Bull. Korean Math. Soc. **56** (2019), No. 5, pp. 1143–1157
https://doi.org/10.4134/BKMS.b180914
pISSN: 1015-8634 / eISSN: 2234-3016

THE NUMBER OF REPRESENTATIONS OF A POSITIVE INTEGER BY TRIANGULAR, SQUARE AND DECAGONAL NUMBERS

Uha Isnaini, Ray Melham, and Pee Choon Toh

Abstract. Let $T_aD_b(n)$ and $T_aD'_b(n)$ denote respectively the number of representations of a positive integer n by $a(x^2-x)/2+b(4y^2-3y)$ and $a(x^2-x)/2+b(4y^2-y)$. Similarly, let $S_aD_b(n)$ and $S_aD'_b(n)$ denote respectively the number of representations of n by $ax^2+b(4y^2-3y)$ and $ax^2+b(4y^2-y)$. In this paper, we prove 162 formulas for these functions.

1. Introduction

Consider a positive definite binary quadratic form $ax^2+bxy+cy^2$ where a,b,c are integers with $a>0$ and discriminant $d=b^2-4ac<0$. We let $R_{(a,b,c)}(n)$ denote the number of representations of an integer n by this quadratic form as x and y range over all integers. In other words, we have

$$R_{(a,b,c)}(n)=\left|\{(x,y)\in\mathbb{Z}\times\mathbb{Z}:n=ax^2+bxy+cy^2\}\right|. \tag{1}$$

Jacobi's celebrated two squares theorem [9] can be stated as

$$R_{(1,0,1)}(n)=4\sum_{d|n}\left(\frac{-4}{d}\right)\text{ for }n\geqslant 1, \tag{2}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

The problem of finding the number of representations of an integer by sums of squares has been studied by many mathematicians throughout history. Formulas for $R_{(1,0,2)}(n)$ and $R_{(1,0,3)}(n)$ are usually attributed respectively to Dirichlet and Lorenz.

$$R_{(1,0,2)}(n)=2\sum_{d|n}\left(\frac{-8}{d}\right), \tag{3}$$

Received September 27, 2018; Revised January 2, 2019; Accepted February 7, 2019.
2010 *Mathematics Subject Classification.* Primary 11E25, 11E16.
Key words and phrases. representations by binary quadratic forms.
U. Isnaini was supported by the National Institute of Education (Singapore) PhD scholarship.

$$R_{(1,0,3)}(n) = 2\sum_{d|n}\left(\frac{-3}{d}\right) + 4\sum_{4d|n}\left(\frac{-3}{d}\right). \tag{4}$$

More details on the above formulas can be found in the accompanying notes to [3, Chpt. 3]. More recently, mathematicians have been interested in finding the number of representations in terms of polygonal numbers. The formula for the k-th polygonal number is given by

$$F_k(n) = \frac{n^2\,(k-2) - n\,(k-4)}{2} \quad \text{for } k \geqslant 3. \tag{5}$$

In particular,

$$F_3(n) = \frac{n^2+n}{2},\ F_4(n) = n^2,\ F_5(n) = \frac{3n^2-n}{2} \ \text{ and } \ F_7(n) = \frac{5n^2-3n}{2}, \tag{6}$$

giving us respectively, the triangular, square, pentagonal and heptagonal numbers.

The results that we briefly survey below all concern formulas for the number of representations of n as a sum of a copies of a polygonal number and b copies of another polygonal number. In other words, they are all closely related to representations by binary quadratic forms.

In [5], Hirschhorn used elementary methods to prove 14 formulas for representations in terms of various combinations of triangular and square numbers. Subsequently, he [6] proved another 27 formulas, where each formula contained at least a pentagonal or an octagonal number. Baruah and Sarmah [1] used one of Ramanujan's theta function identity, namely [2, p. 48, Entry 31], to prove another 25 formulas. Each of their formulas contained at least a heptagonal, decagonal, hendecagonal, dodecagonal or octadecagonal number.

Meanwhile, in a series of papers [14] to [17], Sun used results on binary quadratic forms that he obtained together with Williams [18] to prove 191 formulas involving triangular, square and pentagonal numbers.

Independently, Melham released an unpublished manuscript [10] in 2007 that contained a total of 298 conjectured formulas involving polygonal numbers from triangular to dodecagonal numbers. It is interesting to note that none of the 298 conjectured formulas coincides with the 66 formulas proved in [1, 5, 6]. Melham [11] subsequently extracted and published 21 of these conjectures which involved triangular, pentagonal and heptagonal numbers. He noted in his paper that three of these conjectures involving triangular numbers were equivalent to formulas proved by Sun [15]. In fact, another of these conjectures involving the sum of one pentagonal number and five copies of another pentagonal number was already proved in [14]. A further eight can be found in [16] which appeared in print in 2011. So 12 of the 21 conjectures in [11], in addition to another 21 conjectures in [10] had in fact been proved by Sun. Toh [20] subsequently proved all 21 conjectures in [11]. He also described a uniform approach to proving the remaining 277 conjectures in [10], and in doing so provided proofs for 13 of these 277 as examples. In Toh's paper, he mentioned that Hirschhorn had already proved three of the 21 conjectures in [11]. Hirschhorn's proofs had remained unpublished until recently, and they may now be found in [7, Chpt. 29]. Using Ramanujan's ${}_1\psi_1$ summation formula, Sarmah [12] also proved three of the 21 conjectures in [11]. He concluded his paper by remarking that the rest of Melham's conjectures may be formulated in terms of the ${}_1\psi_1$ summation formula but these "might be too complicated to actually have a proof." We note that two of the three conjectures proved by Sarmah coincided with the three by Hirschhorn and these four conjectures were a subset of those proved by Sun. Finally Humby, in his unpublished Masters thesis [8], also proved all of Melham's 21 conjectures in [11] by using the theory of modular forms.

In this paper, we focus on representations of integers by a combination of triangular or square numbers, and decagonal numbers. We shall denote these numbers respectively as

$$T(n) = F_3(-n) = \frac{n^2-n}{2},\ S(n) = F_4(n),\ D(n) = F_{10}(n) = 4n^2 - 3n, \tag{7}$$

and adopt the following notation

$$T_aD_b(n) = \left|\left\{(x,y) \in \mathbb{Z}^+ \times \mathbb{Z} : n = a\left(\frac{x^2-x}{2}\right) + b(4y^2-3y)\right\}\right|, \tag{8}$$

$$S_aD_b(n) = \left|\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : n = ax^2 + b(4y^2-3y)\right\}\right|, \tag{9}$$

where a and b are positive integers. It turns out that every formula for $T_aD_b(n)$ that we found has a companion formula for $T_aD'_b(n)$ where

$$T_aD'_b(n) = \left|\left\{(x,y) \in \mathbb{Z}^+ \times \mathbb{Z} : n = a\left(\frac{x^2-x}{2}\right) + b(4y^2-y)\right\}\right|. \tag{10}$$

The same holds for every $S_aD_b(n)$ and the corresponding $S_aD'_b(n)$ defined by

$$S_aD'_b(n) = \left|\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : n = ax^2 + b(4y^2-y)\right\}\right|. \tag{11}$$

We summarise our main findings below. First of all, we relate $T_aD_b(n)$ and $S_aD_b(n)$ to representations by binary quadratic forms.

Theorem 1.1. *Let $a,b,n \in \mathbb{Z}^+$. If b is odd, then*

$$4T_aD_b(n) = R_{(8a,8a,2a+b)}(16n + 2a + 9b).$$

The following is a consequence of Theorem 1.1.

Corollary 1.2. *If $4 \nmid a$ and b is odd, then*

$$4T_aD_b(n) = R_{(2a,0,b)}(16n + 2a + 9b).$$

Likewise for $S_aD_b(n)$, we have:

Theorem 1.3. *Let $a,b,n \in \mathbb{Z}^+$. If both a and b are odd, and $4 \nmid (a-b)$, then*

$$2S_aD_b(n) = R_{(a,0,b)}(16n + 9b).$$

We also have the following corresponding results for the companion functions $T_aD'_b(n)$ and $S_aD'_b(n)$.

Theorem 1.4. *Let $a, b, n \in \mathbb{Z}^+$. If b is odd, then*

$$4T_aD'_b(n) = R_{(8a,8a,2a+b)}(16n + 2a + b).$$

Corollary 1.5. *If b is odd, and $4 \nmid a$, then*

$$4T_aD'_b(n) = R_{(2a,0,b)}(16n + 2a + b).$$

Theorem 1.6. *Let $a, b, n \in \mathbb{Z}^+$. If a and b are odd, and $4 \nmid (a - b)$, then*

$$2S_aD'_b(n) = R_{(a,0,b)}(16n + b).$$

Corollary 1.2 and Theorem 1.3 allow us to find the explicit formulas for $T_aD_b(n)$ and $S_aD_b(n)$ through utilizing existing formulas for $R_{(a,b,c)}(n)$ from [19] and [4]. In total, we proved formulas for $T_aD_b(n)$ for 72 values of (a, b) and $S_aD_b(n)$ for 9 values of (a, b). These include the seven cases already proved by Baruah and Sarmah [1] and all the eight conjectures for $T_aD_b(n)$ in [10, Chpt. 11]. The remaining 66 formulas are new. Likewise, all the 72 formulas for $T_aD'_b(n)$ and 9 formulas for $S_aD'_b(n)$ are also new. The complete list is given in the following table.

Formula	(a, b)	Location of formula
$T_aD_b(n)$ or $T_aD'_b(n)$	$(1, 1)$	Theorem 4.1 and [1, (37)]
	$(1, p), (p, 1)$, $p = 3, 5, 11, 29$	Theorem 4.2 and [10, Chpt. 11]
	$(2, 1)$	Theorem 4.3 and [1, (30)]
	$(1, 9), (9, 1)$	Theorem 4.4
	$(6, 1), (2, 3)$	Theorem 4.5 and [1, (32), (29)]
	$(14, 1), (2, 7)$	Theorem 4.5
	$(30, 1), (2, 15), (10, 3), (6, 5)$	Theorem 4.6
	$(1, 15), (3, 5), (5, 3), (15, 1)$	Theorem 4.7
	$(1, 21), (3, 7), (7, 3), (21, 1)$	Theorem 4.8
	$(1, 35), (5, 7), (7, 5), (35, 1)$	Theorem 4.9
	$(1, 39), (3, 13), (13, 3), (39, 1)$	Theorem 4.10
	$(1, 51), (3, 17), (17, 3), (51, 1)$	Theorem 4.11
	$(1, 65), (5, 13), (13, 5), (65, 1)$	Theorem 4.12
	$(1, 95), (5, 19), (19, 5), (95, 1)$	Theorem 4.13
	$(1, 105), (3, 35), (5, 21), (7, 15)$ $(15, 7), (21, 5), (35, 3), (105, 1)$	Theorem 4.14
	$(1, 165), (3, 55), (5, 33), (11, 15)$ $(15, 11), (33, 5), (55, 3), (165, 1)$	Theorem 4.15
	$(1, 231), (3, 77), (7, 33), (11, 21)$ $(21, 11), (33, 7), (77, 3), (231, 1)$	Theorem 4.16
$S_aD_b(n)$ or $S_aD'_b(n)$	$(1, 1)$	Theorem 4.17 and [1, (31)]
	$(1, 3), (3, 1)$	Theorem 4.18 and [1, (28), (33)]
	$(1, 7), (7, 1)$	Theorem 4.18
	$(1, 15), (15, 1), (3, 5), (5, 3)$	Theorem 4.19

In the next section, we recall known results required for our proofs. In Section 3, we prove Theorems 1.1 to 1.6. The 162 explicit formulas are presented in Section 4.

2. Preliminary results

In this section, we first recall some results from the literature. It is known that the associated L-series of a genus character of an imaginary quadratic field with discriminant d can be decomposed into a product of two Dirichlet L-series [13, p. 62, Th. 4]. Consequently, for certain quadratic forms $ax^2 + bxy + cy^2$ with discriminant d, it is possible to write $R_{(a,b,c)}(n)$ as a convolution of two divisor sums with characters [19]. Toh used this property to deduce $R_{(a,0,c)}(n)$ for 11 pairs of (a, c) which are associated with imaginary quadratic fields with class number 2 [19, p. 232].

Theorem 2.1 (Toh [19]). *If $p = 5, 13$ or 37, then*

$$R_{(1,0,p)}(n) = \sum_{d|n} \left(\frac{-4p}{d}\right) + \sum_{d|n} \left(\frac{p}{d}\right) \left(\frac{-4}{n/d}\right). \tag{12}$$

If $p = 3, 5, 11$ or 29, then

$$R_{(1,0,2p)}(n) = \sum_{d|n} \left(\frac{-8p}{d}\right) + \sum_{d|n} \left(\frac{d}{p}\right) \left(\frac{-\left(\frac{-1}{p}\right)2}{n/d}\right) \tag{13}$$

and

$$R_{(2,0,p)}(n) = \sum_{d|n} \left(\frac{-8p}{d}\right) - \sum_{d|n} \left(\frac{d}{p}\right) \left(\frac{-\left(\frac{-1}{p}\right)2}{n/d}\right). \tag{14}$$

We remark that formulas equivalent to Theorem 2.1 also can be found in [18]. In the following, we recall several theorems for $R_{(a,0,c)}(n)$ that were proved by Chan and Toh [4].

Theorem 2.2 (Theorem 2.1 from [4]). *The following identities hold.*

$$\begin{aligned}
R_{(1,0,15)}(n) &= \sum_{d|n} \left(\frac{-15}{d}\right) + \sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{5}{n/d}\right) - 2\sum_{2d|n} \left(\frac{-60}{d}\right) \\
&\quad + 2\sum_{2d|n} \left(\frac{-3}{d}\right)\left(\frac{20}{n/(2d)}\right), \\
R_{(3,0,5)}(n) &= \sum_{d|n} \left(\frac{-15}{d}\right) - \sum_{d|n} \left(\frac{-3}{d}\right)\left(\frac{5}{n/d}\right) - 2\sum_{2d|n} \left(\frac{-60}{d}\right) \\
&\quad - 2\sum_{2d|n} \left(\frac{-3}{d}\right)\left(\frac{20}{n/(2d)}\right).
\end{aligned}$$

Theorem 2.3 (Theorem 3.2 from [4]). *The following identity holds.*

$$R_{(1,0,4)}(n) = 2\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{-4}{n/d}\right) + 4\sum_{4d|n}\left(\frac{-4}{d}\right).$$

Theorem 2.4 (Theorem 4.2 from [4]). *The following identities hold.*

$$R_{(1,0,18)}(n) = \sum_{d|n}\left(\frac{9}{d}\right)\left(\frac{-72}{n/d}\right) + \sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{24}{n/d}\right) + 2\sum_{9d|n}\left(\frac{-8}{d}\right),$$

$$R_{(2,0,9)}(n) = \sum_{d|n}\left(\frac{9}{d}\right)\left(\frac{-72}{n/d}\right) - \sum_{d|n}\left(\frac{-3}{d}\right)\left(\frac{24}{n/d}\right) + 2\sum_{9d|n}\left(\frac{-8}{d}\right).$$

Theorem 2.5 (Theorem 5.1 from [4]). *If $p = 3$ or 7, set $D = -p$, $N_3 = 6$ and $N_7 = 2$, then*

$$\begin{aligned} R_{(1,0,4p)}(n) = &\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{4D}{n/d}\right) + 2\sum_{4d|n}\left(\frac{4}{d}\right)\left(\frac{4D}{n/(4d)}\right) + N_p\sum_{16d|n}\left(\frac{D}{d}\right) \\ &+ \sum_{d|n}\left(\frac{-4}{d}\right)\left(\frac{-4D}{n/d}\right), \end{aligned}$$

$$\begin{aligned} R_{(4,0,p)}(n) = &\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{4D}{n/d}\right) + 2\sum_{4d|n}\left(\frac{4}{d}\right)\left(\frac{4D}{n/(4d)}\right) + N_p\sum_{16d|n}\left(\frac{D}{d}\right) \\ &- \sum_{d|n}\left(\frac{-4}{d}\right)\left(\frac{-4D}{n/d}\right). \end{aligned}$$

Theorem 2.6 (Theorem 5.3 from [4]). *The following identities hold.*

$$\begin{aligned} R_{(1,0,60)}(n) = &\frac{1}{2}\left(1+\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/d}\right) + \sum_{4d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/(4d)}\right) \\ &+ \sum_{16d|n}\left(\frac{-15}{d}\right) + \frac{1}{2}\left(1+\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/d}\right) \\ &+ \sum_{4d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/(4d)}\right) + \sum_{16d|n}\left(\frac{-3}{d}\right)\left(\frac{5}{n/(16d)}\right), \end{aligned}$$

$$\begin{aligned} R_{(3,0,20)}(n) = &\frac{1}{2}\left(1-\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/d}\right) + \sum_{4d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/(4d)}\right) \\ &+ \sum_{16d|n}\left(\frac{-15}{d}\right) - \frac{1}{2}\left(1-\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/d}\right) \\ &- \sum_{4d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/(4d)}\right) - \sum_{16d|n}\left(\frac{-3}{d}\right)\left(\frac{5}{n/(16d)}\right), \end{aligned}$$

$$\begin{aligned}R_{(4,0,15)}(n) &= \frac{1}{2}\left(1-\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/d}\right)+\sum_{4d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/(4d)}\right)\\ &\quad+\sum_{16d|n}\left(\frac{-15}{d}\right)+\frac{1}{2}\left(1-\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/d}\right)\\ &\quad+\sum_{4d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/(4d)}\right)+\sum_{16d|n}\left(\frac{-3}{d}\right)\left(\frac{5}{n/(16d)}\right),\end{aligned}$$

$$\begin{aligned}R_{(5,0,12)}(n) &= \frac{1}{2}\left(1+\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/d}\right)+\sum_{4d|n}\left(\frac{4}{d}\right)\left(\frac{-60}{n/(4d)}\right)\\ &\quad+\sum_{16d|n}\left(\frac{-15}{d}\right)-\frac{1}{2}\left(1+\left(\frac{-1}{n}\right)\right)\sum_{d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/d}\right)\\ &\quad-\sum_{4d|n}\left(\frac{-12}{d}\right)\left(\frac{20}{n/(4d)}\right)-\sum_{16d|n}\left(\frac{-3}{d}\right)\left(\frac{5}{n/(16d)}\right).\end{aligned}$$

We also require a formula for $R_{(1,0,7)}(n)$ which was known to Ramanujan [2, p. 302].

$$R_{(1,0,7)}(n) = 2\sum_{d|n}\left(\frac{-7}{d}\right)-4\sum_{2d|n}\left(\frac{-28}{d}\right). \tag{15}$$

We end this section with the following lemma from Sun [16, Lemma 2.2].

Lemma 2.7. *Let $a,b,n\in\mathbb{Z}^+$, with $2\nmid n$.*

i) *If $2\nmid a$ and $4\nmid(a-b)b$, then*

$$R_{(a,0,4b)}(n) = \begin{cases}R_{(a,0,b)}(n), & \text{if } n\equiv a \pmod 4;\\ 0, & \text{otherwise.}\end{cases} \tag{16}$$

ii) *If $2\nmid a$, $2\mid b$ and $8\nmid b$, then*

$$R_{(a,0,4b)}(n) = \begin{cases}R_{(a,0,b)}(n), & \text{if } n\equiv a \pmod 8;\\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

iii) *If $2\nmid(a+b)$ and $8\nmid ab$, then*

$$R_{(4a,4a,a+b)}(n) = \begin{cases}R_{(a,0,b)}(n), & \text{if } n\equiv a+b \pmod 8;\\ 0, & \text{otherwise.}\end{cases} \tag{18}$$

3. Relating $T_aD_b(n)$ and $S_aD_b(n)$ to $R_{(a,b,c)}(n)$

In this section, we prove our main results by relating $T_aD_b(n)$ and $S_aD_b(n)$ to $R_{(a,b,c)}(n)$.

Proof of Theorem 1.1. Since $T(n+1) = T(-n)$, we have

$$\begin{aligned}
4T_aD_b(n) &= 2\left|\left\{(x,y)\in\mathbb{Z}^2 : n = a\left(\frac{x^2-x}{2}\right) + b(4y^2-3y)\right\}\right| \\
&= 2\left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2a(2x-1)^2 + b(8y-3)^2\right\}\right| \\
&= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2ax^2+by^2,\right.\right. \\
&\qquad\qquad \left.\left. 2\nmid x, y\equiv \pm 3 \pmod 8\right\}\right|. \qquad (19)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
4T_aD'_b(n) = \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+b = 2ax^2+by^2,\right.\right. \\
\left.\left. 2\nmid x, y\equiv \pm 1 \pmod 8\right\}\right|. \qquad (20)
\end{aligned}$$

Combining (19) and (20) gives us

$$\begin{aligned}
&4T_aD_b(n) + 4T_aD'_b\left(n+\frac{b}{2}\right) \\
&= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2ax^2+by^2, 2\nmid xy\right\}\right| \\
&= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2ax^2+by^2, 2\mid (x-y)\right\}\right| \\
&\quad - \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2ax^2+by^2, 2\mid x, 2\mid y\right\}\right| \\
&= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2a(2x+y)^2+by^2\right\}\right| \\
&\quad - \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2a(2x)^2+b(2y)^2\right\}\right| \\
&= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 8ax^2+8axy+(2a+b)y^2\right\}\right| \\
&\quad - \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 8ax^2+4by^2\right\}\right| \\
&= R_{(8a,8a,2a+b)}(16n+2a+9b) - R_{(8a,0,4b)}(16n+2a+9b). \qquad (21)
\end{aligned}$$

When b is odd, we get $T_aD'_b(n+\frac{b}{2}) = 0$ and there are no solutions for

$$16n+2a+9b = 8ax^2+4by^2,$$

which implies $R_{(8a,0,4b)}(16n+2a+9b)$ equals zero. This completes the proof. □

We now deduce Corollary 1.2.

Proof. From Theorem 1.1, when b is odd we have

$$4T_aD_b(n) = R_{(8a,8a,2a+b)}(16n+2a+9b).$$

If we further assume $4\nmid a$, then $2\nmid(2a+b)$, $8\nmid(2a)b$ and

$$16n+2a+9b \equiv 2a+b \pmod 8.$$

By (18),

$$R_{(4(2a),4(2a),2a+b)}(16n+2a+9b) = R_{(2a,0,b)}(16n+2a+9b)$$

and the proof follows. □

The proofs of Theorem 1.4 and Corollary 1.5 for $T_aD'_b(n)$ follow in an analogous manner. We now proceed to the proof of Theorem 1.3.

Proof. We have

$$\begin{aligned} 2S_aD_b(n) &= 2\left|\left\{(x,y)\in\mathbb{Z}^2 : n = ax^2 + b(4y^2-3y)\right\}\right| \\ &= 2\left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+9b = 16ax^2 + b(8y-3)^2\right\}\right| \\ &= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+9b = 16ax^2 + by^2, y\equiv \pm 3 \pmod 8\right\}\right|. \end{aligned} \tag{22}$$

Similarly,

$$2S_aD'_b(n) = \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+b = 16ax^2+by^2, y\equiv\pm 1 \pmod 8\right\}\right|. \tag{23}$$

Combining (22) and (23) gives us

$$\begin{aligned} &2S_aD_b(n) + 2S_aD'_b\left(n+\frac{b}{2}\right) \\ &= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+9b = 16ax^2+by^2, 2\nmid y\right\}\right| \\ &= \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+9b = 16ax^2+by^2\right\}\right| \\ &\quad - \left|\left\{(x,y)\in\mathbb{Z}^2 : 16n+2a+9b = 2ax^2+b(2y)^2\right\}\right| \\ &= R_{(16a,0,b)}(16n+9b) - R_{(16a,0,4b)}(16n+9b). \end{aligned} \tag{24}$$

When b is odd, there are no solutions for

$$16n+9b = 16ax^2+4by^2,$$

which implies $R_{(16a,0,4b)}(16n+9b)$ equals zero. In other words,

$$2S_aD_b(n) = R_{(16a,0,b)}(16n+9b). \tag{25}$$

If we further assume that a is also odd, then all the conditions of (17) are satisfied and we get

$$R_{(b,0,4(4a))}(16n+9b) = R_{(b,0,4a)}(16n+9b).$$

We can simplify further by adding another assumption that $4\nmid(b-a)$. Then by (16) we get

$$R_{(b,0,4a)}(16n+9b) = R_{(b,0,a)}(16n+9b),$$

which completes the proof. □

4. Explicit formulas

In this section, we present the explicit formulas for $T_aD_b(n)$ and $S_aD_b(n)$. As mentioned in the introduction, each formula has a companion formula. The formulas for $T_aD_b(n)$ are stated as a divisor sum for $m = 16n+2a+9b$. To obtain the companion formula for $T_aD'_b(n)$, one simply replaces m by $m' = 16n+2a+b$.

Theorem 4.1. *Let* $m = 16n + 11$. *Then*

$$2T_1D_1(n) = \sum_{d|m} \left(\frac{-8}{d}\right).$$

Proof. From Corollary 1.2, we have

$$4T_1D_1(n) = R_{(2,0,1)}(16n + 11),$$

and the result follows from Dirichlet's formula (3). □

Theorem 4.2. *If* $p = 3, 5, 11$ *or* 29, *then*

$$2T_1D_p(n) = \sum_{d|m} \left(\frac{-8p}{d}\right), \text{ where } m = 16n + 2 + 9p;$$

$$2T_pD_1(n) = \sum_{d|m} \left(\frac{-8p}{d}\right), \text{ where } m = 16n + 2p + 9.$$

Proof. We consider the case $p = 3$ or 11. Corollary 1.2 and (14) give,

$$\begin{aligned}
4T_1D_p(n) &= R_{(2,0,p)}(16n + 9p + 2) \\
&= \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right) - \sum_{d|16n+9p+2} \left(\frac{d}{p}\right)\left(\frac{2}{d}\right)^2 \left(\frac{2}{\frac{16n+9p+2}{d}}\right) \\
&= \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right) - \sum_{d|16n+9p+2} \left(\frac{-2p}{d}\right)\left(\frac{2}{16n+9p+2}\right) \\
&= \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right)\left(1 - \left(\frac{2}{16n+9p+2}\right)\right) \\
&= 2 \sum_{d|16n+9p+2} \left(\frac{-8p}{d}\right).
\end{aligned}$$

The other cases can be proved in a similar manner. □

The following theorem is obtained from Corollary 1.2 and Theorem 2.3.

Theorem 4.3. *Let* $m = 16n + 13$. *Then*

$$2T_2D_1(n) = \sum_{d|m} \left(\frac{-4}{d}\right).$$

The following theorem is an immediate consequence from Corollary 1.2 and Theorem 2.4.

Theorem 4.4. *If* $(a, b, \alpha) = (1, 9, -1)$ *or* $(9, 1, 1)$, *then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{9}{d}\right)\left(\frac{-72}{m/d}\right) + \alpha \sum_{d|m} \left(\frac{-3}{d}\right)\left(\frac{24}{m/d}\right) + 2 \sum_{9d|m} \left(\frac{-8}{d}\right),$$

where $m = 16n + 2a + 9b$.

By Theorem 2.5, we can employ computations similar to those in the proof of Theorem 4.2 to obtain the following.

Theorem 4.5. *If* $p = 3$ *or* 7, *then*

$$2T_{2p}D_1(n) = \sum_{d|m} \left(\frac{-p}{d}\right), \text{ where } m = 16n + 4p + 9;$$

$$2T_2D_p(n) = \sum_{d|m} \left(\frac{-p}{d}\right), \text{ where } m = 16n + 4 + 9p.$$

Likewise, by Theorem 2.6 we can prove the following.

Theorem 4.6. *If* $(a, b, \alpha) = (30, 1, 1), (2, 15, 1), (10, 3, -1)$ *or* $(6, 5, -1)$, *then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{4}{d}\right)\left(\frac{-60}{m/d}\right) + \alpha \sum_{d|m} \left(\frac{-12}{d}\right)\left(\frac{20}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

The proofs of the remaining theorems follow in an analogous manner by using Corollary 1.2 and the corresponding $R_{(a,b,c)}(n)$ formulas in [19] or [4].

Theorem 4.7. *If* $(a, b, \alpha) = (15, 1, 1), (5, 3, 1), (3, 5, -1)$ *or* $(1, 15, -1)$, *then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-120}{d}\right) + \alpha \sum_{d|m} \left(\frac{10}{d}\right)\left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.8. *If* $(a, b, \alpha) = (21, 1, 1), (3, 7, 1), (7, 3, -1)$ *or* $(1, 21, -1)$, *then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-168}{d}\right) + \alpha \sum_{d|m} \left(\frac{14}{d}\right)\left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.9. *If* $(a, b, \alpha) = (35, 1, 1), (7, 5, 1), (5, 7, -1)$ *or* $(1, 35, -1)$, *then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-280}{d}\right) + \alpha \sum_{d|m} \left(\frac{-14}{d}\right)\left(\frac{5}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.10. *If* $(a, b, \alpha) = (39, 1, 1), (3, 13, 1), (13, 3, -1)$ *or* $(1, 39, -1)$, *then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-312}{d}\right) + \alpha \sum_{d|m} \left(\frac{26}{d}\right)\left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.11. *If* $(a, b, \alpha) = (51, 1, 1), (17, 3, 1), (3, 17, -1)$ *or* $(1, 51, -1)$*, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-408}{d}\right) + \sum_{d|m} \left(\frac{34}{d}\right) \left(\frac{-3}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.12. *If* $(a, b, \alpha) = (65, 1, 1), (13, 5, 1), (5, 13, -1)$ *or* $(1, 65, -1)$*, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-520}{d}\right) + \sum_{d|m} \left(\frac{-26}{d}\right) \left(\frac{5}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.13. *If* $(a, b, \alpha) = (95, 1, 1), (5, 19, 1), (19, 5, -1)$ *or* $(1, 95, -1)$*, then*

$$4T_aD_b(n) = \sum_{d|m} \left(\frac{-760}{d}\right) + \sum_{d|m} \left(\frac{-38}{d}\right) \left(\frac{5}{m/d}\right),$$

where $m = 16n + 2a + 9b$.

Theorem 4.14. *Let* $m = 16n + 2a + 9b$*. Then*

$$8T_aD_b(n) = \sum_{d|m} \left(\frac{-840}{d}\right) + \alpha_2 \sum_{d|m} \left(\frac{280}{d}\right) \left(\frac{-3}{m/d}\right) + \alpha_3 \sum_{d|m} \left(\frac{-168}{d}\right) \left(\frac{5}{m/d}\right) + \alpha_4 \sum_{d|m} \left(\frac{120}{d}\right) \left(\frac{-7}{m/d}\right),$$

where the values for a, b, α_i *are listed in the following table.*

a	b	α_2	α_3	α_4
1	105	−1	−1	1
3	35	−1	1	−1
5	21	1	1	−1
7	15	−1	1	1
15	7	1	−1	1
21	5	−1	−1	−1
35	3	1	−1	−1
105	1	1	1	1

Theorem 4.15. *Let* $m = 16n + 2a + 9b$*. Then*

$$8T_aD_b(n) = \sum_{d|m} \left(\frac{-1320}{d}\right) + \alpha_2 \sum_{d|m} \left(\frac{440}{d}\right) \left(\frac{-3}{m/d}\right) + \alpha_3 \sum_{d|m} \left(\frac{-264}{d}\right) \left(\frac{5}{m/d}\right) + \alpha_4 \sum_{d|m} \left(\frac{120}{d}\right) \left(\frac{-11}{m/d}\right),$$

where the values for a, b, α_i *are listed in the following table.*

a	b	α_2	α_3	α_4
1	165	−1	−1	−1
3	55	1	1	−1
5	33	1	−1	−1
11	15	1	−1	1
15	11	−1	1	−1
33	5	−1	1	1
55	3	−1	−1	1
165	1	1	1	1

Theorem 4.16. *Let $m = 16n + 2a + 9b$. Then*

$$8T_aD_b(n) = \sum_{d|m} \left(\frac{-1848}{d}\right) + \alpha_2 \sum_{d|m} \left(\frac{616}{d}\right)\left(\frac{-3}{m/d}\right) + \alpha_3 \sum_{d|m} \left(\frac{264}{d}\right)\left(\frac{-7}{m/d}\right) + \alpha_4 \sum_{d|m} \left(\frac{168}{d}\right)\left(\frac{-11}{m/d}\right),$$

where the values for a, b, α_i are listed in the following table.

a	b	α_2	α_3	α_4
1	231	−1	1	−1
3	77	−1	−1	−1
7	33	−1	−1	1
11	21	1	1	−1
21	11	−1	1	1
33	7	1	−1	−1
77	3	1	−1	1
231	1	1	1	1

Formulas for $S_aD_b(n)$ and $S_aD'_b(n)$ are presented in the next three theorems. Although we only list the formulas for $S_aD_b(n)$ with divisors sum over $m = 16n + 9b$, each entry has a corresponding formula for $S_aD'_b(n)$ where m is replaced by $m' = 16n + b$.

Theorem 4.17. *Let $m = 16n + 9$. Then*

$$S_1D_1(n) = \sum_{d|m} \left(\frac{-4}{d}\right).$$

Proof. From (25), we have

$$2S_1D_1(n) = R_{(1,0,16)}(16n + 9).$$

From (17), we get

$$R_{(1,0,16)}(16n + 9) = R_{(1,0,4)}(16n + 9).$$

The proof follows from Theorem 2.3. □

The next theorem follows from Theorem 1.3 and (4) and (15).

Theorem 4.18. *If* $p = 3$ *or* 7*, then*

$$S_1 D_p(n) = \sum_{d|m} \left(\frac{-p}{d}\right), \text{ where } m = 16n + 9p;$$

$$S_p D_1(n) = \sum_{d|m} \left(\frac{-p}{d}\right), \text{ where } m = 16n + 9.$$

The following theorem is an immediate consequence of Theorem 1.3 and Theorem 2.2.

Theorem 4.19. *If* $(a, b, \alpha) = (1, 15, 1), (15, 1, 1), (3, 5, -1)$ *or* $(5, 3, -1)$*, then*

$$2S_a D_b(n) = \sum_{d|m} \left(\frac{-15}{d}\right) + \alpha \sum_{d|m} \left(\frac{-3}{d}\right) \left(\frac{5}{m/d}\right),$$

where $m = 16n + 9b$.

References

[1] N. D. Baruah and B. K. Sarmah, *The number of representations of a number as sums of various polygonal numbers*, Integers **12** (2012), Paper No. A54, 16 pp.

[2] B. C. Berndt, *Ramanujan's Notebooks. Part III*, Springer-Verlag, New York, 1991. https://doi.org/10.1007/978-1-4612-0965-2

[3] ———, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, **34**, American Mathematical Society, Providence, RI, 2006. https://doi.org/10.1090/stml/034

[4] H. H. Chan and P. C. Toh, *Theta series associated with certain positive definite binary quadratic forms*, Acta Arith. **169** (2015), no. 4, 331–356. https://doi.org/10.4064/aa169-4-3

[5] M. D. Hirschhorn, *The number of representations of a number by various forms*, Discrete Math. **298** (2005), no. 1-3, 205–211. https://doi.org/10.1016/j.disc.2004.08.045

[6] ———, *The number of representations of a number by various forms involving triangles, squares, pentagons and octagons*, in Ramanujan rediscovered, 113–124, Ramanujan Math. Soc. Lect. Notes Ser., 14, Ramanujan Math. Soc., Mysore, 2010.

[7] ———, *The power of q*, Developments in Mathematics, **49**, Springer, Cham, 2017. https://doi.org/10.1007/978-3-319-57762-3

[8] G. Humby, *A proof of Melham's identities*, Unpublished Masters' Thesis, University of Exeter, 2017.

[9] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, Sumptibus fratrum Bornträger, 1829.

[10] R. S. Melham, *Analogues of Jacobi's two-square theorem*, Research Report R07-01, Department of Mathematical Sciences, University of Technology, Sydney.

[11] ———, *Analogues of Jacobi's two-square theorem: an informal account*, Integers **10** (2010), A8, 83–100. https://doi.org/10.1515/INTEG.2010.008

[12] B. K. Sarmah, *Proof of some conjectures of Melham using Ramanujan's* ${}_1\psi_1$ *formula*, Int. J. Math. Math. Sci. **2014** (2014), Art. ID 738948, 6 pp. https://doi.org/10.1155/2014/738948

[13] C. L. Siegel, *Advanced Analytic Number Theory*, second edition, Tata Institute of Fundamental Research Studies in Mathematics, **9**, Tata Institute of Fundamental Research, Bombay, 1980.

[14] Z.-H. Sun, *The expansion of* $\prod_{k=1}^{\infty}(1-q^{ak})(1-q^{bk})$, Acta Arith. **134** (2008), no. 1, 11–29. https://doi.org/10.4064/aa134-1-2

[15] ———, *On the number of representations of* n *by* $ax(x-1)/2+by(y-1)/2$, J. Number Theory **129** (2009), no. 5, 971–989. https://doi.org/10.1016/j.jnt.2008.11.007

[16] ———, *Binary quadratic forms and sums of triangular numbers*, Acta Arith. **146** (2011), no. 3, 257–297. https://doi.org/10.4064/aa146-3-5

[17] ———, *On the number of representations of* n *by* $ax^2+by(y-1)/2$, $ax^2+by(3y-1)/2$ *and* $ax(x-1)/2+by(3y-1)/2$, Acta Arith. **147** (2011), no. 1, 81–100. https://doi.org/10.4064/aa147-1-5

[18] Z.-H. Sun and K. S. Williams, *On the number of representations of* n *by* $ax^2+bxy+cy^2$, Acta Arith. **122** (2006), no. 2, 101–171. https://doi.org/10.4064/aa122-2-1

[19] P. C. Toh, *Representations of certain binary quadratic forms as Lambert series*, Acta Arith. **143** (2010), no. 3, 227–237. https://doi.org/10.4064/aa143-3-3

[20] ———, *On representations by figurate numbers*: *a uniform approach to the conjectures of Melham*, Int. J. Number Theory **9** (2013), no. 4, 1055–1071. https://doi.org/10.1142/S1793042113500127

Uha Isnaini
Mathematics & Mathematics Education
National Institute of Education
Nanyang Technological University
1 Nanyang Walk, 637616, Singapore
Email address: uhaisnaini@yahoo.co.id

Ray Melham
School of Mathematical and Physical Sciences
University of Technology, Sydney
Broadway NSW 2007, Australia
Email address: ray.melham@uts.edu.au

Pee Choon Toh
Mathematics & Mathematics Education
National Institute of Education
Nanyang Technological University
1 Nanyang Walk, 637616, Singapore
Email address: peechoon.toh@nie.edu.sg