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w-MATLIS COTORSION MODULES AND w-MATLIS DOMAINS

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Abstract. Let R be a domain with its field Q of quotients. An R-module M is said to be weak w-projective if $\mathrm{Ext}_R^1(M,N)=0$ for all $N\in\mathcal{P}_w^\dagger$, where $\mathcal{P}_w^\dagger$ denotes the class of GV-torsionfree R-modules N with the property that $\mathrm{Ext}_R^k(M,N)=0$ for all w-projective R-modules M and for all integers $k\geq 1$. In this paper, we define a domain R to be w-Matlis if the weak w-projective dimension of the R-module Q is ≤ 1. To characterize w-Matlis domains, we introduce the concept of w-Matlis cotorsion modules and study some basic properties of w-Matlis modules. Using these concepts, we show that R is a w-Matlis domain if and only if $\mathrm{Ext}_R^k(Q,D)=0$ for any $\mathcal{P}_w^\dagger$-divisible R-module D and any integer $k\geq 1$, if and only if every $\mathcal{P}_w^\dagger$-divisible module is w-Matlis cotorsion, if and only if w.w-$\mathrm{pd}_R Q/R\leq 1$.

1. Introduction

Throughout this paper, R will denote a commutative domain with 1 and $Q(\neq R)$ its field of quotients. For a ring R and an R-module M, we use $\mathrm{pd}_R M$ to denote the classical projective dimension of M.

Now, we review some definitions and notation. Let J be an ideal of R. Following [23], J is called a *Glaz-Vasconcelos ideal* (a GV-*ideal* for short) if J is finitely generated and the natural homomorphism φ: $R\to J^*=\mathrm{Hom}_R(J,R)$ is an isomorphism. Note that the set $\mathrm{GV}(R)$ of GV-ideals of R is a multiplicative system of ideals of R. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M)=\{x\in M\mid Jx=0 \text{ for some } J\in \mathrm{GV}(R)\}.$$

Thus $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. Now M is said to be GV-torsion (resp., GV-tosionfree) if $\mathrm{tor}_{\mathrm{GV}}(M)=M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M)=0$). A GV-torsionfree module M is called a *w-module* if $\mathrm{Ext}_R^1(R/J,M)=0$ for any $J\in\mathrm{GV}(R)$. Then projective modules and reflexive modules are both w-modules. In [24], it was shown that all flat modules are w-modules. In [20], a GV-torsionfree

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module M is said to be a *strong w-module* if $\text{Ext}_R^i(N, M) = 0$ for any integer $i \geq 1$ and for all GV-torsion modules N. Then all GV-torsionfree injective modules are strong w-modules. Clearly all strong w-modules are w-modules. Let w-Max(R) denote the set of w-ideals of R maximal among proper integral w-ideals of R and we call $m \in w$-Max(R) a maximal w-ideal of R. Then by [23, Proposition 3.8] every maximal w-ideal is prime. Notice that an R-module M is GV-torsion if and only if $M_m = 0$ for all $m \in w$-Max(R) (see [21, Theorem 2.7]). For any GV-torsionfree R-module M,

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \text{GV}(R)\}$$

is a w-submodule of $E(M)$ containing M and is called the *w-envelope* of M, where $E(M)$ denotes the injective hull (or envelope) of M. It is clear that a GV-torsionfree module M is a w-module if and only if $M_w = M$. Following [14], let M and N be R-modules and let $f\colon M \to N$ be a homomorphism, then f is called a *w-monomorphism* (resp., *w-epimorphism*, *w-isomorphism*) if $f_m\colon M_m \to N_m$ is a monomorphism (resp., an epimorphism, an isomorphism) for all $m \in w$-Max(R). A sequence $A \to B \to C$ of modules and homomorphisms is called *w-exact* if the sequence $A_m \to B_m \to C_m$ is exact for all $m \in w$-Max(R).

In recent years, homological theoretic characterization of w-modules has received attention in several papers in the literature (for example see [1, 19, 20, 22]). The notion of w-projective modules and w-flat modules appeared first in [15] when R is an integral domain and was extended to an arbitrary commutative ring in [6,17]. In [6], an R-module M is said to be *w-flat* if for any w-monomorphism $f\colon A \to B$, the induced sequence $1 \otimes f\colon M \otimes_R A \to M \otimes_R B$ is a w-monomorphism. In [17], F. G. Wang and H. Kim generalized projective modules to w-projective modules by the w-operation. An R-module M is said to be *w-projective* if $\text{Ext}_R^1(L(M), N)$ is GV-torsion for any torsionfree w-module N, where $L(M) = (M/\text{tor}_{\text{GV}}(M))_w$. Following [20], throughout this paper, $\mathcal{P}_w^\dagger$ denotes the class of GV-torsionfree R-modules N with the property that $\text{Ext}_R^k(M, N) = 0$ for all w-projective R-modules M and for all integers $k \geq 1$. Clearly, every module in $\mathcal{P}_w^\dagger$ is a strong w-module. Then an R-module M is said to be *weak w-projective* if $\text{Ext}_R^1(M, N) = 0$ for all $N \in \mathcal{P}_w^\dagger$. Then projective modules are weak w-projective. An R-module M is said to have *weak w-projective dimension* less than or equal to an integer $n \geq 0$, denoted by w.w-pd$_R M \leq n$, if there is a w-exact sequence

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is a weak w-projective module for every $i \in \{0, \ldots, n\}$. This w-exact sequence is called a *weak w-projective w-resolution* of length n of M. If n is the length of a shortest weak w-projective w-resolution of M, then w.w-pd$_R M = n$, and otherwise, we set w.w-pd$_R M = \infty$.

Proposition 1.1 ([20, Proposition 3.1])**.** *The following statements are equivalent for an R-module M.*

(1) w.w-pd$_R M \leq n$;

(2) $\text{Ext}_R^{n+k}(M, N) = 0$ *for all* $N \in \mathcal{P}_w^\dagger$ *and for any integer* $k > 0$;
(3) $\text{Ext}_R^{n+1}(M, N) = 0$ *for all* $N \in \mathcal{P}_w^\dagger$;
(4) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is an exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are projective* R*-modules, then* P_n *is weak* w*-projective*;
(5) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a* w*-exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are weak* w*-projective* R*-modules, then* P_n *is weak* w*-projective*;
(6) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is an exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are weak* w*-projective* R*-modules, then* P_n *is weak* w*-projective*;
(7) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a* w*-exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are projective* R*-modules, then* P_n *is weak* w*-projective*;
(8) *If* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a* w*-exact sequence, where* $P_0, P_1, \ldots, P_{n-1}$ *are weak* w*-projective* w*-modules over* R*, then* P_n *is weak* w*-projective.*

Recall that, in [20], the *global weak* w*-projective dimension* of a ring R is defined by

$$\text{gl.w.}w\text{-dim}(R) = \sup\{\text{w.}w\text{-pd}_R M \mid M \text{ is an } R\text{-module}\}.$$

Proposition 1.2 ([20, Proposition 3.2 and Theorem 4.3])**.** *The following statements are equivalent for an integral domain* R*.*
(1) R *is a Krull domain*;
(2) $\text{gl.w.}w\text{-dim}(R) \leq 1$;
(3) *Every ideal of* R *is weak* w*-projective.*

Recall that a domain R is said to *Matlis* if the projective dimension of the R-module Q is ≤ 1. It is well-known that Matlis domains can also be characterized in term of homological algebra. Lee, in [7], has shown that Matlis domains are those over which h-divisible R-modules are Matlis cotorsion. (Recall that an R-module M is called *Matlis cotorsion* [10] if $\text{Ext}_R^1(Q, M) = 0$, and an R-module D is called *h-divisible* if it is an epic image of an injective R-module.) This notion has been considered by many authors, for example see [2, 4, 5, 7–10, 12]. In this paper, a domain R is called *w-Matlis* if the weak w-projective dimension of the R-module Q is ≤ 1. The main purpose of this article is to extend the aforementioned homological theoretic methods to investigate w-Matlis domains.

In Section 2 we study $\mathcal{P}_w^\dagger$-divisible modules. It is shown that for a $\mathcal{P}_w^\dagger$-divisible module D, $\text{tor}(D)$ is $\mathcal{P}_w^\dagger$-divisible. Moreover, it is shown that for a divisible strong w-module D, if every $\mathcal{P}_w^\dagger$-divisible torsion R-submodule of D is a direct summand of D, then $\text{w.}w\text{-pd}_R Q \leq 1$. In Section 3 we introduce and study the concept of w-Matlis cotorsion modules (see Definition 3.1). It is shown that for an h-divisible w-module D, we have the exact sequence $0 \to K \to E \to D \to 0$, where E is a Q-vector space and K is an h-reduced w-Matlis cotorsion w-module. Moreover, for a projective R-module P if F is a

w-strongly flat R-submodule of P, then F is weak w-projective. In Section 4 we introduce and study the concept of w-Matlis domains (see Definition 4.1). Firstly, we construct an example of a w-Matlis domain R which is not a Matlis domain. And, by pullbacks, we construct an example of a w-Matlis domain R which is not a Krull domain. Moreover, for a domain R, R is a w-Matlis domain if and only if $\mathrm{Ext}_R^k(Q, D) = 0$ for any $\mathcal{P}_w^\dagger$-divisible R-module D and any integer $k \geq 1$, if and only if every $\mathcal{P}_w^\dagger$-divisible module is w-Matlis cotorsion, if and only if w.w-pd$_R Q/R \leq 1$.

2. $\mathcal{P}_w^\dagger$-divisible modules

Recall that, in [20], an R-module D is said to be $\mathcal{P}_w^\dagger$-divisible if it is isomorphic to E/N, where E is a GV-torsionfree injective R-module and $N \in \mathcal{P}_w^\dagger$ is a submodule of E. It is obvious that every $\mathcal{P}_w^\dagger$-divisible R-module is h-divisible. Moreover, it follows from [20] that $M \in \mathcal{P}_w^\dagger$ for any $\mathcal{P}_w^\dagger$-divisible R-module M.

Lemma 2.1. (1) *Let M be an R-module and any integer $m \geq 1$. Then* w.w-pd$_R M \leq m$ *if and only if* $\mathrm{Ext}_R^m(M, D) = 0$ *for all $\mathcal{P}_w^\dagger$-divisible R-modules D.*

(2) *For any $\mathcal{P}_w^\dagger$-divisible R-modules D,* $\mathrm{id}_R D + 1 \leq w$.w.gl.dim$(R)$.

Proof. (1) If $m = 1$, it is just [20, Proposition 4.2]. Let $X \in \mathcal{P}_w^\dagger$. We have an exact sequence $0 \to X \to E \to H \to 0$, where E is a GV-torsionfree injective R-module. Then H is a $\mathcal{P}_w^\dagger$-divisible R-module. So we have the induced exact sequence $\mathrm{Ext}_R^m(M, H) \to \mathrm{Ext}_R^{m+1}(M, X) \to \mathrm{Ext}_R^{m+1}(M, E) = 0$. Here the first term is zero by hypothesis. Thus $\mathrm{Ext}_R^{m+1}(M, X) = 0$. Therefore, by Proposition 1.1, w.w-pd$_R M \leq m$.

For the converse, assume w.w-pd$_R M \leq m$. Let D be a $\mathcal{P}_w^\dagger$-divisible module. Then we have an exact sequence $0 \to K \to E \to D \to 0$, where E is a GV-torsionfree injective R-module and $K \in \mathcal{P}_w^\dagger$. Thus we have the induced exact sequence $0 = \mathrm{Ext}_R^m(M, E) \to \mathrm{Ext}_R^m(M, D) \to \mathrm{Ext}_R^{m+1}(M, D)$. Here the third term is zero by Proposition 1.1. Therefore $\mathrm{Ext}_R^m(M, D) = 0$, proving the result.

(2) Let w.w.gl.dim$(R) = m$ and D be a $\mathcal{P}_w^\dagger$-divisible R-module. Then we have an exact sequence $0 \to K \to E \to D \to 0$, where E is a GV-torsionfree injective R-module and $K \in \mathcal{P}_w^\dagger$. For any R-module M, we have the induced exact sequence $0 = \mathrm{Ext}_R^m(M, E) \to \mathrm{Ext}_R^m(M, D) \to \mathrm{Ext}_R^{m+1}(M, K)$. Here the third term is zero by Proposition 1.1. So $\mathrm{Ext}_R^m(M, D) = 0$. Therefore $\mathrm{id}_R D \leq m - 1$, completing the proof. □

Recall that an R-module M is said to be *torsionfree*, if given any nonzero element $r \in R$, the left multiplication by r on M is a monomorphism. In general we will let tor(M) denote the *torsion R-submodule* of M; i.e.,

$$\mathrm{tor}(M) = \{x \in M \mid rx = 0 \text{ for some nonzero } r \in R\}.$$

Then $M/\mathrm{tor}(M)$ is torsionfree, and so M is torsionfree if and only if $\mathrm{tor}(M) = 0$. We will say that M is a torsion module if $\mathrm{tor}(M) = M$.

Theorem 2.2. *Let D be a $\mathcal{P}_w^\dagger$-divisible module. Then* $\operatorname{tor}(D)$ *is $\mathcal{P}_w^\dagger$-divisible.*

Proof. Obviously, every $\mathcal{P}_w^\dagger$-divisible R-module is h-divisible. So, by [9, Theorem 1.1], $\operatorname{tor}(D)$ is a direct summand of D. Then we have an exact sequence $0 \to D/\operatorname{tor}(D) \to D \to \operatorname{tor}(D) \to 0$, where $D/\operatorname{tor}(D)$ is torsionfree divisible. Then $D/\operatorname{tor}(D)$ is a Q-vector space. So $D/\operatorname{tor}(D) \in \mathcal{P}_w^\dagger$. By hypothesis, we have an exact sequence $0 \to K \to E \to D \to 0$, where E is a GV-torsionfree injective R-module and $K \in \mathcal{P}_w^\dagger$. Consider the following pullback:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & D/\operatorname{tor}(D) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{tor}(D) & = & \operatorname{tor}(D) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $K, D/\operatorname{tor}(D) \in \mathcal{P}_w^\dagger$, by the exact sequence of the first row, we have $X \in \mathcal{P}_w^\dagger$. Therefore, by the exact sequence of the second column, $\operatorname{tor}(D)$ is $\mathcal{P}_w^\dagger$-divisible, establishing the result. $\square$

Theorem 2.3. *Let D be a divisible strong w-module. Suppose that every $\mathcal{P}_w^\dagger$-divisible torsion R-submodule of D is a direct summand of D. Then* w.w-$\operatorname{pd}_R Q \le 1$.

Proof. Let $A \in \mathcal{P}_w^\dagger$ and $E = E(A)$ be the injective envelope of A. Then E/A is $\mathcal{P}_w^\dagger$-divisible torsion. Consider an exact sequence $0 \to E/A \to G \to Q \to 0$. Then G is a divisible strong w-module. Obviously, E/A is the $\mathcal{P}_w^\dagger$-divisible torsion R-submodule of G. By hypothesis, E/A is a direct summand of G. So $\operatorname{Ext}_R^1(Q, E/A) = 0$. In the exact sequence $\operatorname{Ext}_R^1(Q, E/A) \to \operatorname{Ext}_R^2(Q, A) \to \operatorname{Ext}_R^2(Q, E) = 0$ induced by an exact sequence $0 \to A \to E \to E/A \to 0$, the first term is zero by the first part of the proof. Then $\operatorname{Ext}_R^2(Q, A) = 0$. Therefore, by Proposition 1.1, w.w-$\operatorname{pd}_R Q \le 1$. $\square$

3. *w*-Matlis cotorsion modules

We start this section with the definition of w-Matlis cotorsion modules, which is generation of that of Matlis cotorsion modules.

Definition 3.1. An R-module M is called *w-Matlis cotorsion* if $\operatorname{Ext}_R^1(Q, L(M)) = 0$, where $L(M) = (M/\operatorname{tor}_{\mathrm{GV}}(M))_w$.

Recall that, in [16], an R-module E is said to be w-*injective* if $\mathrm{Ext}_R^1(M, L(E))$ is GV-torsion for any R-modules M, where $L(E) = (E/\mathrm{tor}_{\mathrm{GV}}(E))_w$. It is obvious that every w-injective R-module is w-Matlis cotorsion. Particularly, every GV-torsion R-module is w-Matlis cotorsion. In what follows, we denote by w-$\mathcal{MC}$ the class of all w-Matlis cotorsion R-modules.

Proposition 3.2. (1) *Let M and M' be R-modules and let $f\colon M \to M'$ be a w-isomorphism. Then M is w-Matlis cotorsion if and only if M' is w-Matlis cotorsion. Thus, for a* GV*-torsionfree R-module M, M is w-Matlis cotorsion if and only if M_w is w-Matlis cotorsion.*

(2) *Let M be a w-module. Then M is Matlis cotorsion if and only if M is w-Matlis cotorsion.*

(3) *Let E be a* GV*-torsionfree injective R-module. Then* $\mathrm{Hom}_R(A, E)$ *is w-Matlis cotorsion.*

Proof. (1) Since f is a w-isomorphism, the induced map $\overline{f}\colon M/\mathrm{tor}_{\mathrm{GV}}(M) \to M'/\mathrm{tor}_{\mathrm{GV}}(M')$ defined by $\overline{f}(\overline{x}) = \overline{f(x)}$ is also a w-isomorphism. By [18, Theorem 6.3.2], $L(M) \cong L(M')$. Thus $\mathrm{Ext}_R^1(Q, L(M)) \cong \mathrm{Ext}_R^1(Q, L(M'))$. Therefore, M is w-Matlis cotorsion if and only if M' is w-Matlis cotorsion.

(2) and (3) are obvious. □

Recall that, in [9], an R-module M is said to be h-reduced if M has no nonzero h-divisible submodules.

Lemma 3.3. *Let B be a w-module and let $0 \to A \to B \to C \to 0$ be an exact sequence of* GV*-torsionfree R-modules. If $B \in w$-$\mathcal{MC}$ and C is h-reduced, then $A \in w$-$\mathcal{MC}$.*

Proof. By [23, Theorem 2.7], A is a w-module. Since C is h-reduced, so $\mathrm{Hom}_R(Q, C) = 0$. Thus, we have an exact sequence $0 \to \mathrm{Ext}_R^1(Q, A) \to \mathrm{Ext}_R^1(Q, B) = 0$. Therefore $\mathrm{Ext}_R^1(Q, A) = 0$, completing the proof. □

Lemma 3.4. *Let A be a torsion R-module. Then* $\mathrm{Hom}_R(A, X)$ *is an h-reduced w-Matlis cotorsion w-module for any w-module X.*

Proof. Let B be an R-module. By Adjoint Isomorphism Theorem, $\mathrm{Hom}_R(Q, \mathrm{Hom}_R(A, B)) = 0$. Then $\mathrm{Hom}_R(A, B)$ is h-reduced. Consider an exact sequence $0 \to X \to E \to Y \to 0$, where E is the injective envelope of X. By [23, Theorem 2.7], $\mathrm{Hom}_R(A, X)$ and $\mathrm{Hom}_R(A, E)$ are w-modules. So we have an exact sequence $0 \to \mathrm{Hom}_R(A, X) \to \mathrm{Hom}_R(A, E) \to \mathrm{Hom}_R(A, Y)$. Here $\mathrm{Hom}_R(A, Y)$ is h-reduced. Therefore, by Proposition 3.2 and Lemma 3.3, $\mathrm{Hom}_R(A, X)$ is an h-reduced w-Matlis cotorsion w-module. □

Lemma 3.5. *Let D be an h-divisible w-module. Then we have the exact sequence $0 \to K \to E \to D \to 0$, where E is a Q-vector space and K is an h-reduced w-Matlis cotorsion w-module. Particularly, if D is a $\mathcal{P}_w^{\dagger}$-divisible module, then we have the exact sequence $0 \to K \to E \to D \to 0$, where E is a Q-vector space and K is an h-reduced w-Matlis cotorsion w-module.*

Proof. Let D be an h-divisible w-module. Then we have an exact sequence

$$0 \to \mathrm{Hom}_R(Q/R, D) \to \mathrm{Hom}_R(Q, D) \to D \to 0$$

induced by an exact sequence $0 \to R \to Q \to Q/R \to 0$. Thus we can assume that $E := \mathrm{Hom}_R(Q, D)$ and $K := \mathrm{Hom}_R(Q/R, D)$, completing the proof by Lemma 3.4. □

Lemma 3.6. *Let M be a torsionfree strong w-module. Then $\mathrm{Ext}^1_R(Q/R, M)$ is an h-reduced w-Matlis cotorsion w-module.*

Proof. Let M be a torsionfree strong w-module. By [18, Theorem 3.6.13], we have an exact sequence $0 \to M \to Q \otimes_R M \to (Q/R) \otimes_R M \to 0$. Here, M and $Q \otimes_R M$ are torsionfree, and $Q \otimes_R M$ is GV-torsionfree injective. Then $Q \otimes_R M$ is a strong w-module. So $(Q/R) \otimes_R M$ is a strong w-module. So we have the induced the exact sequence $0 \to \mathrm{Hom}_R(Q/R, (Q/R) \otimes_R M) \to \mathrm{Ext}^1_R(Q/R, M) \to \mathrm{Ext}^1_R(Q/R, Q \otimes_R M) = 0$. Thus $\mathrm{Hom}_R(Q/R, (Q/R) \otimes_R M) \cong \mathrm{Ext}^1_R(Q/R, M)$. Then the assertion by Lemma 3.4. □

Proposition 3.7. *Let M ba an h-reduced w-module.*

(1) *$M \in w$-$\mathcal{MC}$ if and only if $M \cong \mathrm{Ext}^1_R(Q/R, M)$.*

(2) *Let M be a torsionfree strong w-module. Then we have an exact sequence $0 \to M \to C \to E \to 0$, where E is a Q-vector space and C is an h-reduced w-Matlis cotorsion w-module.*

Proof. (1) By hypothesis, $\mathrm{Hom}_R(Q, M) = 0$. So we have an exact sequence

$$0 \to M \to \mathrm{Ext}^1_R(Q/R, M) \to \mathrm{Ext}^1_R(Q, M) \to 0.$$

Therefore, $M \in w$-$\mathcal{MC}$ if and only if $\mathrm{Ext}^1_R(Q, M) = 0$ if and only if $M \cong \mathrm{Ext}^1_R(Q/R, M)$.

(2) In the proof of (1), we may assume that $C := \mathrm{Ext}^1_R(Q/R, M)$ and $E := \mathrm{Ext}^1_R(Q, M)$. Then the assertion follows by Lemma 3.6. □

Recall that, in [3], an R-module M is *strongly flat* if $\mathrm{Ext}^1_R(M, N) = 0$ for every Matlis cotorsion R-module N. It was shown in [8, Theorem 3.2] that strongly flat submodules of projective modules over any domain are projective. Then an R-module M is called *w-strongly flat* if $\mathrm{Ext}^1_R(M, N) = 0$ for any w-Matlis cotorsion R-module N. In fact, we have:

Theorem 3.8. *Let P be a projective R-module. If F is a w-strongly flat R-submodule of P, then F is weak w-projective.*

Proof. By Proposition 1.1, we establish the weak w-projectivity of F by proving that w.w-$\mathrm{pd}_R P/F \leq 1$. Let D be any $\mathcal{P}^\dagger_w$-divisible R-module. This will be done by showing that $\mathrm{Ext}^1_R(P/F, D) = 0$ by Lemma 2.1.

By Lemma 3.5, we have an exact sequence $0 \to K \to E \to D \to 0$, where E is a Q-vector space and K is an h-reduced w-Matlis cotorsion w-module. So we have the induced exact sequence

$$0 = \mathrm{Ext}^1_R(P/F, E) \to \mathrm{Ext}^1_R(P/F, D) \to \mathrm{Ext}^2_R(P/F, K) \to \mathrm{Ext}^2_R(P/F, E) = 0.$$

Thus it suffices to verify that $\mathrm{Ext}^2_R(P/F, N) = 0$ for arbitrary w-Matlis cotorsion R-module N. Now in the exact sequence

$$0 = \mathrm{Ext}^1_R(P, N) \to \mathrm{Ext}^1_R(F, N) \to \mathrm{Ext}^2_R(P/F, N) \to \mathrm{Ext}^2_R(P, N) = 0$$

induced by the exact sequence $0 \to F \to P \to P/F \to 0$, we have $\mathrm{Ext}^1_R(F, N) = 0$, since F is w-strongly flat. This establishes the result. □

It was shown in [20, Proposition 2.12] that if I is a nonzero nil ideal of R, then I is never weak w-projective ideal. In fact, we have:

Corollary 3.9. *Let R be a domain. If I is a w-strongly flat ideal of R, then I is weak w-projective.*

Corollary 3.10. *Let R be a domain. If every ideal of R is w-strongly flat, then R is a Krull domain.*

Proof. This follows easily from Proposition 1.2 and Theorem 3.8. □

4. w-Matlis domains

We start this section with the definition of w-Matlis domains.

Definition 4.1. A domain R is called *w-Matlis* if the weak w-projective dimension of the R-module Q is ≤ 1.

Obviously, all Krull domains and all Matlis domains are w-Matlis. But this raises the question: is there a w-Matlis domain that is not a Krull domain (resp., a Matlis domain)? The answer of this question is positive. In the following, to explore a more interesting question, we can construct an example of a w-Matlis domain R which is not a Matlis domain.

Example 4.2. Let K be an uncountable field. Consider the polynomial domain $R = K[x_1, x_2]$ over K in two indeterminates x_1 and x_2. Let Q be the quotient field of R. Then, by [5, Theorem 2], $\mathrm{pd}_R Q = 2$. Thus, R is not Matlis. Since R is a unique factorization domain, and so R is a Krull domain. Therefore, R is a w-Matlis domain.

In the following, to explore a more interesting question, by pullbacks, we can construct an example of a w-Matlis domain R which is not a Krull domain.

Example 4.3. Let $\mathbb{Z}$ denote the ring of integers and let $\mathbb{R}$ denote the field of real numbers. Consider the power series ring $T = \mathbb{R}[[X]]$ in the indeterminate X with coefficients in the field $\mathbb{R}$. Then, for the pullback ring $R = \mathbb{Z} + XT$, we have: the domain R is a w-Matlis domain, but not a Krull domain. Indeed, by [13, Corollary 2] and [13, Theorem 6], R is a Matlis domain. So R is a w-Matlis domain. Clearly, T is an SM domain. Then, by [18, Theorem 8.4.15], R is not an SM domain. Therefore, R is not a Krull domain.

Proposition 4.4. *Let R be a domain. If $\mathrm{Ext}^2_R(Q, L(M)) = 0$ for every w-Matlis cotorsion R-module M, then R is a w-Matlis domain.*

Proof. Let D be a $\mathcal{P}_w^\dagger$-divisible R-module. By Lemma 3.5, we have the exact sequence $0 \to K \to E \to D \to 0$, where E is a Q-vector space and K is an h-reduced w-Matlis cotorsion w-module. Then in view of the induced exact sequence

$$0 \to \mathrm{Ext}_R^1(Q, D) \to \mathrm{Ext}_R^2(Q, K) \to 0,$$

the hypothesis implies $\mathrm{Ext}_R^1(Q, D) = 0$. It follows from Lemma 2.1 that w.w-pd$_R Q \le 1$, and so R is a w-Matlis domain. □

Theorem 4.5. *If a domain R has the property that the torsion submodule in any $\mathcal{P}_w^\dagger$-divisible R-module is a summand, then R is a w-Matlis domain. Particularly, if a domain R has the property that every divisible strong w-module D is $\mathcal{P}_w^\dagger$-divisible, then* $\mathrm{tor}(D)$ *is a direct summand of D and R is a w-Matlis domain.*

Proof. Let D be a $\mathcal{P}_w^\dagger$-divisible R-module. Then $D/\mathrm{tor}(D)$ is torsionfree divisible. So $D/\mathrm{tor}(D)$ is a direct sum of copies of Q. In the exact sequence $0 \to \mathrm{tor}(D) \to D \to \bigoplus Q \to 0$, we have $\mathrm{Ext}_R^1(\bigoplus Q, \mathrm{tor}(D)) = 0$ by hypothesis. Thus $\mathrm{Ext}_R^1(Q, D) = 0$. Therefore, by Lemma 2.1, w.w-pd$_R Q \le 1$, completing the proof. □

Theorem 4.6. *Let R be a domain. Then the following are equivalent*:

(1) *R is a w-Matlis domain*;

(2) $\mathrm{Ext}_R^k(Q, D) = 0$ *for any $\mathcal{P}_w^\dagger$-divisible R-module D and any integer $k \ge 1$*;

(3) *Every $\mathcal{P}_w^\dagger$-divisible module is w-Matlis cotorsion*;

(4) w.w-pd$_R Q/R \le 1$.

Proof. (1)⇒(2) Let D be a $\mathcal{P}_w^\dagger$-divisible R-module. Then we have an exact sequence $0 \to K \to E \to D \to 0$, where E is a GV-torsionfree injective R-module and $K \in \mathcal{P}_w^\dagger$. So in view of the induced exact sequence

$$0 = \mathrm{Ext}_R^k(Q, E) \to \mathrm{Ext}_R^k(Q, D) \to \mathrm{Ext}_R^{k+1}(Q, K), \qquad k \ge 1,$$

we have $\mathrm{Ext}_R^{k+1}(Q, K) = 0$ by Proposition 1.1. Therefore $\mathrm{Ext}_R^k(Q, D) = 0$.

(2)⇒(3) Obvious.

(3)⇒(1) See Lemma 2.1.

(1)⇔(4) Let D be a $\mathcal{P}_w^\dagger$-divisible R-module. Then we have an exact sequence

$$0 \to \mathrm{Ext}_R^1(Q/R, D) \to \mathrm{Ext}_R^1(Q, D) \to 0$$

induced by the exact sequence $0 \to R \to Q \to Q/R \to 0$. Therefore R is a w-Matlis domain if and only if w.w-pd$_R Q/R \le 1$, completing the proof. □

Corollary 4.7. *All torsion-free R-modules are of weak w-projective dimension ≤ 1 if and only if* w.w-pd$_R Q \le 1$ *and* gl.w.w-dim$(R) \le 2$.

Proof. We have only to prove the sufficiency. Let M be a torsionfree R-module. Then we have an exact sequence $0 \to M \to \bigoplus Q \to C \to 0$. Let D be a $\mathcal{P}_w^\dagger$-divisible R-module. So in the induced exact sequence $\mathrm{Ext}_R^1(\bigoplus Q, D) \to \mathrm{Ext}_R^1(M, D) \to \mathrm{Ext}_R^2(C, D)$, the first term is zero by Lemma 2.1(1), and the third term is zero by Lemma 2.1(2). Thus $\text{Ext}_R^1(M, D) = 0$, completing the proof. □

Corollary 4.8. *Let R be a PVMD. If $\text{w.}w\text{-pd}_R F \leq 1$ for any w-flat R-module F, then* $\text{gl.w.}w\text{-dim}(R) \leq 2$.

Proof. By hypothesis, any torsionfree R-module M is w-flat. Then $\text{w.}w\text{-pd}_R M \leq 1$. Therefore, by Corollary 4.7, $\text{gl.w.}w\text{-dim}(R) \leq 2$. □

Recall that, in [11], a ring R is called a *DW-ring* if every ideal of R is a w-ideal. Recently, rings of this type have received attention in several papers in the literature (for example see [1, 14]).

Theorem 4.9. *Let R be a domain. Then the following are equivalent*:
(1) *R is a DW-domain*;
(2) *Every* GV*-ideal is projective*;
(3) *Every h-divisible R-module is $\mathcal{P}_w^\dagger$-divisible.*

Proof. (1)⇔(2) Clear.

(1)⇒(3) Let X be an R-module. Then we have an exact sequence $0 \to X \to E \to E/X \to 0$, where E is the injective envelope of X. Let P be any w-projective module. Then $\text{Ext}_R^1(P, X)$ is GV-torsion. By hypothesis, $\text{Ext}_R^1(P, X)$ is GV-torsionfree. So $\text{Ext}_R^1(P, X) = 0$. Thus $\text{Ext}_R^1(P, E/X) = 0$. It follows from this that $\text{Ext}_R^n(P, X) = 0$, $n \geq 1$. Therefore, $X \in \mathcal{P}_w^\dagger$, completing the proof.

(3)⇒(2) Let $J \in \text{GV}(R)$. Then J is weak w-projective. Thus, by Proposition 1.1, $\text{w.}w\text{-pd}_R R/J \leq 1$. Let X be an R-module. Then we have an exact sequence $0 \to X \to E \to E/X \to 0$, where E is the injective envelope of X. By hypothesis, E/X is $\mathcal{P}_w^\dagger$-divisible. Thus in the reduced exact sequence

$$\text{Ext}_R^1(R/J, E/X) \to \text{Ext}_R^2(R/J, X) \to \text{Ext}_R^2(R/J, E) = 0,$$

we have $\text{Ext}_R^1(R/J, E/X) = 0$ by Lemma 2.1. So $\text{Ext}_R^2(R/J, X) = 0$. Therefore $\text{pd}_R R/J \leq 1$, completing the result. □

Corollary 4.10. *Let R be a DW domain. Then R is a w-Matlis domain if and only if R is a Matlis domain.*

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