$$

The proof of the theorem is almost complete. It only remains to estimate the term  $\mathbf{E}_{x,x} \left( \tau \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right)$  on the right-hand side of the inequality (21) from above. To establish this estimate, we need the following elementary inequality. Let  $L > 1$ , then

$$(22) \qquad \qquad \qquad ae^b \leq e^a + Le^b - bL - 1 - L$$

for  $a \geq 0$ , and  $b \geq 0$ .

Suppose that there are positive constants  $M, N$  and  $L$  such that  $MN = \frac{\alpha^2}{\theta}$  for  $0 < \theta \leq 1$  and  $0 < \alpha < \infty$ , and  $1 < L < \frac{1}{N}$  for  $0 < N < 1$ . Now set  $a = M\tau$  and  $b = \alpha \max_{0 \leq t \leq \tau} X_t$  in (22). This immediately implies that

$$\frac{\alpha^2}{\theta} \mathbf{E}_{x,x} \left( \tau \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) = N \mathbf{E}_{x,x} \left( M\tau \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right)$$

$$\begin{aligned}
 &\leq N L \mathbf{E}_{x,x} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) + N \mathbf{E}_{x,x} \left( e^{M\tau} \right) \\
 (23) \quad &\quad - \alpha N L \mathbf{E}_{x,x} \left( \max_{0 \leq t \leq \tau} X_t \right) - N(1 + L)
 \end{aligned}$$

using (22).

Combining (21) and (23), we obtain

$$\begin{aligned}
 \mathbf{E}_{x,x} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) &\leq N L \mathbf{E}_{x,x} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) + N \mathbf{E}_{x,x} \left( e^{M\tau} \right) \\
 &\quad + \frac{2}{2-\theta} \left( \frac{\alpha^2}{\theta^2} x^2 - \frac{\alpha^2}{\theta^2} x^{2-\theta} \left( x - \frac{\theta}{\alpha} \right)_+^\theta - \alpha x + 1 \right) e^{\alpha x} \\
 &\quad - \alpha N L \mathbf{E}_{x,x} \left( \max_{0 \leq t \leq \tau} X_t \right) - N(1 + L) \\
 (24) \quad &\leq N L \mathbf{E}_{x,x} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau} X_t \right] \right) + N \mathbf{E}_{x,x} \left( e^{M\tau} \right).
 \end{aligned}$$

The desired result (2) now follows immediately from (24). We complete the proof of the theorem by proving the sharpness of (2). Let  $B = (B_t)_{t \geq 0}$  be a standard Brownian motion, and let

$$(25) \quad \tau_\beta = \inf \{ t > 0 : |B_t| = \beta \} \quad (\beta > 0)$$

be a stopping time for  $B$ . Then, it is well known (see [6], p. 669) that

$$(26) \quad \mathbf{E} \left( e^{M\tau_\beta} \right) = \frac{1}{\cos \sqrt{2M}\beta}$$

for  $0 < \beta < \frac{\pi}{2\sqrt{2M}}$ . On the other hand, we have

$$(27) \quad \mathbf{E} \left( \exp \left[ \alpha \max_{0 \leq t \leq \tau_\beta} |B_t| \right] \right) = e^{\alpha\beta}.$$

Choose  $N$  and  $L$  to be positive constants such that

$$(28) \quad N < \frac{e^{\alpha\beta} \cos \sqrt{2M}\beta}{1 + e^{\alpha\beta} \cos \sqrt{2M}\beta}$$

and

$$(29) \quad L = \frac{1}{N} - \frac{1}{e^{\alpha\beta} \cos \sqrt{2M}\beta}$$

for positive constants  $M$  and  $N$  satisfying  $MN = \alpha^2$  with  $0 < \alpha < \infty$ .

It follows from the right-hand side of the inequality (2) that

$$(30) \quad \frac{N}{1 - NL} \mathbf{E} \left( e^{M\tau_\beta} \right) = \frac{N}{(1 - NL) \cos \sqrt{2M}\beta} = e^{\alpha\beta}$$

using (26) and (29). This proves that we have equality in (2) which is attained from (27) and (30). The proof of the theorem is now complete.  $\square$

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