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DOUBLE COVERS OF PLANE CURVES OF DEGREE SIX WITH ALMOST TOTAL FLEXES

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Abstract. In this paper, we study plane curves of degree 6 with points whose multiplicities of the tangents are 5. We determine all the Weierstrass semigroups of ramification points on double covers of the plane curves when the genera of the covering curves are greater than 29 and the ramification points are on the points with multiplicity 5 of the tangent.

1. Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is a finite set. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. For a numerical semigroup H we denote by $d_2(H)$ the set consisting of the elements h with $2h \in H$, which is a numerical semigroup.

In this article a *curve* means a projective 1-dimensional algebraic (not necessarily irreducible) variety over an algebraically closed field k of characteristic 0. Let C be a smooth irreducible curve of genus g. For a point P of C we define $H(P)$ as the set

$$\{s \in \mathbb{N}_0 \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = sP\},$$

where $(f)_\infty$ means the polar divisor of f. Let $g(C)$ be the genus of the curve. Then the set $H(P)$ becomes a numerical semigroup of genus $g(C)$, which is called the *Weierstrass semigroup* of P. Such a numerical semigroup is said to be *Weierstrass.* If $\pi : \tilde{C} \longrightarrow C$ is a double covering of a curve with a ramification point $\tilde{P}$ over P, then we have $d_2(H(\tilde{P})) = H(P)$. Such a numerical semigroup $H = H(\tilde{P})$ is said to be *of double covering type.* In this article a double covering $\pi : \tilde{C} \longrightarrow C$ of a curve means that C and $\tilde{C}$ are smooth and irreducible. We

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are interested in the Weierstrass semigroups of ramification points on double covers of smooth plane curves of degree d. Such a numerical semigroup H, i.e., $H = H(\tilde{P})$, is said to be *of double covering type of a plane curve of degree* d, which is abbreviated to *DCP of degree* d. We consider the following problem:

DCP Hurwitz Problem. Let d be a positive integer. Then determine all the Weierstrass semigroups which are DCP of degree d.

For the known facts of DCP Hurwitz Problem for $d \leqq 5$, refer to [3]. We treat the case $d = 6$ in this article. Let C be a smooth plane curve of degree 6 and P its total flex, i.e., $\mathrm{ord}_P\ C.T_P = 6$ where T_P is the tangent line at P on C and $\mathrm{ord}_P\ C.T_P$ is the multiplicity at P of the intersection divisor $C.T_P$ of C with T_P. Then we have $H(P) = \langle 5, 6\rangle$ where $\langle a_1, \ldots, a_s\rangle$ is the additive monoid generated by $a_1, \ldots, a_s$ for positive integers $a_1, \ldots, a_s$. When H is a numerical semigroup with $d_2(H) = \langle 5, 6\rangle$, DCP Hurwitz Problem is solved in [4]. Namely, if $g(H) \geqq 30$ with $d_2(H) = \langle 5, 6\rangle$, then H is DCP of degree 6. We consider the case where P is an almost total flex on C, i.e., $\mathrm{ord}_P\ C.T_P = 5$, in this case we have $H(P) = \langle 5, 9, 13, 17, 21\rangle$, and vice versa. The following is the main result of this article:

Main Theorem. *We determine all the numerical semigroups* H *with* $d_2(H) = \langle 5, 9, 13, 17, 21\rangle$ *which are DCP of degree* 6*. The number of the DCP numerical semigroups* H *is* 70*, and the number of the non-DCP numerical semigroups* H *is* 20*.*

We note that there are many numerical semigroups H which are not DCP even if $d_2(H) = \langle 5, 9, 13, 17, 21\rangle$. This is different from the result (Main Theorem in [4]) in the case of numerical semigroups H with $d_2(H) = \langle 5, 6\rangle$. We do not know whether these twenty numerical semigroups are of double covering type or not. More widely we do not know even whether they are Weierstrass or not.

2. Proof of Main Theorem

In this section, let H be a numerical semigroup with $g(H) \geqq 30$, $d_2(H) = J_6$ and $n \geqq 25$ where we set $J_6 = \langle 5, 9, 13, 17, 21\rangle$ and $n = \min\{h \in H \mid h \text{ is odd}\}$. Let $\delta(H)$ be the number of the odd elements of $\mathbb{N}_0 \backslash H$ which are larger than n and less than $n + 34$. We set $r(H) = 10 - \delta(H)$. Let $t(H)$ be the cardinality of the set

$$\{u \in M(H) \mid u \text{ is an odd integer distinct from } n\},$$

where $M(H)$ denotes the minimal set of generators for the monoid H. Here we prepare the diagram where we only draw its frame, and later associated to H we fill in the blanks by the symbols $\odot$, $\circ$ and $\times$ which indicate an integer in $M(H)$, $H \backslash M(H)$ and $\mathbb{N}_0 \backslash H$, respectively.

$$
\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & & & & & \downarrow \\
(n) & & & & \bullet & +10 \\
 & & & \bullet & (n+18) & \\
 & & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}
$$

Hence, we note that $0 \leqq \delta(H) \leqq 10$, $0 \leqq r(H) \leqq 10$ and

$$g(H) = 20 + \frac{n-1}{2} - r(H)$$

(for example, see Lemma 3.1 in [1]). For example, we associate the following diagram to the numerical semigroup $H_0 = 2J_6 + \langle n, n+4, n+8, n+16 \rangle$ where we set $J_6 = \langle 5, 9, 13, 17, 21 \rangle$:

$$
\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \times & \odot & \downarrow \\
(n) & \times & \circ & \odot & \bullet & +10 \\
 & \circ & \circ & \bullet & (\mathrm{n}{+}18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}
$$

In this case we have $t(H_0) = 3$, $r(H_0) = 7$ and $g(H_0) = 20 + \frac{n-1}{2} - 7$.

The proof of Main Theorem is divided into ninety cases classified by the value of $t(H)$ and the generators which are odd. In this section we take a pointed non-singular plane curve (C, P) of degree 6 with $T_P.C = 5P + R$ where R is a point distinct from P. Then we have $H(P) = J_6$. In the proof of Main Theorem we use the following lemma and theorem many times which are stated in Lemma 2.1 and Theorem 2.3 in [4] respectively.

Lemma 2.1 ([4])**.** i) 2 *points impose independent condition on the system of lines.*

ii) 3 *points fail to impose independent condition on the system of of lines if and only if the three points are collinear.*

iii) 3 *points impose independent condition on the system of conics.*

iv) 4 *points fail to impose independent condition on the system of conics if and only if the four points are collinear.*

v) 5 *points fail to impose independent condition on the system of conics if and only if there are four collinear points among them.*

vi) 6 *points fail to impose independent condition on the system of conics if and only if there are four collinear points among them or the six points are on a conic.*

vii) 4 *points impose independent condition on the system of cubics.*

viii) 5 *points fail to impose independent condition on the system of cubics if and only if the five points are collinear.*

ix) 6 *points fail to impose independent condition on the system of cubics if and only if there are five collinear points among them.*

x) 7 *points fail to impose independent condition on the system of cubics if and only if there are five collinear points among them.*

xi) 8 *points fail to impose independent condition on the system of cubics if and only if there are five collinear points among them or the eight points are on a conic.*

Theorem 2.2 ([4])**.** *Let (C, P) be a pointed non-singular plane curve of degree 6 and H a numerical semigroup with $d_2(H) = H(P)$ and $g(H) \geqq 30$. Set*

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

We note that

$$g(H) = 20 + \frac{n-1}{2} - r$$

with some non-negative integer r. Let $Q_1, \ldots, Q_r$ be points of C different from P with $h^0(Q_1 + \cdots + Q_r) = 1$. Moreover, assume that H has an expression

$$H = 2d_2(H) + \langle n, n + 2l_1, \ldots, n + 2l_s\rangle$$

of generators with positive integers $l_1, \ldots, l_s$ such that for any cubic C_3 the inequality $C_3.C \geqq (l_i - 1)P + Q_1 + \cdots + Q_r$ implies that $C_3.C \geqq l_iP + Q_1 + \cdots + Q_r$, i.e.,

$$h^0(K - (l_i - 1)P - Q_1 - \cdots - Q_r) = h^0(K - l_iP - Q_1 - \cdots - Q_r),$$

where K is a canonical divisor on C. Then the complete linear system $|nP - 2Q_1 - \cdots - 2Q_r|$ is base point free and there is a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$, i.e., H is DCP of degree 6.

We begin the proof of Main Theorem case by case.

(I) *The case $t(H) = 0$.* Then $H = 2J_6 + \langle n\rangle$, which is DCP by Proposition 2.3 in [2].

From now on, we set $E_r = Q_1 + \cdots + Q_r$ with $r = r(H)$ where $Q_1, \ldots, Q_r$ are points of C defined in each item and different from P. For simplicity, we use the following notations: For a conic C_2 and a line L we denote by C_2L or LC_2 the cubic defined by the product of the equations of C_2 and L. If $L = T_P$ where T_P denotes the tangent line at P on C for a pointed non-singular plane curve (C, P), then we use the notation C_2T_P so as not to be confused with the tangent line to C_2. For lines L_1, L_2 and L_3 we also define the cubic $L_1L_2L_3$ and the conic L_1L_2 in a similar way. For a line L we set $L^2 = LL$ and $L^3 = LLL$.

(II) *The case $t(H) = 1$.* There are ten kinds of numerical semigroups. We will show that half of the numerical semigroups with $t(H) = 1$ are DCP. But we will prove that any of the remaining half is not DCP.

II-1) $H = 2J_6 + \langle n, n + 2\rangle$. Then $r(H) = 4$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Then we have $h^0(K - E_4) = 10 - 4 = 6$. Let C_3 be a cubic with $C_3.C \geqq P + E_4$. Then we get $C_3 = L_1C_2$ with a conic C_2, which implies that $h^0(K - P - E_4) = 6$. Hence, H is DCP.

II-2) $H = 2J_6 + \langle n, n+4 \rangle$.

$$
\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \times & \times & \downarrow \\
(n) & \times & \circ & \times & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (\mathrm{n+26}) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}
$$

Assume that H is DCP. Then there are five points $Q_1, \dots, Q_5$ distinct from P such that

$$
\begin{aligned}
&h^0(K-P-E_5) = h^0(K-2P-E_5) = 4,\ h^0(K-3P-E_5) = 3,\\
&h^0(K-4P-E_5) = h^0(K-5P-E_5) = 2,\\
&h^0(K-6P-E_5) = h^0(K-7P-E_5) = 1, \text{ and } h^0(K-8P-E_5) = 0.
\end{aligned}
$$

Let C_3 be a unique cubic with $C_3.C \geqq 7P + Q_1 + \cdots + Q_5$. Then $C_3 = C_2T_P$ with a conic C_2 with $C_2.C \geqq 2P$ and $C_2.C \not\geqq 3P$.
Case: $E_5 \geqq R$. We set $D_4 = E_5 - R$. Let C_3' be a cubic with $C_3'.C \geqq 3P + E_5 = 3P + R + D_4$. Then we get $C_3' = C_2'T_P$ with a conic C_2' containing $Q_1, \dots, Q_4$. This contradicts

$$h^0(K-3P-E_5) \neq h^0(K-5P-E_5).$$

Case: $E_5 \not\geqq R$. Let C_3' be a cubic distinct from C_3 with $C_3'.C \geqq 5P + E_5$ and $C_3'.C \not\geqq 6P$. Then we get $C_3' = C_2'T_P$ with a conic C_2' such that $C_2'.C \geqq E_5$ and $C_2'.C \not\geqq P$. We have $C_2.C \geqq 2P + E_5$, which implies that $C_2.C_2' \geqq E_5$. Hence C_2 and C_2' have a common component L_0. Namely, we have $C_2 = L_0L_1$ and $C_2' = L_0L'$. Since $C_2'.C \not\geqq P$, we have $L_1.C \geqq 2P$. Hence $L_1 = T_P$. Thus we have $C_2 = L_0T_P$, which contradicts $h^0(K-8P-E_5) = 0$.
Thus, H is not DCP.

II-3) $H = 2H_6 + \langle n, n+6 \rangle$.

$$
\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \odot & \times & \downarrow \\
(n) & \times & \times & \circ & \bullet & +10 \\
 & \times & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}
$$

Assume that H is DCP. Then there are four points Q_1, Q_2, Q_3 and Q_4 distinct from P such that

$$
\begin{aligned}
&h^0(K-E_4) = 6,\ h^0(K-P-E_4) = 5,\\
&h^0(K-2P-E_4) = h^0(K-3P-E_4) = 4,\\
&h^0(K-4P-E_4) = h^0(K-5P-E_4) = 3,\ h^0(K-6P-E_4) = 2,\\
&h^0(K-7P-E_4) = h^0(K-10P-E_4) = 1, \text{ and } h^0(K-11P-E_4) = 0.
\end{aligned}
$$

There is a unique cubic C_3 such that $C_3.C \geqq 10P + E_4$ and $C_3.C \ngeqq 11P$. Hence we get $C_3 = L_0T_P^2$ with a unique line L_0 such that $L_0 \not\ni P$.
Case: $E_4 \ngeqq R$. We have $L_0.C \geqq E_4$. Let $C_3'.C \geqq 3P + E_4$. Then $C_3'.L_0 \geqq E_4$, which implies that $C_3' = L_0C_2$ with a conic C_2 satisfying $C_2.C \geqq 3P$ because $L_0 \not\ni P$. Hence we get $C_2 = LT_P$ with a line L, which implies that $C_3'.C \geqq 5P + E_4$. Thus, we have $h^0(K - 3P - E_4) = h^0(K - 5P - E_4)$, which is a contradiction.
Case: $E_4 \geqq R$. We set $D_3 = E_4 - R$. Let $C_3'.C \geqq 3P + E_4 = 3P + R + D_3$. Then $C_3' = T_PC_2$ with a conic C_2 such that $C_2.C \geqq D_3$. Hence, $h^0(K - 3P - E_4) = h^0(K - 5P - E_4)$, which is a contradiction.
Therefore, H is not DCP.

II-4) $H = 2J_6 + \langle n, n+8\rangle$. Then $r(H) = 1$. We set $Q_1 = R$. Let C_3 be a cubic with $C_3.C \geqq 3P + Q_1$. Then we have $C_3 = C_2T_P$ with a conic C_2. Hence we get

$$h^0(K - 3P - Q_1) = h^0(K - 5P - Q_1) = 6.$$

Hence, H is DCP.

II-5) $H = 2J_6 + \langle n, n+12\rangle$. Then $r(H) = 3$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 on the intersection of C and L_1. Then we have $h^0(K - E_3) = 7$. Let C_3 be a cubic with $C_3.C \geqq 4P + E_3$. Then we have $C_3 = L_1LT_P$ with a line L. Hence we get

$$h^0(K - 4P - E_3) = h^0(K - 6P - E_3) = 3.$$

Hence, H is DCP.

II-6) $H = 2J_6 + \langle n, n+14\rangle$. By Theorem 3.1 in [3] H is not DCP.

II-7) $H = 2J_6 + \langle n, n+16\rangle$.

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \times & \times & \odot & \bullet & +10 \\
 & \times & \times & \bullet & (n+18) & \\
 & \times & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there exists some point Q_1 distinct from P such that

$$h^0(K - 4P - Q_1) = 5 \quad \text{and} \quad h^0(K - 12P - Q_1) = 1.$$

Case: $Q_1 = R$. Let C_3 be a cubic with $C_3.C \geqq 4P + R$. Then $C_3 = C_2T_P$ with a conic C_2, which means that $h^0(K - 4P - Q_1) = 6$. This is a contradiction.
Case: $Q_1 \neq R$. There exists a unique cubic C_3 with $C_3.C \geqq 12P + Q_1$. Then $C_3 = T_P^3$, but $T_P^3.C \ngeqq 12P + Q_1$. This is also a contradiction.
Thus, H is not DCP.

II-8) $H = 2J_6 + \langle n, n+22\rangle$. Then $r(H) = 2$. Let L_1 be a line through P different from the tangent line T_P. Let Q_1 and Q_2 be distinct points belonging to $L_1 \cap C$. Let C_3 be a cubic with $C_3.C \geqq 8P + E_2$. Then $C_3 = L_1T_P^2$, which implies that

$$h^0(K-8P-E_2)=h^0(K-11P-E_2)=1.$$

Thus, H is DCP.

II-9) $H=2J_6+\langle n,n+24\rangle$. By Theorem 3.1 in [3] H is not DCP.

II-10) $H=2J_6+\langle n,n+32\rangle$. Then $r(H)=1$. Let Q_1 be a point of C distinct from R. Let L_1 be the line through P and Q_1. Let C_3 be a cubic with $C_3.C\geqq 11P+Q_1$. Then we have $C_3=L_1T_P^2$. Hence we get $h^0(K-12P-Q_1)=0$. Thus, H is DCP.

(III) *The case* $t(H)=2$. There are thirty kinds of numerical semigroups. We will show that half of the numerical semigroups are not DCP. Moreover, we will prove that the remaining half, i.e., fifteen numerical semigroups are DCP.

III-1) $H=2J_6+\langle n,n+2,n+4\rangle$. Then $r(H)=7$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1,Q_2,Q_3 and Q_4 (resp. Q_5,Q_6 and Q_7) on the intersection of C and L_1 (resp. L_2) such that $Q_i\notin L_1$ for $i=5,6,7$. Let C_3 be a cubic with $C_3.C\geqq E_7$. Then $C_3=L_1L_2L$ with a line L, which implies that

$$h^0(K-E_7)=h^0(K-P-E_7)=h^0(K-2P-E_7)=3.$$

Hence, H is DCP.

III-2) $H=2J_6+\langle n,n+2,n+6\rangle$.

$(\to +2)$	$(n+2)$	$(n+4)$	$(n+6)$	$(n+8)$	
$\bullet$	$\odot$	$\times$	$\odot$	$\times$	$\downarrow$
(n)	$\circ$	$\times$	$\circ$	$\bullet$	$+10$
	$\circ$	$\circ$	$\bullet$	$(n+18)$	
	$\circ$	$\bullet$	$(n+26)$	$\swarrow +8$ $(\downarrow +10)$	
	$\bullet$	$(n+34)$			
	$(n+42)$				

Assume that H is DCP. Then there are seven points $Q_1,\dots,Q_7$ distinct from P such that

$$\begin{aligned}&h^0(K-E_7)=h^0(K-P-E_7)=3,\\&h^0(K-2P-E_7)=h^0(K-3P-E_7)=2,\\&h^0(K-4P-E_7)=h^0(K-6P-E_7)=1,\ \text{and}\ h^0(K-7P-E_7)=0.\end{aligned}$$

There is a unique cubic C_3 such that $C_3.C\geqq 6P+E_7$ and $C_3.C\ngeqq 7P$. Hence we get $C_3=C_2T_P$ with a unique conic C_2 such that $C_2.C\geqq P$ and $C_2.C\ngeqq 2P$. Moreover, there is a cubic C_3' with $C_3'\neq C_3$ such that $C_3'.C\geqq 3P+E_7$ and $C_3'.C\ngeqq 4P$. Then $E_7\ngeqq R$, because $C_3'.C\ngeqq 4P$. Hence, we get $C_2.C\geqq P+E_7$. Since $h^0(K-P-E_7)=3$, a cubic C_3'' with $C_3''.C\geqq P+E_7$ must be equal to C_2L with a line L. Thus, we get $C_3'=C_2L$ with a line L, which implies that $C.L\geqq 2P$. Hence, we get $L=T_P$. Therefore, $C_3'=C_3$, which is a contradiction. Thus, H is not DCP.

III-3) $H=2J_6+\langle n,n+2,n+8\rangle$. Then $r(H)=5$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1,Q_2,Q_3 and Q_4 on the intersection of C and L_1. We set $Q_5 = R$. Let C_3 be a cubic with $C_3.C \geqq E_5$. Then $C_3 = L_1 C_2$ with a conic $C_2 \ni Q_5$, which implies that

$$h^0(K - E_5) = h^0(K - P - E_5) = 5.$$

Moreover, let $C_3.C \geqq 3P + E_5$. Then $C_3 = L_1 L T_P$ with a line L, which implies that

$$h^0(K - 3P - E_5) = h^0(K - 6P - E_5) = 3.$$

Thus, H is DCP.

III-4) $H = 2J_6 + \langle n, n+2, n+14 \rangle$. Then $r(H) = 6$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6$. Let C_3 be a cubic with $C_3.C \geqq E_6$. Then $C_3 = L_1 C_2$ with a conic $C_2 \ni Q_5, Q_6$, which implies that

$$h^0(K - E_6) = h^0(K - P - E_6) = 4.$$

Moreover, let $C_3.C \geqq 4P + E_6$. Then $C_3 = L_1 L_2 T_P$, which means that

$$h^0(K - 4P - Q_1 - \cdots - Q_6) = h^0(K - 7P - E_6) = 1.$$

Hence, H is DCP.

III-5) $H = 2J_6 + \langle n, n+2, n+16 \rangle$.

$(\to +2)$	$(n+2)$	$(n+4)$	$(n+6)$	$(n+8)$	
$\bullet$	$\odot$	$\times$	$\times$	$\times$	$\downarrow$
(n)	$\circ$	$\times$	$\odot$	$\bullet$	$+10$
	$\circ$	$\times$	$\bullet$	$(n+18)$	
	$\circ$	$\bullet$	$(n+26)$	$\swarrow +8$ $(\downarrow +10)$	
	$\bullet$	$(n+34)$			
	$(n+42)$				

Assume that H is DCP. Then there are five points $Q_1, \ldots, Q_5$ distinct from P such that

$$h^0(K - E_5) = h^0(K - P - E_5) = 5,$$
$$h^0(K - 4P - E_5) = h^0(K - 6P - E_5) = 2,$$
$$h^0(K - 7P - E_5) = h^0(K - 11P - E_5) = 1, \text{ and } h^0(K - 12P - E_5) = 0.$$

There exists a unique conic C_3 with $C_3.C \geqq 11P + E_5$. Then $C_3 = L_0 T_P^2$ with a line $L_0 \neq T_P$ and $L_0 \ni P$.

Case 1. $Q_1, \ldots, Q_5$ are distinct from R. Then we have $L_0 \ni Q_1, \ldots, Q_5$, which means that $h^0(K - E_5) = 6$. This is a contradiction.

Case 2. $Q_1, \ldots, Q_4$ are distinct from R, and $Q_5 = R$. We have $L_0 \ni P, Q_1, \ldots, Q_4$, which implies that $L_0 L T_P \geqq 6P + Q_1 + \cdots + Q_5$ with a line L. Hence we get $h^0(K - 6P - E_5) = 3$, which is a contradiction.

Case 3. $Q_4 = Q_5 = R$. We get $L_0.C \geqq Q_1 + Q_2 + Q_3 + P$. If a cubic C_3' has $C_3'.C \geqq 3P + E_5$, then $C_3' = L_0 C_2'$ with $C_2'.C \geqq 2P + 2R$, because $L_0 \neq T_P$. Hence, we get $C_2' = L T_P$ with a line L satisfying $L.C \geqq R$. This contradicts $h^0(K - 3P - E_5) = 3$. Hence H is not DCP.

III-6) $H = 2J_6 + \langle n, n+2, n+24\rangle$. Then $r(H) = 5$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let Q_5 be a point different from R which does not lie in L_1. Let C_3 be a cubic with $C_3.C \geqq E_5$. Then we get $C_3 = L_1C_2$ with a conic $C_2 \ni Q_5$, which implies that

$$h^0(K - E_5) = h^0(K - P - E_5) = 5.$$

Let C_3' be a cubic with $C_3'.C \geqq 7P + E_5$. Then we get $C_3' = L_1LT_P$ with a line $L \ni P, Q_5$, which implies that $h^0(K - 8P - E_5) = 0$. Hence, H is DCP.

III-7) $H = 2J_6 + \langle n, n+4, n+6\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \odot & \times & \downarrow \\
(n) & \times & \circ & \circ & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. There are seven points $Q_1, \dots, Q_7$ distinct from P such that

$$h^0(K - E_7) = 3,\ h^0(K - P - E_7) = h^0(K - 3P - E_7) = 2,$$
$$h^0(K - 4P - E_7) = h^0(K - 5P - E_7) = 1, \text{ and } h^0(K - 6P - E_7) = 0.$$

Case: $Q_7 = R$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_7$. Then $C_3 = C_2T_P$ with a conic C_2 satisfying $C_2.C \geqq Q_1 + \cdots + Q_6$. Hence $C_3.C \geqq 5P + E_7$. This is a contradiction.

Case: $Q_i \neq R$ for all i. There exists a unique cubic C_3 with $C_3.C \geqq 5P + E_7$. Then $C_3 = C_2T_P$ with a conic C_2 such that $C_2.C \ngeqq P$ and $C_2C \geqq E_7$. Let C_3' be a cubic with $C_3'.C \geqq E_7$. Since $h^0(K - E_7) = 3$ and $C_2C \geqq E_7$, we have $C_3 = C_2L$ with a line L. Moreover, assume that $C_3.C \geqq 2P + E_7$. Then $C_3 = C_2T_P$, because $P \notin C_2$. Hence we get $h^0(K - 2P - E_7) = 1$. This is a contradiction. Hence, H is not DCP.

III-8) $H = 2J_6 + \langle n, n+4, n+8\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \times & \odot & \downarrow \\
(n) & \times & \circ & \times & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are six points $Q_1, \dots, Q_6$ distinct from P such that

$$h^0(K - E_6) = 4,\ h^0(K - P - E_6) = h^0(K - 2P - E_6) = 3,$$
$$h^0(K - 3P - E_6) = h^0(K - 5P - E_6) = 2,$$
$$h^0(K - 6P - E_6) = h^0(K - 7P - E_6) = 1, \text{ and } h^0(K - 8P - E_6) = 0.$$

There is a unique cubic C_3 such that $C_3.C \geqq 7P+E_6$ and $C_3.C \not\geqq 8P$. Then we get $C_3 = C_2T_P$ with a unique conic C_2 such that $C_2.C \geqq 2P$ and $C_2.C \not\geqq 3P$. Moreover, there is a cubic C'_3 with $C'_3 \neq C_3$ such that $C'_3.C \geqq 5P + E_6$ and $C'_3.C \not\geqq 6P$. Then $C'_3 = C'_2T_P$ with a conic C'_2 such that $C'_2 \neq C_2$ and $C'_2.C \not\geqq P$.

Case 1. $E_6 \geqq R$. We set $D_5 = E_6 - R$. Then we get $C_2.C'_2 \geqq D_5$, which implies that C_2 and C'_2 have a common line L_1. We set $C_2 = L_1L_2$ with a line L_2 satisfying $L_2.C \geqq 2P$. Hence, we get $L_2 = T_P$. This is a contradiction.

Case 2. $E_6 \not\geqq R$. Then we get $C_2.C'_2 \geqq E_6$, which implies that C_2 and C'_2 have a common line L_1. By the same way as in the above we get a contradiction. Thus, H is not DCP.

III-9) $H = 2J_6 + \langle n, n+4, n+12\rangle$. Then $r(H) = 6$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5, 6$. Then we have $h^0(K - E_6) = 4$. Let C_3 be a cubic with $C_3.C \geqq P + E_6$. Then we get $C_3 = L_1L_2L$ with a line L which implies that

$$h^0(K - P - E_6) = h^0(K - 2P - E_6) = 3.$$

Moreover, let $C_3.C \geqq 4P + E_6$. Then we get $C_3 = L_1L_2T_P$, which means that

$$h^0(K - 4P - E_6) = h^0(K - 7P - E_6) = 1.$$

Thus, H is DCP.

III-10) $H = 2J_6 + \langle n, n+4, n+16\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \odot & \times & \times & \downarrow \\
(n) & \times & \circ & \odot & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are six points $Q_1, \ldots, Q_6$ distinct from P such that

$$h^0(K - E_6) = 4,\ h^0(K - P - E_6) = h^0(K - 2P - E_6) = 3, \text{ and}$$
$$h^0(K - 6P - E_6) = 0.$$

Since $h^0(K - 2P - E_6) = 3$, some five points of $P, P, Q_1, \ldots, Q_6$ are collinear or the eight points are on a conic C_2. If $Q_1, \ldots, Q_5$ are collinear, then $h^0(K - E_6) \geqq 5$, which is a contradiction. If $P, Q_1, \ldots, Q_4$ are collinear, then $h^0(K - P - E_6) \geqq 4$, which is a contradiction. If $P, P, Q_1, \ldots, Q_3$ are collinear, then the line is T_P, which is a contradiction. If the eight points are on a conic C_2, then $C_2T_P.C \geqq 7P + E_6$, which contradicts $h^0(K - 6P - E_6) = 0$. Hence, H is not DCP.

III-11) $H = 2J_6 + \langle n, n+6, n+8\rangle$. Then $r(H) = 5$. Let L_1 be a line neither through P nor R. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. We set $Q_5 = R$. Then we have $h^0(K - E_5) = 5$. Let C_3 be a cubic with $C_3.C \geqq 2P+Q_1+\cdots+Q_5$. Then $C_3 = L_1T_PL$ with a line L, which implies that

$$h^0(K-2P-E_5) = \cdots = h^0(K-5P-E_5) = 3.$$

Thus, H is DCP.

III-12) $H = 2J_6 + \langle n, n+6, n+12\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \odot & \times & \downarrow \\
(n) & \odot & \times & \circ & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow{+8}\ (\downarrow{+10}) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are six points $Q_1, \dots, Q_6$ distinct from P such that

$$\begin{aligned}
&h^0(K-E_6) = 4,\ h^0(K-P-E_6) = 3,\\
&h^0(K-2P-E_6) = h^0(K-3P-E_6) = 2,\\
&h^0(K-4P-E_6) = h^0(K-6P-E_6) = 1, \text{ and } h^0(K-7P-E_6) = 0.
\end{aligned}$$

There is a unique cubic C_3 such that $C_3.C \geqq 6P+E_6$ and $C_3.C \ngeqq 7P$. Hence we get $C_3 = T_PC_2$ with a unique conic C_2 such that $C_2.C \geqq P$ and $C_2.C \ngeqq 2P$. Moreover, there is a cubic C_3' with $C_3' \neq C_3$ such that $C_3'.C \geqq 3P+E_6$ and $C_3'.C \ngeqq 4P$. Then $E_6 \ngeqq R$, because $C_3'.C \ngeqq 4P$. Hence we get $C_3'.C_2 \geqq P+E_6$. Thus, C_3' and C_2 have a common component.

Case 1. C_2 is irreducible. We have $C_3' = C_2L_0$ with a line L_0 satisfying $L_0.C \geqq 2P$. Hence $L_0 = T_P$, which contradicts $C_3'.C \ngeqq 4P$.

Case 2. C_2 is not irreducible. We have $C_2 = L_0L_1$ and $C_3' = L_0C_2'$ with lines L_0, L_1 and a conic C_2'. First, assume that $L_0 \not\ni P$. Then $C_2'.C \geqq 3P$, which implies that $C_2' = T_PL$ with a line L. Thus, $C_3'.C \geqq 5P$, which is a contradiction. Secondly, we assume that $L_0 \ni P$, which implies that $L_1 \not\ni P$. Then we have $L_0.C \ngeqq P+D_4$ for any divisor D_4 of degree 4 with $D_4 < E_6$, because we have $h^0(K-P-E_6) = 3$. Hence we get $C_2'.C \geqq 2P+D_3$ and $L_1.C \geqq D_3$ for some divisor D_3 of degree with $D_3 < E_6$. Hence, we get $C_2' = L_1L_2$ with a line L_2 satisfying $L_2.C \geqq 2P$. Thus, $L_2 = T_P$. This is a contradiction.

Therefore, H is not DCP.

III-13) $H = 2H_6 + \langle n, n+6, n+14\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \odot & \times & \downarrow \\
(n) & \times & \odot & \circ & \bullet & +10 \\
 & \times & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow{+8}\ (\downarrow{+10}) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are five points $Q_1, \dots, Q_5$ such that

$$h^0(K-E_5)=5,\ h^0(K-P-E_5)=4,$$
$$h^0(K-2P-E_5)=h^0(K-3P-E_5)=3,$$
$$h^0(K-4P-E_5)=h^0(K-5P-E_5)=2,$$
$$h^0(K-6P-E_5)=h^0(K-10P-E_5)=1 \text{ and } h^0(K-11P-E_5)=0.$$

There is a unique cubic C_3 such that $C_3.C \geqq 10P+E_5$ and $C_3.C \ngeqq 11P$. Hence we get $C_3 = T_P^2 L_0$ with a unique line L_0 such that $L_0 \not\ni P$.
Case: $E_5 \ngeqq R$. We have $L_0.C \geqq E_5$. Then we get $h^0(K-E_5)=6$, which is a contradiction.
Case: $E_5 \geqq R$. We set $D_4 = E_5 - R$. Let C_3' be a cubic with $C_3'.C \geqq 3P+E_5 = 3P+R+D_4$. Then $C_3' = T_P C_2$ with a conic C_2 satisfying $C_2.C \geqq D_4$. Hence, we get $h^0(K-3P-E_5)=h^0(K-5P-E_5)$, which is a contradiction. Therefore, H is not DCP.

III-14) $H = 2J_6 + \langle n, n+6, n+22\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \odot & \times & \downarrow \\
(n) & \times & \times & \circ & \bullet & +10 \\
 & \odot & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow{+8}\ (\downarrow{+10}) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are five points $Q_1, \dots, Q_5$ distinct from P such that

$$h^0(K-E_5)=5,\ h^0(K-P-E_5)=4,$$
$$h^0(K-2P-E_5)=h^0(K-3P-E_5)=3, \text{ and}$$
$$h^0(K-4P-E_5)=h^0(K-5P-E_5)=2.$$

Since $h^0(K-3P-E_5)=3$, some five points of $P,P,P,Q_1,\dots,Q_5$ are collinear or the eight points $P,P,P,Q_1,\dots,Q_5$ are on a conic C_2. If $Q_1,\dots,Q_5$ are collinear, this contradicts $h^0(K-E_5)=5$. If $P,Q_1,\dots,Q_4$ are collinear, this contradicts $h^0(K-P-E_5)=4$. If P,P,Q_1,Q_2,Q_3 are collinear, the line is T_P, which is a contradiction. If P,P,P,Q_1,Q_2 are collinear, the line is T_P, which is a contradiction. Hence, the eight points $P,P,P,Q_1,\dots,Q_5$ are on a conic C_2. Then $C_2 = T_P L$ with a line L. Hence, we get $h^0(K-5P-E_5)=3$. This is a contradiction. Hence, H is not DCP.

III-15) $H = 2J_6 + \langle n, n+8, n+12\rangle$. Then $r(H)=4$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 on the intersection of C and L_1. We set $Q_4 = R$. Then we have $h^0(K-E_4)=6$. Let C_3 be a cubic with $C_3.C \geqq 3P+E_4$. Then we get $C_3 = L_1 T_P L$ with a line L, which implies that

$$h^0(K-3P-E_4)=h^0(K-6P-E_4)=3.$$

Hence, H is DCP.

III-16) $H = 2J_6 + \langle n, n+8, n+14 \rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \odot & \downarrow \\
(n) & \times & \odot & \times & \bullet & +10 \\
 & \times & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are four points $Q_1, \dots, Q_4$ distinct from P such that

$$h^0(K - E_4) = 6,\ h^0(K - P - E_4) = 5,$$
$$h^0(K - 2P - E_4) = 4,\ h^0(K - 3P - E_4) = h^0(K - 5P - E_4) = 3,$$
$$h^0(K - 6P - E_4) = h^0(K - 7P - E_4) = 2,$$
$$h^0(K - 8P - E_4) = h^0(K - 10P - E_4) = 1 \text{ and } h^0(K - 11P - E_4) = 0.$$

There is a unique cubic C_3 such that $C_3.C \geqq 10P + E_4$ and $C_3.C \ngeqq 11P$. Hence we get $C_3 = T_P^2 L_0$ with a unique line L_0 such that $L_0.C \ngeqq P$. Moreover, there is a cubic C_3' with $C_3' \neq C_3$ such that $C_3'.C \geqq 7P + E_4$ and $C_3'.C \ngeqq 8P$. Then $C_3' = T_P C_2$ with a conic C_2 such that $C_2.C \geqq 2P$.

Case 1. $E_4 \geqq 2R$. We have $C_2.C \geqq 2P + R$, which means that $C_2 = T_P L$. Hence, we get $C_3' = T_P^2 L$. We set $D_2 = E_4 - 2R$. Then we have $L_0.L \geqq D_2$, which implies that $L = L_0$. Hence we get $C_3 = C_3'$, which is a contradiction.

Case 2. $E_4 \geqq R$ *and* $E_4 \ngeqq 2R$. We set $D_3 = E_4 - R$. Then we have $L_0.C \geqq D_3$ and $C_2.C \geqq 2P + D_3$. Hence, we get $L_0.C_2 \geqq D_3$, which implies that $C_2 = L_0 L$ with a line L. Since $L_0 \not\ni P$, we have $L.C \geqq 2P$, which means that $L = T_P$. Hence we get $C_3' = T_P^2 L_0 = C_3$. This is a contradiction.

Case 3. $E_4 \ngeqq R$. We have $L_0.C \geqq E_4$ and $C_2.C \geqq 2P + E_4$. Hence we get $C_2 = L_0 L$ with a line L with $L.C \geqq 2P$. Hence we get $L = T_P$. This is a contradiction.

Thus, H is not DCP.

III-17) $H = 2J_6 + \langle n, n+8, n+16 \rangle$. Then $r(H) = 2$. We set $Q_1 = Q_2 = R$. Let C_3 be a cubic with $C_3.C \geqq 3P + 2R$. Then we have $C_3 = T_P C_2$ with a conic $C_2 \ni R$. Hence we get

$$h^0(K - 3P - E_2) = h^0(K - 5P - E_2) = 5.$$

Similarly we get

$$h^0(K - 7P - E_2) = h^0(K - 10P - E_2) = 3.$$

Thus, H is DCP.

III-18) $H = 2J_6 + \langle n, n+8, n+22 \rangle$. Then $r(H) = 3$. Let L_1 be a line through P different from T_P. Take two distinct points Q_1 and Q_2 on the intersection of C and L_1. We set $Q_3 = R$. Then we have $h^0(K - E_3) = 7$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_3$. Then we get $C_3 = T_P C_2$ with a conic $C_2 \ni Q_1, Q_2$.

Hence we get

$$h^0(K-5P-E_3)=h^0(K-3P-E_3)=6-2=4.$$

Moreover, let $C_3.C \geqq 8P+E_3$. Then we have $C_3=T_P^2L_1$, which means that

$$h^0(K-8P-E_3)=h^0(K-11P-E_3)=1.$$

Hence, H is DCP.

III-19) $H=2J_6+\langle n,n+8,n+24\rangle$.

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \odot & \downarrow \\
(n) & \times & \times & \times & \bullet & +10 \\
 & \times & \odot & \bullet & (n+18) & \\
 & \times & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are two points Q_1 and Q_2 distinct from P such that

$$h^0(K-11P-E_2)=h^0(K-15P-E_2)=1.$$

Hence there is a unique cubic C_3 with $C_3.C \geqq 15P+E_2$. Then $C_3=T_P^3$, which implies that $Q_1=Q_2=R$. On the other hand, $T_P^2L.C \geqq 11P+2R$ with a line $L \ni P$, which means that $h^0(K-11P-E_2)=2$. This is a contradiction. Hence H is not DCP.

III-20) $H=2J_6+\langle n,n+8,n+32\rangle$. Then $r(H)=2$. Let Q_1 be a point distinct from P and R. We set $Q_2=R$. Let C_3 be a cubic with $C_3.C \geqq 3P+Q_1+R$. Then $C_3=T_PC_2$ with a conic $C_2 \ni Q_1$. Hence, we get

$$h^0(K-5P-E_2)=h^0(K-3P-E_2)=5.$$

Moreover, let $C_3.C \geqq 11P+Q_1+R$. Then $C_3=T_P^2L_0$ with the line L_0 through P and Q_1, which implies that $h^0(K-12P-E_2)=0$. Thus, H is DCP.

III-21) $H=2J_6+\langle n,n+12,n+14\rangle$. Then $r(H)=5$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1,Q_2 and Q_3 (resp. Q_4 and Q_5) on the intersection of C and L_1 (resp, L_2) such that $Q_i \notin L_1$ for $i=4,5$. Then we have $h^0(K-E_5)=5$. Let C_3 be a cubic with $C_3.C \geqq 4P+E_5$. Then $C_3=L_1L_2T_P$, which implies that

$$h^0(K-4P-E_5)=h^0(K-7P-E_5)=1.$$

Hence, H is DCP.

III-22) $H=2J_6+\langle n,n+12,n+16\rangle$.

$$\begin{array}{cccccc}
(\rightarrow +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \odot & \times & \odot & \bullet & +10 \\
 & \circ & \times & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are four points $Q_1, \ldots, Q_4$ such that

$$\begin{aligned}&h^0(K-P-E_4)=5,\ h^0(K-5P-E_4)=h^0(K-6P-E_4)=2,\\ &\text{and } h^0(K-11P-E_4)=1.\end{aligned}$$

Hence there exists a cubic C_3 with $C_3.C \geqq 11P+E_4$. We have $C_3 = T_P^2 L_0$ with the line $L_0 \ni P$.

Case 1. $Q_1, \ldots, Q_4$ are distinct from R. The line L_0 contains the five points $P, Q_1, \ldots, Q_4$, which is a contradiction.

Case 2. $Q_1 \neq R, Q_2 \neq R, Q_3 \neq R$ and $Q_4 = R$. We have $L_0 T_P L.C \geqq 6P+E_4$ with a line L, which is a contradiction.

Case 3. $Q_3 = Q_4 = R$. We have $T_P C_2.C \geqq 5P+R+Q_1+Q_2+R$ with a conic $C_2 \ni Q_1, Q_2, R$, which is a contradiction.

Thus H is not DCP.

III-23) $H = 2J_6 + \langle n, n+12, n+24\rangle$. Then $r(H)=4$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 on the intersection of C and L_1. Let Q_4 be a point with $Q_4 \neq R$ which does not lie in L_1. Then we have $h^0(K-E_4)=6$. Let C_3 be a cubic with $C_3.C \geqq 4P+E_4$. Then we have $C_3 = L_1 T_P L$ with a line $L \ni Q_4$. Hence we get

$$h^0(K-4P-E_4)=h^0(K-6P-E_4)=2.$$

Moreover, let $C_3.C \geqq 8P+E_4$. Then $C_3 = T_P^2 L_1$ with the line $L_1 \ni Q_4$, which is a contradiction. Hence, we get $h^0(K-8P-E_4)=0$. Thus, H is DCP.

III-24) $H = 2J_6 + \langle n, n+14, n+16\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \times & \odot & \odot & \bullet & +10 \\
 & \times & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow{+8}\ (\downarrow{+10}) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are four points $Q_1, \ldots, Q_4$ distinct from P such that

$$h^0(K-6P-E_4)=h^0(K-10P-E_4)=1.$$

There is a unique cubic C_3 with $C_3.C \geqq 10P+E_4$. Then $C_3 = T_P^2 L_0$ with the line L_0 such that $T_P \cup L_0 \ni Q_1, \ldots, Q_4$.

Case: $Q_1, \ldots, Q_4$ *are different from* R. We have $T_P L_0 L.C \geqq 6P+E_4$ with a line $L \ni P$. This contradicts $h^0(K-6P-E_4)=1$.

Case: $Q_4 = R$. We have $T_P C_2.C \geqq 6P+E_4$ with a conic $C_2.C \geqq P+Q_1+Q_2+Q_3$. This is a contradiction.

Thus, H is not DCP.

III-25) $H = 2J_6 + \langle n, n+14, n+22\rangle$. Then $r(H)=4$. Let L_1 and L_2 be distinct lines through P different from T_P. Take distinct points Q_1 and Q_2 (resp. Q_3 and Q_4) on the intersection of C and L_1 (resp. L_2) such that

$Q_i \notin L_1$ for $i = 3, 4$. Then we have $h^0(K - E_4) = 6$. Let C_3 be a cubic with $C_3.C \geqq 6P + E_4$. Then we get $C_3 = T_P L_1 L_2$. Thus we obtain

$$h^0(K - 6P - E_4) = h^0(K - 7P - E_4) = 1 \text{ and } h^0(K - 8P - E_4) = 0.$$

Thus, H is DCP.

III-26) $H = 2J_6 + \langle n, n + 16, n + 22\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \times & \times & \odot & \bullet & +10 \\
 & \odot & \times & \bullet & (\mathrm{n}{+}18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are three points Q_1, Q_2 and Q_3 such that

$$h^0(K - 7P - E_3) = h^0(K - 11P - E_3) = 1 \text{ and } h^0(K - 12P - E_3) = 0.$$

Let C_3 be a unique cubic with $C_3.C \geqq 11P + Q_1 + Q_2 + Q_3$. Then we have $C_3 = T_P^2 L_0$ with the line L_0 which contains at least P and Q_3 by renumbering Q_1, Q_2 and Q_3.

Case 1. Q_1, Q_2 and Q_3 are distinct from R. Then $L_0.C \geqq P + Q_1 + Q_2 + Q_3$. Hence $T_P L_0 L_P.C \geqq 7P + Q_1 + Q_2 + Q_3$ with a line $L_P \ni P$, which contradicts $h^0(K - 7P - E_3) = 1$.

Case 2. Let $Q_1 \neq R$, $Q_2 \neq R$ and $Q_3 = R$. Then $L_0.C \geqq P + Q_1 + Q_2$. Hence $T_P L_0 L_P.C \geqq 7P + E_3$ with a line $L_P \ni P$. This is a contradiction.

Case 3. Let $Q_1 \neq R$ and $Q_2 = Q_3 = R$. Then $T_P^2 L_{Q_1}.C \geqq 10P + E_3$ with a line $L_{Q_1} \ni Q_1$, which contradicts $h^0(K - 10P - E_3) = 1$.

Case 4. Let $Q_1 = Q_2 = Q_3 = R$. We have $L_0 \ni P, R$, which implies that $L_0 = T_P$. Hence, $T_P^3.C \geqq 15P + E_3$, which is a contradiction.

Thus, H is not DCP.

III-27) $H = 2J_6 + \langle n, n + 16, n + 24\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \times & \times & \odot & \bullet & +10 \\
 & \times & \odot & \bullet & (\mathrm{n}{+}18) & \\
 & \times & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then we have

$$h^0(K - 7P - Q_1 - Q_2) = 2 \text{ and } h^0(K - 15P - Q_1 - Q_2) = 1.$$

Let C_3 be a unique cubic with $C_3.C \geqq 15P + Q_1 + Q_2$. Then we have $C_3 = T_P^3$, which implies that $Q_1 = Q_2 = R$. We obtain $T_P^2 L.C \geqq 10P + 2R$ with a line L, which means that $h^0(K - 10P - Q_1 - Q_2) = 3$. This is a contradiction. Hence, H is not DCP.

III-28) $H = 2J_6 + \langle n, n + 16, n + 32\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \times & \times & \times & \times & \downarrow \\
(n) & \times & \times & \odot & \bullet & +10 \\
 & \times & \times & \bullet & (n+18) & \\
 & \odot & \bullet & (n+26) & \swarrow{+8}\ (\downarrow{+10}) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Assume that H is DCP. Then there are two points Q_1 and Q_2 distinct from P such that

$$h^0(K-5P-E_2)=4,\ h^0(K-7P-E_2)=h^0(K-10P-E_2)=2,$$
$$\text{and } h^0(K-11P-E_2)=1.$$

Let C_3 be a unique cubic with $C_3.C \geqq 11P+E_2$. Then we have $C_3=T_P^2L_0$ with a line $L_0 \ni P$. Here, we may assume that $Q_1 \neq R$. Assume that $Q_2 \neq R$. Then $L_0.C \geqq P+E_2$. Let $C_3'.C \geqq 10P+E_2$. Then $C_3'=T_P^2L_0$. This is a contradiction. Hence we get $Q_2=R$. Let $C_3''=T_PC_2$ with a conic C_2 satisfying $C_2.C \geqq Q_1$. Then we have $C_3''.C \geqq 5P+E_2$. This is a contradiction. Hence H is not DCP.

III-29) $H=2H_6+\langle n,n+22,n+24\rangle$. Then $r(H)=3$. Let $Q_1.Q_2$ and Q_3 be three points different from P and R which are not collinear. We have $h^0(K-E_3)=7$. Let C_3 be a cubic with $C_3.C \geqq 8P+E_3$. Then $C_3=T_P^2L$ with a line $L \ni Q_1,Q_2$ and Q_3. This is a contradiction. Hence, we obtain $h^0(K-8P-E_3)=0$. Thus H is DCP.

III-30) $H=2J_6+\langle n,n+24,n+32\rangle$. Then $r(H)=2$. Let L_1 be a line passing through neither P nor R. Take two distinct points Q_1 and Q_2 on L_1. Let C_3 be a cubic with $C_3.C \geqq 10P+E_2$. Then we get $C_3=T_P^2L_1$, which means that $h^0(K-11P-E_2)=0$. Hence, H is DCP.

(IV) *The case* $t(H)=3$. There are thirty five kinds of numerical semigroups. We will show that all the thirty five numerical semigroups are DCP.

IV-1) $H=2J_6+\langle n,n+2,n+4,n+6\rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \odot & \odot & \odot & \times & \downarrow \\
(n) & \circ & \circ & \circ & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow{+8}\ (\downarrow{+10}) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Let L_1,L_2 and L_3 be distinct three lines through P different from T_P. Take distinct points Q_1,Q_2,Q_3 and Q_4 (resp. Q_5,Q_6 and Q_7, Q_8 and Q_9) on the intersection of C and L_1 (resp. L_2, L_3) such that $Q_i \notin L_1$ for $i=5,6,7$ and $Q_i \notin L_1 \cup L_2$ for $i=8,9$. Let C_3 be a cubic with $C_3.C \geqq E_9$. Then $C_3=L_1L_2L_3$, which implies that

$$h^0(K-E_9)=h^0(K-3P-E_9)=1 \text{ and } h^0(K-4P-E_9)=0.$$

IV-2) $H = 2J_6 + \langle n, n+2, n+4, n+8 \rangle$. Then $r(H) = 8$. Let L_1 and L_2 be distinct lines through P different from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6, 7$. We set $Q_8 = R$. Let C_3 be a cubic with $C_3.C \geqq E_8$. Then $C_3 = L_1 L_2 L$ with a line $L \ni Q_8$, which implies that

$$h^0(K - E_8) = h^0(K - 2P - E_8) = 2 \text{ and } h^0(K - 3P - E_8) = h^0(K - 7P - E_8) = 1.$$

IV-3) $H = 2J_6 + \langle n, n+2, n+4, n+16 \rangle$. Then $r(H) = 8$. Let L_1 and L_2 be distinct lines through P different from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6, 7$. Let Q_8 be a point different from R which lies on neither L_1 nor L_2. Let C_3 be a cubic with $C_3.C \geqq Q_1 + \cdots + Q_8$. Then $C_3 = L_1 L_2 L$ with a line $L \ni Q_8$, which implies that

$$h^0(K - E_8) = h^0(K - 2P - E_8) = 2 \text{ and } h^0(K - 4P - E_8) = 0.$$

IV-4) $H = 2J_6 + \langle n, n+2, n+6, n+8 \rangle$. Then $r(H) = 8$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let L_2 be a line through neither P nor R. Take distinct points Q_5, Q_6 and Q_7 on the intersection of C and L_2 such that $Q_i \notin L_1$ for $i = 5, 6, 7$. We set $Q_8 = R$. Let C_3 be a cubic with $C_3.C \geqq E_8$. Then $C_3 = L_1 L_2 L$ with a line $L \ni Q_8$, which implies that

$$h^0(K - E_8) = h^0(K - P - E_8) = 2.$$

Moreover, let $C_3.C \geqq 2P + Q_1 + \cdots + Q_8$. Then $C_3 = L_1 L_2 T_P$, which implies that

$$h^0(K - 2P - E_8) = h^0(K - 6P - E_8) = 1.$$

IV-5) $H = 2J_6 + \langle n, n+2, n+6, n+14 \rangle$. Then $r(H) = 8$. Let L_1, L_2 and L_3 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6, Q_7 and Q_8) on the intersection of C and L_1 (resp. L_2, L_3) such that $Q_i \notin L_1$ for $i = 5, 6$ and $Q_i \notin L_1 \cup L_2$ for $i = 7, 8$. Let C_3 be a cubic with $C_3.C \geqq E_8$. Then $C_3 = L_1 C_2$ with a conic $C_2 \ni Q_5, Q_6, Q_7, Q_8$, which implies that

$$h^0(K - E_8) = h^0(K - P - E_8) = 2,$$

$$h^0(K - 2P - E_8) = h^0(K - 3P - E_8) = 1 \text{ and } h^0(K - 4P - E_8) = 0.$$

IV-6) $H = 2J_6 + \langle n, n+2, n+8, n+14 \rangle$. Then $r(H) = 7$. Let L_1 and L_2 be distinct lines through P different from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6$. We set $Q_7 = R$. Let C_3 be a cubic with $C_3.C \geqq E_7$. Then $C_3 = L_1 C_2$ with a conic $C_2 \ni Q_5, Q_6, Q_7$, which implies that

$$h^0(K - E_7) = h^0(K - P - E_7) = 3.$$

Let $C_3'.C \geqq 3P + E_7$. Then we have $C_3' = L_1 L_2 T_P$, which implies that

$$h^0(K - 3P - E_7) = h^0(K - 7P - E_7) = 1 \text{ and } h^0(K - 8P - E_7) = 0.$$

IV-7) $H = 2J_6 + \langle n, n+2, n+8, n+16\rangle$. Then $r(H) = 6$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. We set $Q_5 = Q_6 = R$. Let C_3 be a cubic with $C_3.C \geqq E_6$. Then $C_3 = L_1C_2$ with a conic $C_2.C \geqq 2R$, which implies that

$$h^0(K - E_6) = h^0(K - P - E_6) = 4.$$

Let $C_3'.C \geqq 3P + E_6$. Then we have $C_3' = L_1T_PL$ with a line $L \ni R$, which implies that

$$h^0(K-3P-E_6) = h^0(K-6P-E_6) = 2 \text{ and } h^0(K-7P-E_6) = h^0(K-11P-E_6).$$

IV-8) $H = 2J_6 + \langle n, n+2, n+8, n+24\rangle$. Then $r(H) = 6$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let Q_5 be a point different from R which does not lie in L_1. We set $Q_6 = R$. Let C_3 be a cubic with $C_3.C \geqq E_6$. Then $C_3 = L_1C_2$ with a conic $C_2 \ni Q_5, Q_6$, which implies that

$$h^0(K - E_6) = h^0(K - P - E_6) = 4.$$

Moreover, let $C_3.C \geqq 3P + E_6$. Then $C_3 = L_1T_PL$ with a line $L \ni Q_5$, which means that

$$h^0(K - 3P - E_6) = h^0(K - 6P - E_6) = 2 \text{ and } h^0(K - 8P - E_6) = 0.$$

IV-9) $H = 2J_6 + \langle n, n+2, n+14, n+16\rangle$. Then $r(H) = 7$. Let L_1, L_2 and L_3 be distinct lines through P different from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6, Q_7) on the intersection of C and L_1 (resp. L_2, L_3) such that $Q_i \notin L_1$ for $i = 5, 6$ and $Q_7 \notin L_1 \cup L_2$. Let C_3 be a cubic with $C_3.C \geqq E_7$. Then $C_3 = L_1C_2$ with a conic $C_2 \ni Q_5, Q_6, Q_7$, which implies that

$$h^0(K - E_7) = h^0(K - P - E_7) = 3 \text{ and } h^0(K - 4P - E_7) = 0.$$

IV-10) $H = 2J_6 + \langle n, n+2, n+16, n+24\rangle$. Then $r(H) = 6$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let L_2 be a line not through P. Let Q_5 and Q_6 be distinct points different from R which lie on the intersection of C and L_2 such that $Q_i \notin L_1$ for $i = 5, 6$. Let C_3 be a cubic with $C_3.C \geqq E_6$. Then we get $C_3 = L_1C_2$ with a conic $C_2 \ni Q_5, Q_6$, which implies that

$$h^0(K - E_6) = h^0(K - P - E_6) = 6 - 2 = 4.$$

Moreover, let $C_3.C \geqq 4P + E_6$. Then we have $C_3 = L_1T_PL_2$, which means that $h^0(K - 7P - E_6) = 0$.

IV-11) $H = 2J_6 + \langle n, n+4, n+6, n+8\rangle$. Then $r(H) = 8$. Let L_1 and L_2 be distinct lines not through P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6, 7$. We set $Q_8 = R$. Let C_3 be a cubic with $C_3.C \geqq E_8$. Then $C_3 = L_1L_2L$ with a line $L \ni Q_8$, which implies that

$$h^0(K - E_8) = 2 \text{ and } h^0(K - P - E_8) = h^0(K - 5P - E_8) = 1.$$

IV-12) $H = 2J_6 + \langle n, n+4, n+6, n+12\rangle$. Then $r(H) = 8$. Let L_1, L_2 and L_3 be distinct lines through P different from T_P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6, Q_7 and Q_8) on the intersection of C and L_1 (resp, L_2, L_3) such that $Q_i \notin L_1$ for $i = 4, 5, 6$ and $Q_i \notin L_1 \cup L_2$ for $i = 7, 8$. Then we have $h^0(K - E_8) = 2$. Let C_3 be a cubic with $C_3.C \geqq P + E_8$. Then we get $C_3 = L_1L_2L_3$, which implies that

$$h^0(K - P - E_8) = h^0(K - 3P - E_8) = 1 \text{ and } h^0(K - 4P - E_8) = 0.$$

IV-13) $H = 2J_6 + \langle n, n+4, n+8, n+12\rangle$. Then $r(H) = 7$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5, 6$. We set $Q_7 = R$. Since any five points of $Q_1, \ldots, Q_7$ are not collinear, we have $h^0(K-E_7) = 3$. Let C_3 be a cubic with $C_3.C \geqq P+E_7$. Then we get $C_3 = L_1L_2L$ with a line $L \ni Q_7$, which implies that

$$h^0(K - P - E_7) = h^0(K - 2P - E_7) = 2 \text{ and}$$
$$h^0(K - 3P - E_7) = h^0(K - 7P - E_7) = 1.$$

IV-14) $H = 2J_6 + \langle n, n+4, n+8, n+16\rangle$. Then $r(H) = 7$. Let L_1 and L_2 be distinct lines not through P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5, 6$. Let $Q_7 = R$. Since any five points of $Q_1, \ldots, Q_7$ are not collinear, we have $h^0(K - E_7) = 3$. Moreover, any five points of $P, Q_1, \ldots, Q_7$ are not collinear and the eight points $P, Q_1, \ldots, Q_7$ are not on a conic, hence we get $h^0(K-P-E_7) = 2$. Let C_3 be a cubic with $C_3.C \geqq 3P+Q_1+\cdots+Q_7$. Then we get $C_3 = T_PL_1L_2$, which implies that

$$h^0(K - 3P - E_7) = h^0(K - 5P - E_7) = 1 \text{ and } h^0(K - 6P - E_7) = 0.$$

We need to prove that $h^0(K - 2P - E_7) = 2$. It suffices to give two distinct cubics C_{31} and C_{32} on some plane curve of degree 6 with $C_{31}.C \geqq 2P + E_7$ and $C_{32.C} \geqq 2P + E_7$. Let C be a non-singular plane curve of degree 6 whose equation is

$$\begin{aligned} z^3(yz^2 - y^3) + ax^3(x^2z + y(-(c+d)x^2 + cy^2 - yz + dz^2)) \\ + by^3(x^2z + y(-2x^2 + y^2 - yz + z^2)) = 0, \end{aligned}$$

where a, b, c and d are general constants. Let $P = (0 : 0 : 1)$ and T_P the line defined by $y = 0$. Then we have $R = (1 : 0 : 0)$. Let L_1 and L_2 be the lines defined by the equations $z + y = 0$ and $z - y = 0$ respectively. We set $C_{31} = T_PL_1L_2$. Let C_{32} be the cubic defined by the equation $x^2z + y(-2x^2 + y^2 - yz + z^2) = 0$. We set $Q_1 = (1 : -1 : 1), Q_2 = (-1 : -1 : 1), Q_3 = R, Q_4 = (1 : 1 : 1), Q_5 = (-1 : 1 : 1)$ and $Q_6 = R$. Then we obtain

$$C_{31}.C_{32} = 2P + Q_1 + Q_2 + Q_4 + Q_5 + 3R.$$

IV-15) $H = 2J_6 + \langle n, n+4, n+12, n+16\rangle$. Then $r(H) = 7$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2

and Q_3 (resp. Q_4, Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5, 6$. Let Q_7 be a point different from R which lies on neither L_1 nor L_2. Since any 5 points of $Q_1, \ldots, Q_7$ are not collinear, we have $h^0(K - E_7) = 3$. Let C_3 be a cubic with $C_3.C \geqq P + Q_1 + \cdots + Q_7$. Then we have $C_3 = L_1 L_2 L$ with a line $L \ni Q_7$. Hence we get

$$h^0(K - P - E_7) = h^0(K - 2P - E_7) = 2 \text{ and } h^0(K - 4P - E_7) = 0.$$

IV-16) $H = 2J_6 + \langle n, n+6, n+8, n+12 \rangle$. Then $r(H) = 7$. Let L_1 be a line through neither P nor R. Let L_2 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Take distinct points Q_5 and Q_6 on the intersection of C and L_2 with $Q_i \notin L_1$ for $i = 5, 6$. We set $Q_7 = R$. Since any 5 points of $Q_1, \ldots, Q_7$ are not collinear, we have $h^0(K - E_7) = 3$. Let C_3 be a cubic with $C_3.C \geqq 2P + E_7$. Then we get $C_3 = L_1 L_2 T_P$, which implies that

$$h^0(K - 2P - E_7) = h^0(K - 6P - E_7) = 1.$$

IV-17) $H = 2J_6 + \langle n, n+6, n+8, n+14 \rangle$. Then $r(H) = 6$. Let L_1 be a line through neither P nor R. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. We set $Q_5 = Q_6 = R$. Let C_3 be a cubic with $C_3.C \geqq E_6$. Then $C_3 = L_1 C_2$ where C_2 is a conic with $C_2.C \geqq 2R$, which implies that $h^0(K - E_6) = 4$. Moreover, let $C_3.C \geqq 2P + E_6$. Then $C_3 = L_1 T_P L$ with a line $L \ni R$, which implies that

$$h^0(K - 2P - E_6) = h^0(K - 5P - E_6) = 2 \text{ and}$$
$$h^0(K - 6P - E_6) = h^0(K - 10P - E_6) = 1.$$

IV-18) $H = 2J_6 + \langle n, n+6, n+8, n+22 \rangle$. Then $r(H) = 6$. Let L_1 be a line through neither P nor R. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let Q_5 be a point distinct from P and R which does not lie in L_1. We set $Q_6 = R$. Then we have $h^0(K - E_6) = 4$. Let C_3 be a cubic with $C_3.C \geqq 2P + E_6$. Then we have $C_3 = L_1 T_P L$ with a line $L \ni Q_5$. Hence we get

$$h^0(K - 2P - E_6) = h^0(K - 5P - E_6) = 2 \text{ and } h^0(K - 7P - E_6) = 0.$$

IV-19) $H = 2J_6 + \langle n, n+6, n+12, n+14 \rangle$. Then $r(H) = 7$. Let L_1, L_2 and L_3 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4 and Q_5, Q_6 and Q_7) on the intersection of C and L_1 (resp, L_2, L_3) such that $Q_i \notin L_1$ for $i = 4, 5, 6$ and $Q_i \notin L_1 \cup L_2$ for $i = 6, 7$. Then we have $h^0(K - E_7) = 3$. Let C_3 be a cubic with $C_3.C \geqq 2P + E_7$. Then we get $C_3 = L_1 L_2 L_3$, which implies that

$$h^0(K - 2P - E_7) = h^0(K - 3P - E_7) = 1 \text{ and } h^0(K - 4P - E_7) = 0.$$

IV-20) $H = 2J_6 + \langle n, n+6, n+14, n+22 \rangle$. Then $r(H) = 6$. Let L_1 and L_2 be distinct lines not through P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5, 6$. Since any five points of $Q_1, \ldots, Q_6$ are not collinear, we have

$h^0(K-E_6)=4$. Any five points of $P,P,Q_1,\ldots,Q_6$ are not collinear and the eight points $P,P,Q_1,\ldots,Q_6$ are not on a conic. Hence we get $h^0(K-2P-E_6)=2$. If a cubic C_3 satisfies that $C_3.C \geqq 4P+E_6$, then $C_3=T_PL_1L_2$ and $C_3.C \ngeqq 6P$. Hence, we get $h^0(K-6P-E_6)=0$. We need to prove that $h^0(K-3P-E_6)=2$. Hence it suffices to give two distinct cubics C_{31} and C_{32} on some non-singular plane curve of degree 6 with $C_{31}.C_{32}=3P+E_6$. Let C be a curve whose equation is

$$\begin{aligned} z^3(yz^2-y^3)&+ax^3(x^2z+y(-(c+d)x^2+cy^2-yz+dz^2)) \\ &+by^3(x^3+y((-1-d)x^2-xy+y^2+dz^2))=0, \end{aligned}$$

where a,b,c and d are general constants. Let $P=(0:0:1)$ and T_P the line defined by $y=0$. Then we have $R=(1:0:0)$. Let L_1 and L_2 be the lines defined by the equations $x+z=0$ and $x-z=0$ respectively. We set $Q_1=(-1:1:1), Q_2=(-1:-1:1), Q_3=Q_2, Q_4=(1:1:1), Q_5=(1:-1:1)$ and $Q_6=Q_5$. Then we have $L_1.C \geqq Q_1+Q_2+Q_3$ and $L_2.C \geqq Q_4+Q_5+Q_6$. We set $C_{31}=T_PL_1L_2$. Let C_{32} be the cubic defined by the equation

$$x^3+y((-1-d)x^2-xy+y^2+dz^2)=0.$$

Then we obtain $C_{31}.C_{32}=3P+E_6$.

IV-21) $H=2J_6+\langle n,n+8,n+12,n+14\rangle$. Then $r(H)=6$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1,Q_2 and Q_3 (resp. Q_4 and Q_5) on the intersection of C and L_1 (resp, L_2) such that $Q_i \notin L_1$ for $i=4,5$. We set $Q_6=R$. Then we have $h^0(K-E_6)=4$. Let C_3 be a cubic with $C_3.C \geqq 3P+E_6$. Then we get $C_3=L_1L_2T_P$, which implies that

$$h^0(K-3P-E_6)=h^0(K-7P-E_6)=1 \text{ and } h^0(K-8P-E_6)=0.$$

IV-22) $H=2J_6+\langle n,n+8,n+12,n+16\rangle$. Then $r(H)=5$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1,Q_2 and Q_3 on the intersection of C and L_1. We set $Q_4=Q_5=R$. Then we have $h^0(K-E_5)=5$. Let C_3 be a cubic with $C_3.C \geqq 3P+E_5$. Then we get $C_3=L_1T_PL$ with a line $L \ni R$, which implies that

$$\begin{aligned} &h^0(K-3P-E_5)=h^0(K-6P-E_5)=2 \text{ and} \\ &h^0(K-7P-E_5)=h^0(K-11P-E_5)=1. \end{aligned}$$

IV-23) $H=2J_6+\langle n,n+8,n+12,n+24\rangle$. Then $r(H)=5$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1,Q_2 and Q_3 on the intersection of C and L_1. Let Q_4 be a point different from R which does not lie in L_1. We set $Q_5=R$. Then we get $h^0(K-E_5)=5$. Let C_3 be a cubic with $C_3.C \geqq 3P+E_5$. Then we have $C_3=L_1T_PL$ with a line $L \ni Q_4$. Hence we get

$$h^0(K-3P-E_5)=h^0(K-6P-E_5)=2 \text{ and } h^0(K-8P-E_5)=0.$$

IV-24) $H = 2J_6 + \langle n, n+8, n+14, n+16 \rangle$. Then $r(H) = 5$. Let L_1 be a line through neither P nor R. Take distinct points Q_1, Q_2 and Q_3 on the intersection of C and L_1. We set $Q_4 = Q_5 = R$. Then we have $h^0(K - E_5) = 5$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_5$. Then we have $C_3 = L_1 T_P L$ with a line $L \ni Q_5$. Hence we get

$$h^0(K - 3P - E_5) = h^0(K - 5P - E_5) = 2.$$

Let C_3' be a cubic with $C_3'.C \geqq 6P + E_5$. Then we have $C_3' = L_1 T_P^2$. Hence we get

$$h^0(K - 6P - E_5) = h^0(K - 10P - E_5) = 1 \text{ and } h^0(K - 11P - E_5) = 0.$$

IV-25) $H = 2J_6 + \langle n, n+8, n+14, n+22 \rangle$. Then $r(H) = 5$. Let L_1 and L_2 be distinct lines through P different from T_P. Take distinct points Q_1 and Q_2 (resp. Q_3 and Q_4) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 3, 4$. We set $Q_5 = R$. Since the five points $Q_1, \ldots, Q_5$ are not collinear, we have $h^0(K - E_5) = 5$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_5$. Then we get $C_3 = T_P C_2$ with a conic $C_2 \ni Q_1, \ldots, Q_4$. Hence we have

$$h^0(K - 5P - E_5) = h^0(K - 3P - E_5) = 6 - 4 = 2.$$

Moreover, let $C_3.C \geqq 6P + E_5$. Then we must have $C_3 = T_P L_1 L_2$, which means that

$$h^0(K - 6P - E_5) = h^0(K - 7P - E_5) = 1 \text{ and } h^0(K - 8P - E_5) = 0.$$

IV-26) $H = 2J_6 + \langle n, n+8, n+16, n+22 \rangle$. Then $r(H) = 4$. Let L_1 be a line through P different from T_P. Take distinct points Q_1 and Q_2 on the intersection of C and L_1. We set $Q_3 = Q_4 = R$. Then we have $h^0(K - E_4) = 10 - 4 = 6$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_4$. Then $C_3 = T_P C_2$ with a conic $C_2 \ni Q_1, Q_2, Q_3$, which implies that

$$h^0(K - 5P - E_4) = h^0(K - 3P - E_4) = 6 - 3 = 3.$$

Moreover, let $C_3.C \geqq 7P + E_4$. Then we must have $C_3 = T_P^2 L_1$, which means that

$$h^0(K - 7P - E_4) = h^0(K - 11P - E_4) = 1.$$

IV-27) $H = 2J_6 + \langle n, n+8, n+16, n+24 \rangle$. Then $r(H) = 3$. We set $Q_1 = Q_2 = Q_3 = R$. Then we get

$$\begin{aligned}
&h^0(K - E_3) = 10 - 3 = 7, \\
&h^0(K - 3P - E_3) = h^0(K - 5P - E_3) = 6 - 2 = 4, \\
&h^0(K - 7P - E_3) = h^0(K - 10P - E_3) = 2, \text{ and} \\
&h^0(K - 11P - E_3) = h^0(K - 15P - E_3) = 1.
\end{aligned}$$

IV-28) $H = 2J_6 + \langle n, n+8, n+16, n+32 \rangle$. Then $r(H) = 3$. Let Q_1 be a point of C distinct from P and R. We set $Q_2 = Q_3 = R$. We have

$h^0(K-E_3)=10-3=7$. Let C_3 be a cubic with $C_3.C \geqq 3P+E_3$. Then $C_3=T_PC_2$ with a conic $C_2 \ni Q_1, Q_2$. Hence, we get

$$h^0(K-5P-E_3)=h^0(K-3P-E_3)=6-2=4.$$

Moreover, let $C_3.C \geqq 7P+Q_1+2R$. Then $C_3=T_P^2L$ with a line $L \ni Q_1$. Thus, we obtain

$$h^0(K-7P-E_3)=h^0(K-10P-E_3)=2 \text{ and } h^0(K-12P-E_3)=0.$$

IV-29) $H=2J_6+\langle n,n+8,n+22,n+24\rangle$. Then $r(H)=4$. Let Q_1, Q_2 and Q_3 be distinct points different from P and R which are not collinear. Let $Q_4=R$. Then we have $h^0(K-E_4)=10-4=6$. Let C_3 be a cubic with $C_3.C \geqq 3P+Q_1+Q_2+Q_3+R$. Then we get $C_3=T_PC_2$ with a conic $C_2 \ni Q_1,Q_2,Q_3$. Hence, we obtain

$$h^0(K-5P-E_4)=h^0(K-3P-E_4)=6-3=3.$$

Moreover, let $C_3.C \geqq 8P+Q_1+\cdots+Q_4$. Then we have $C_3=T_P^2L$ with a line $L \ni Q_1,Q_2,Q_3$. This is impossible. Hence, we get $h^0(K-8P-E_4)=0$.

IV-30) $H=2J_6+\langle n,n+8,n+24,n+32\rangle$. Then $r(H)=3$. Let Q_1 and Q_2 be two points of C distinct from P and R. Let L_1 be the line through Q_1 and Q_2. Let $Q_3=R$. We have $h^0(K-E_3)=10-3=7$. Let C_3 be a cubic with $C_3.C \geqq 3P+E_3$. Then we get $C_3=T_PC_2$ with a conic $C_2 \ni Q_1,Q_2$. Hence, we obtain

$$h^0(K-5P-E_3)=h^0(K-3P-E_3)=6-2=4.$$

Moreover, let $C_3.C \geqq 11P+E_3$. This is impossible.

IV-31) $H=2J_6+\langle n,n+12,n+14,n+16\rangle$. Then $r(H)=6$. Let C_3 be a cubic with $C_3.C \geqq 4P$. Then $C_3=T_PC_2$ with a conic C_2. Hence we get $h^0(K-4P)=6$. Thus if $Q_1,\dots,Q_6$ are general points, then we have

$$h^0(K-E_6)=10-6=4 \text{ and } h^0(K-4P-E_6)=0.$$

IV-32) $H=2J_6+\langle n,n+12,n+16,n+24\rangle$. Then $r(H)=5$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1,Q_2 and Q_3 on the intersection of C and L_1. Let L_2 be a line through neither P nor R. Take distinct points Q_4 and Q_5 which do not lie on L_1. Then we have $h^0(K-E_5)=10-5=5$. Let C_3 be a cubic with $C_3.C \geqq 4P+E_5$. Then we have $C_3=T_PL_1L_2$. Hence we get

$$h^0(K-4P-E_5)=h^0(K-6P-E_5)=1 \text{ and } h^0(K-7P-E_5)=0.$$

IV-33) $H=2J_6+\langle n,n+14,n+16,n+22\rangle$. Then $r(H)=5$. Let L_1 and L_2 be distinct lines not through P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4 and Q_5) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i=4,5$. Since the five points $Q_1,\dots,Q_5$ are not collinear, we have $h^0(K-E_5)=5$. Let C_3 be a cubic with $C_3.C \geqq 6P+E_5$. This is impossible.

IV-34) $H=2J_6+\langle n,n+16,n+22,n+24\rangle$. Then $r(H)=4$. Let L_1 be a line through P and L_2 a line not through P. Take distinct points Q_1 and

Q_2 (resp. Q_3 and Q_4) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 3, 4$. Then we have $h^0(K - E_4) = 10 - 4 = 6$. Let C_3 be a cubic with $C_3.C \geqq 7P + E_4$. This is impossible.

IV-35) $H = 2J_6 + \langle n, n+16, n+24, n+32 \rangle$. Then $r(H) = 3$. Let L_1 be a line not through P. Let Q_1, Q_2 and Q_3 be distinct points on the intersection of C and L_1. Then we have $h^0(K - E_3) = 10 - 3 = 7$. Let C_3 be a cubic with $C_3.C \geqq 7P + E_3$. Then we get $C_3 = T_P^2 L_1$. Hence, we get

$$h^0(K - 7P - E_3) = h^0(K - 10P - E_3) = 1 \text{ and } h^0(K - 11P - E_3) = 0.$$

(V) *The case* $t(H) = 4$. There are fourteen kinds of numerical semigroups. We will prove that all such numerical semigroups are DCP.

V-1) $H = 2J_6 + \langle n, n+2, n+4, n+6, n+8 \rangle$.
If $Q_1, \ldots, Q_{10}$ are general points, then we have $h^0(K - E_{10}) = 0$.

V-2) $H = 2J_6 + \langle n, n+2, n+4, n+8, n+16 \rangle$.

$$\begin{array}{cccccc}
(\to +2) & (n+2) & (n+4) & (n+6) & (n+8) & \\
\bullet & \odot & \odot & \times & \odot & \downarrow \\
(n) & \circ & \circ & \odot & \bullet & +10 \\
 & \circ & \circ & \bullet & (n+18) & \\
 & \circ & \bullet & (n+26) & \swarrow +8\ (\downarrow +10) & \\
 & \bullet & (n+34) & & & \\
 & (n+42) & & & &
\end{array}$$

Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5, Q_6 and Q_7) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6, 7$. Let L_3 be a line not through P. Take distinct points Q_8 and Q_9 on the intersection of C and L_3 such that $Q_i \notin L_1 \cup L_2$ for $i = 8, 9$. Let C_3 be a cubic with $C_3.C \geqq E_9$. Then $C_3 = L_1 L_2 L_3$, which implies that

$$h^0(K - E_9) = h^0(K - 2P - E_9) = 1 \text{ and } h^0(K - 3P - E_9) = 0.$$

V-3) $H = 2J_6 + \langle n, n+2, n+6, n+8, n+14 \rangle$. Then $r(H) = 9$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let L_2 and L_3 be distinct lines not through P. Take distinct points Q_5, Q_6 and Q_7 (resp. Q_8 and Q_9) on the intersection of C and L_2 (resp. L_3) such that $Q_i \notin L_1$ for $i = 5, 6, 7$ and $Q_i \notin L_1 \cup L_2$ for $i = 8, 9$. Let C_3 be a cubic with $C_3.C \geqq E_9$. Then $C_3 = L_1 L_2 L_3$, which implies that

$$h^0(K - E_9) = h^0(K - P - E_9) = 1 \text{ and } h^0(K - 2P - E_9) = 0.$$

V-4) $H = 2J_6 + \langle n, n+2, n+8, n+14, n+16 \rangle$. Then $r(H) = 8$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let L_2 be a line not through P. Take distinct points Q_5, Q_6 and Q_7 on the intersection of C and L_2 which do not lie on L_1. Let Q_8 be a point different from R which lies on neither L_1 nor L_2. Let C_3 be a cubic with $C_3.C \geqq E_8$. Then $C_3 = L_1 L_2 L$ with a line $L \ni Q_8$, which implies that

$$h^0(K-E_8) = h^0(K-P-E_8) = 2 \text{ and } h^0(K-3P-E_8) = 0.$$

V-5) $H = 2J_6 + \langle n, n+2, n+8, n+16, n+24\rangle$. Then $r(H) = 7$. Let L_1 be a line not through P. Take distinct points Q_1, Q_2, Q_3 and Q_4 on the intersection of C and L_1. Let L_2 be a line through P distinct from T_P. Let Q_5, Q_6 and Q_7 be distinct points which lie on the intersection of C and L_2 such that $Q_i \notin L_1$ for $i = 5, 6, 7$. Let C_3 be a cubic with $C_3.C \geqq E_7$. Then we get $C_3 = L_1L_2L$ with a line L, which implies that

$$\begin{aligned}
&h^0(K-E_7) = h^0(K-P-E_7) = 3,\\
&h^0(K-3P-E_7) = h^0(K-6P-E_7) = 1, \text{ and}\\
&h^0(K-7P-E_7) = 0.
\end{aligned}$$

V-6) $H = 2J_6 + \langle n, n+4, n+6, n+8, n+12\rangle$. Then $r(H) = 9$. Let $Q_1, \ldots, Q_9$ be general points of C. Then we get

$$h^0(K-E_9) = 1 \text{ and } h^0(K-P-E_9) = 0.$$

V-7) $H = 2J_6 + \langle n, n+4, n+8, n+12, n+16\rangle$. Then $r(H) = 8$. Let L_1 and L_2 be distinct lines through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4, Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5, 6$. Let L_3 be a line not through P. Take distinct points Q_7 and Q_8 on the intersection of C and L_3 such that any five points of $Q_1, \ldots, Q_8$ are not collinear and $Q_i \notin L_1 \cup L_2$ for $i = 7, 8$. Then we have $h^0(K-E_8) = 10-8 = 2$. Let C_3 be a cubic with $C_3.C \geqq P + E_8$. Then we have $C_3 = L_1L_2L_3$. Hence we get

$$h^0(K-P-E_8) = h^0(K-2P-E_8) = 1 \text{ and } h^0(K-3P-E_8) = 0.$$

V-8) $H = 2J_6 + \langle n, n+6, n+8, n+12, n+14\rangle$. Then $r(H) = 8$. If $Q_1, \ldots, Q_8$ are general points, we have

$$h^0(K-E_8) = 2 \text{ and } h^0(K-2P-E_8) = 0.$$

V-9) $H = 2J_6 + \langle n, n+6, n+8, n+14, n+22\rangle$. Then $r(H) = 7$. Let L_1 and L_2 be distinct lines through neither P nor R. Take distinct points Q_1, Q_2, Q_3 and Q_4 (resp. Q_5 and Q_6) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 5, 6$. We set $Q_7 = R$. We obtain $h^0(K-E_7) = 3$. Let C_3 be a cubic with $C_3.C \geqq 2P + E_7$. Then we must have $C_3 = L_1T_PL_2$. Hence, we get

$$h^0(K-2P-E_7) = h^0(K-5P-E_7) = 1 \text{ and } h^0(K-6P-E_7) = 0.$$

V-10) $H = 2J_6 + \langle n, n+8, n+12, n+14, n+16\rangle$. Then $r(H) = 7$. Since $h^0(K-3P) = 7$, we have

$$h^0(K-E_7) = 3 \text{ and } h^0(K-3P-E_7) = 0$$

for general points $Q_1, \ldots, Q_7$.

V-11) $H = 2J_6 + \langle n, n+8, n+12, n+16, n+24\rangle$. Then $r(H) = 6$. Let L_1 be a line through P distinct from T_P. Take distinct points Q_1, Q_2 and Q_3 on the intersection of C and L_1. Let L_2 be a line not through P. Take distinct points Q_4 and Q_5 on the intersection of C and L_2 such that $Q_i \notin L_1$ for $i = 4, 5$. We set $Q_6 = R$. Then we have $h^0(K - E_6) = 4$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_6$. Then we have $C_3 = L_1 T_P L_2$. Hence we have

$$h^0(K - 3P - E_6) = h^0(K - 6P - E_6) = 1 \text{ and } h^0(K - 7P - E_6) = 0.$$

V-12) $H = 2J_6 + \langle n, n+8, n+14, n+16, n+22\rangle$. Then $r(H) = 6$. Let L_1 and L_2 be distinct lines through neither P nor R. Take distinct points Q_1, Q_2 and Q_3 (resp. Q_4 and Q_5) on the intersection of C and L_1 (resp. L_2) such that $Q_i \notin L_1$ for $i = 4, 5$. We set $Q_6 = R$. We get $h^0(K - E_6) = 4$. Let C_3 be a cubic with $C.C_3 \geqq 3P + E_6$. Then we must have $C_3 = T_P L_1 L_2$. Hence we get

$$h^0(K - 3P - E_6) = h^0(K - 5P - E_6) = 1 \text{ and } h^0(K - 6P - E_6) = 0.$$

V-13) $H = 2J_6 + \langle n, n+8, n+16, n+22, n+24\rangle$. Then $r(H) = 5$. Let L_1 be a line not through P. Take distinct points Q_1, Q_2 and Q_3 on the intersection of C and L_1. Let Q_4 be a point of C not belonging to L_1 with $Q_4 \neq P$ and $Q_4 \neq R$. We set $Q_5 = R$. Since the five points $Q_1, \dots, Q_5$ are not collinear, we get $h^0(K - E_5) = 5$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_5$. Then we get $C_3 = T_P L_1 L$ with a line $L \ni Q_4$. Hence we have

$$h^0(K - 3P - E_5) = h^0(K - 5P - E_5) = 2.$$

Moreover, let $C_3.C \geqq 7P + E_5$. Then we must have $L.C \geqq 2P + Q_4$, which is impossible. Thus, we get $h^0(K - 7P - E_5) = 0$.

V-14) $H = 2J_6 + \langle n, n+8, n+16, n+24, n+32\rangle$. Then $r(H) = 4$. We set $Q_1 = Q_2 = Q_3 = R$. Let Q_4 be a point of C different from P. Then we have $h^0(K - E_4) = 6$. Let C_3 be a cubic with $C_3.C \geqq 3P + E_4$. Then $C_3 = T_P C_2$ where C_2 is a conic with $C_2.C \geqq 2R + Q_4$. Hence, we get

$$h^0(K - 5P - E_4) = h^0(K - 3P - E_4) = 6 - 3 = 3.$$

Let C_3' be a cubic with $C_3'.C \geqq 7P + E_4$. Then $C_3' = T_P^2 L$ where L is a line with $L.C \geqq R + Q_4$. Thus, we obtain

$$h^0(K - 7P - E_4) = h^0(K - 10P - E_4) = 1.$$

Let C_3'' be a cubic with $C_3''.C \geqq 11P + E_4$. This is impossible.

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