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A FINANCIAL MARKET OF A STOCHASTIC DELAY EQUATION

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Abstract. We propose a stochastic delay financial model which describes influences driven by historical events. The underlying is modeled by stochastic delay differential equation (SDDE), and the delay effect is modeled by a stopping time in coefficient functions. While this model makes good economical sense, it is difficult to mathematically deal with this. Therefore, we circumvent this model with similar delay effects but mathematically more tractable, which is by the backward time integration. We derive the option pricing equation and provide the option price and the perfect hedging portfolio.

1. Introduction

While many stock price models such as the Black Scholes assume the Markov property and/or stationary and independent increments of returns, empirical studies suggest that the stock price has short term memories and feedback effects. There are a few approaches to capture this phenomenon. One popular approach is to use a fractional Brownian motion. Fractional Brownian motion still has stationary increments, but increments are no longer independent. While a model based on a fractional Brownian motion is certainly one way to address, a fatal drawback is that the price process is not a semimartingale, so the conventional asset pricing theory cannot be applied. Nevertheless, there are rich collection of literature on asset pricing with fractional Brownian motions, circumventing this difficulty in various ways. There are examples of stochastic processes with stationary but dependent increments, as introduced in [1, 2], [10], but they are a little too much complicated to be used as an asset pricing model.

Models with dependent or non-stationary increments are common in time series studies, although the time is always discrete in time series studies while we are dealing with continuous time models. ARCH, GARCH models fall to this category. So, it seems that it is natural to consider a continuous time version of GARCH model, which may be called 'COGARCH'. There have been several

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attempts to generalize GARCH to a continuous time, and some are potentially useful as an asset pricing model. About detailed history of a continuous time GARCH model, we refer to a nice summary in [3], [11] and [12]. While this COGARCH approach has certain desirable properties, its sampled discrete process does not in general possess GARCH properties. So it becomes just a little complicated stochastic volatility model, and is not so satisfactory in our purpose.

This short term dependency is sometimes called a feedback effect, since past stock prices give feedback to the current price. [13] studied a jump diffusion model where the jump intensity gets instantaneous feedback from the current stock price. [8] studied regime switching models with feedback effects. [9] studied market volatility and feedback effects. Those feedback effect models give the dependency of the certain part of the price process such as jumps to the past price process, but do not give the dependency all the way through.

Our approach is to use a stochastic delay differential equation which, unlike a feedback model, gives continuous effects from the past price processes, as the one in [16]. The main difficulty of this approach is that while the model makes sense, it is mathematically very difficult to deal with. Especially, the proposed model is not Markov and the market is incomplete. In this paper, we study a modified version of the original stochastic delay equation. The main idea is to replace a realization of a random point on the interval with a time integration with respect to a certain density. While this modified one is not same as the original delayed equation, we believe that this still captures the main idea of the delay effect.

The rest of the paper is organized as follows. In Chapter 2, we introduce the model and justifies the existence and uniqueness of the solution. In Chapter 3, we find the equivalent martingale measure. We also discuss a new dynamic of the model under the changed measure. Chapter 4 discusses the Markov property and its infinitesimal generator. Chapter 5 gives the price of European options.

2. Model

For convenience, we assume that the spot rate of interest is zero. Let us consider the price process which follows the stochastic delay model given by

$$\begin{aligned} dS_t &= \mu\alpha(S_{t-\tilde{\tau}_t})dt + \sigma\beta(S_{t-\tilde{\tau}_t})dW_t, \quad 0 < t \le T < \infty, \\ S_t &= \phi(t), \quad -\tilde{s} \le t \le 0 \end{aligned} \tag{2.1}$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions. Here μ, σ, and s are positive constants and W_t is a standard Brownian motion on $\mathbb{P}$. Functions $\alpha(\cdot)$ and $\beta(\cdot)$ are globally Lipschitz continuous. The initial data is given by $\phi(t)$, which is cadlag for $-\tilde{s} \le t \le 0$. Here, we assume that $S_0 = \phi(0) > 0$. We also assume that the stopping process $\tilde{\tau}_t$ is a strong Markov process with initial value τ_0, and $\mathcal{L}_\tau$ is its infinitesimal generator. In addition, the process $\tilde{\tau}_t$ has a probability density $P(\tilde{\tau}_t \in ds) = k_t(s)ds$ for all t. Note that $k_t(s)$ is the continuous transition probability density function on $[0, t+\tilde{s}]$.

This model captures the short term dependency structure through the stopping time $\tilde{\tau}_t$. In other word, the dynamic of the current price process depends on the price at the previous time $t - \tilde{\tau}_t$. The delay stopping time $\tilde{\tau}_t$ should be determined by the market. While this model makes sense, it is practically difficult to even find a solution of the equation. Also, the lack of Markov property makes subsequent mathematical formulation more difficult. Therefore, we try to modify this to make calculation doable while still keeping the spirit of the short term dependency. The idea is to replace $\tilde{\tau}_t$ with the integration on the interval $[-\tilde{s}, t]$.

Instead of (2.1) let us consider the following modified equation.

(2.2)
$$dS_t = \mu\alpha(\int_{-\tilde{s}}^{t} S_u\, p_t(du))dt + \sigma\beta(\int_{-\tilde{s}}^{t} S_u\, p_t(du))dW_t, \quad -\tilde{s} < 0 < t \leq T < \infty,$$

where p_t is a measure at t with a smooth function $f(t,u)$, i.e.,

$$p_t(du) = f(t,u)du$$

for $u \in [-\tilde{s}, t]$. For example, if p_t is a uniform probability measure with density $f(t,u) = 1/(t+\tilde{s})$, then $\int_{-\tilde{s}}^{t} S_u\, p_t(du)$ is represented by $\frac{1}{t+\tilde{s}} \int_{-\tilde{s}}^{t} S_u du$.

Let us define $\tau_t := t - \tilde{\tau}_t$, for the notational simplicity and set $p_t(du)$ be a probability measure of τ_t. Then, the vector process (S_t, A_t) becomes an expected delay time model where

$$A_t := \int_{-\tilde{s}}^{t} S_u\, p_t(du) = \int_{-\tilde{s}}^{t} S_u\, f(t,u)du.$$

The financial meaning of (2.2) is that the model has a delay effect as a weighted average by τ_t. Therefore, the integration with respect to $p_t(u)$ captures the effect of the random delay time $\tilde{\tau}$.

A typical example of the process τ_t is a fixed constant. In this case, $f(t,u)$ becomes a Dirac delta function $\delta_{u_0}(u)$ for $-\tilde{s} \leq u_0 \leq t$. As a choice of the density of τ_t, [4] introduced a rich family of bounded stochastic processes. Motivated by their work, we assume that the dynamic of τ_t follows the stochastic differential equation:

$$d\tau_t = -\gamma\tau_t dt + D(\tau_t)dB_t, \tag{2.3}$$

where B_t is a Brownian motion independent of W_t and

$$D^2(x) = -\frac{2\gamma}{F_X(x)} \int_{-\tilde{s}}^{x} \theta\, F_X(\theta)d\theta,$$

and $F_X(x)$ is a desired density on $[-\tilde{s}, t]$.

By the Itô formula, we derive the infinitesimal generator $\mathcal{L}_{\tau_t}$ of $\tau_t = u$,

$$\mathcal{L}_{\tau_t} := -\gamma u \frac{\partial}{\partial u} + \frac{1}{2} D^2(u) \frac{\partial^2}{\partial u^2}. \tag{2.4}$$

One reasonable choice of $F_X(x)$ is the uniform distribution, when $F_X(x) = \frac{1}{t+\tilde{s}} 1_{(-\tilde{s},t)}(x)$. We refer to Section 3.1 of [4] for detailed procedure to pick a density.

Then $f(t,u)$ becomes the solution of PDE

$$\begin{aligned} &\frac{\partial f}{\partial t}(t,u) + \mathcal{L}_{\tau_t} f(t,u) = 0, \\ &f(0,u) = \delta_{\tau_0}(u). \end{aligned} \tag{2.5}$$

In general, $\frac{\partial^m f}{\partial t^m}$ for $m \geq 1$ is not an element of function space $C_0(\mathbb{R}^2)$. This makes it difficult to prove the Markov property of our problem. Therefore, we impose an additional assumption on $f(t,u)$ below.

Assumption 2.1. There exists an integer $\tilde{m} \geq 2$ such that $\frac{\partial^{\tilde{m}} f}{\partial t^{\tilde{m}}} = 0$ and $\frac{\partial^m f}{\partial t^m} \in C_0([0,\infty) \times [-\tilde{s}, t])$ for all integer $\tilde{m} > m \geq 1$.

Assumption 2.1 implies that we can convert our problem into a finite dimensional problem. Since $f(t,u)$ is a probability density function on $[-\tilde{s}, t]$, Assumption 2.1 is not a strong condition and should be satisfied in most reasonable cases.

The next theorem gives us the existence and uniqueness of the solution of SDDE (2.2).

Theorem 2.1. *SDDE* (2.2) *has an a.s. continuous adapted solution* S_t, $0 < t \leq T$ *and it is unique.*

Proof. The basic idea of the proof is to apply basic existence arguments. Define the process,

$$\begin{aligned} dS_t^{(i+1)} &= \mu\alpha\Big(\int_{-\tilde{s}}^t S_u^{(i)}\, p_t(du)\Big)dt + \sigma\beta\Big(\int_{-\tilde{s}}^t S_u^{(i)}\, p_t(du)\Big)dW_t, \\ S_t^{(i+1)} &= \phi(t), \quad -\tilde{s} \leq t \leq 0, \end{aligned} \tag{2.6}$$

for $0 < t$ and $i = 0, 1, \ldots$. Since α and β are globally Lipschitz continuous, by the Hölder inequality and the Itô isometry, we have

$$\begin{aligned} &E[|S_t^{(i+1)} - S_t^{(i)}|^2] \\ &= E\Big[\Big|\int_0^t \mu\alpha\Big(\int_{-\tilde{s}}^s S_u^{(i)}\, p_s(du)\Big)ds + \int_0^t \sigma\beta\Big(\int_{-\tilde{s}}^s S_u^{(i)}\, p_t(du)\Big)dW_s \\ &\quad - \int_0^t \mu\alpha\Big(\int_{-\tilde{s}}^s S_u^{(i-1)}\, p_s(du)\Big)ds - \int_0^t \sigma\beta\Big(\int_{-\tilde{s}}^s S_u^{(i-1)}\, p_s(du)\Big)dW_s\Big|^2\Big] \\ &\leq C\, E\Big[\Big|\int_0^t \int_{-\tilde{s}}^s (S_u^{(i)} - S_u^{(i-1)})\, p_s(du)ds \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&+\int_0^s \int_{-\tilde{s}}^s (S_u^{(i)} - S_u^{(i-1)})\, p_s(du) dW_s|^2] \\
&\le 2C\, E[|\int_0^t \int_{-\tilde{s}}^s (S_u^{(i)} - S_u^{(i-1)})\, p_s(du) ds|^2 \\
&\quad + |\int_0^t \int_{-\tilde{s}}^s (S_u^{(i)} - S_u^{(i-1)})\, p_s(du) dW_s|^2] \\
&\le 2(t+1)C \int_0^t E[\int_{-\tilde{s}}^s |S_u^{(i)} - S_u^{(i-1)}|^2 p_s(du)] ds,
\end{aligned}
$$

where C is a positive constant which is independent of i. Since $p_t(du)$ is a smooth probability measure and $S_u^{(i)} = S_u^{(i-1)}$ for $u \in [-\tilde{s}, 0]$, the first term of (2.7) satisfies the following inequality,

$$
\begin{aligned}
&E[|S_t^{(i+1)} - S_t^{(i)}|^2] \\
&\le 2(t+1)C \int_0^t E[\int_{-\tilde{s}}^s |S_u^{(i)} - S_u^{(i-1)}|^2 p_s(du)] ds \\
&\le 2(t+1)C \Big(\int_0^t E[\int_{-\tilde{s}}^0 |S_u^{(i)} - S_u^{(i-1)}|^2 p_s(du)] ds \\
&\quad + \int_0^t E[\int_0^s |S_u^{(i)} - S_u^{(i-1)}|^2 p_s(du)] ds \Big) \\
&\le 2(t+1)^2 C \int_0^t \int_s^t E[|S_u^{(i)} - S_u^{(i-1)}|^2] f(s,u) ds\, du \\
&\le 2(t+1)^3 C \int_0^t E[|S_u^{(i)} - S_u^{(i-1)}|^2] du.
\end{aligned} \tag{2.8}
$$

Let us define $v_i(u) = E[|S_u^{(i)} - S_u^{(i-1)}|^2]$ for $i = 1, 2, \ldots$. From the Fubini theorem and (2.8), we get

$$
\begin{aligned}
v_{i+1}(t) &\le 2(1+t)^3 C \int_0^t v_i(s) ds \\
&\le 2(1+T)^3 C \int_0^t v_i(s) ds.
\end{aligned} \tag{2.9}
$$

Similarly, we obtain $v_1(t) \le tD$ for a positive constant D which depends on C, T, and $E[|S_0|^2]$. Therefore, by an induction argument, we obtain

$$
v_i(t) \le \frac{M^i t^{3i}}{i!}, \tag{2.10}
$$

where M is a positive constant which is independent of i. Let us define the norm $\|\cdot\|_{L^2(\mathbb{P})}$, where λ is the Lebesque measure on $[0, T]$. It follows from

(2.10) that

$$\begin{aligned}\|S_t^{(m)} - S_t^{(n)}\|_{L^2(\lambda\times\mathbb{P})} &= \|\sum_{i=n}^{m-1}(S_t^{(i+1)} - S_t^{(i)})\|_{L^2(\lambda\times\mathbb{P})} \\ &\le \sum_{i=n}^{m-1}\|(S_t^{(i+1)} - S_t^{(i)})\|_{L^2(\lambda\times\mathbb{P})} \\ &\le \sum_{i=n}^{m-1}(\int_0^T \frac{M^i t^{3i}}{i!}dt)^{1/2} \\ &\le \sum_{i=n}^{m-1}(\frac{M^i T^{3i}}{i!})^{1/2},\end{aligned} \tag{2.11}$$

where m and n are integers which satisfy $n < m$. Therefore, $\{S_t^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2(\lambda \times \mathbb{P})$. Consequently, $S_t := \lim_{n\to\infty} S_t^{(n)}$ exists in $L^2(\lambda \times \mathbb{P})$ and it satisfies (2.2) for $0 < t \le \tilde{s}$. Moreover, by the Lipschitz condition we imposed on $\alpha(\cdot)$ and $\beta(\cdot)$, the uniqueness of the solution is derived by the standard argument (for instance, we refer to Theorem 5.2.1 in [14]) and the Gronwall's inequality. □

Let M_t denote the martingale part of S_t,

$$M_t = \int_0^t \sigma\beta(\int_{-\tilde{s}}^t S_u\, p_t(du))dW_s.$$

We can easily check that

$$\|[M,M]_T^{1/2}\|_{L^2(\mathbb{P})}^2 < \infty. \tag{2.12}$$

Therefore, S_t becomes a $\mathcal{H}^2$-semimartingale with the canonical decomposition $S_t = M_t + R_t$ and M_t is a square-integrable martingale under $\mathbb{P}$.

3. The equivalent martingale measure

The original model (2.1) is incomplete, because the delay $\tilde{\tau}_t$ is an additional random source. On the other hand, since we replaced the effect of $\tilde{\tau}_t$ with a density function $p_t(dx)$ in (2.2), it is no longer incomplete. In this section, we find the equivalent martingale measure and study dynamics under it.

Let

$$X_t = \int_0^t \frac{\mu\alpha(\int_{-\tilde{s}}^s S_u\, p_s(du))}{\sigma\beta(\int_{-\tilde{s}}^s S_u\, p_s(du))}dW_s, \tag{3.1}$$

and assume that $E[e^{2X_t}] < \infty$ for every $t \le T$. The next theorem provides us the equivalent martingale measure $\mathbb{Q}$ the dynamic of S_t under it, and the new Brownian motion.

Theorem 3.1. *Let*

$$Z_t = 1 - \int_0^t Z_s dX_s. \tag{3.2}$$

Then, Z_t is a $\mathbb{P}$-martingale and the probability measure $\mathbb{Q}$ defined by $d\mathbb{Q} = Z_T d\mathbb{P}$ is the equivalent martingale measure of S.

Moreover, under the equivalent martingale measure $\mathbb{Q}$,

$$\widetilde{W}_t := W_t + \int_0^t \frac{\mu\alpha(\int_{-\tilde{s}}^s S_u\, p_s(du))}{\sigma\beta(\int_{-\tilde{s}}^s S_u\, p_s(du))} ds \tag{3.3}$$

is a Brownian motion. Thus S satisfies the SDE

$$dS_t = \sigma\beta(\int_{-\tilde{s}}^t S_u\, p_t(du)) d\widetilde{W}_t \tag{3.4}$$

under measure $\mathbb{Q}$.

Proof. By the Girsanov-Meyer theorem, $\mathbb{Q}$ is the equivalent martingale measure and

$$W_t - \int_0^t \frac{1}{Z_s} d\langle Z, W\rangle_s \tag{3.5}$$

is a $\mathbb{Q}$-local martingale. Thus, we get

$$\int_0^t \frac{1}{Z_s} d\langle Z, W\rangle_s = -\int_0^t \frac{\mu\alpha(\int_{-\tilde{s}}^s S_u\, p_s(du))}{\sigma\beta(\int_{-\tilde{s}}^s S_u\, p_s(du))} ds. \tag{3.6}$$

Moreover,

$$[\widetilde{W}, \widetilde{W}]_t = [W, W]_t$$

and $\widetilde{W}_t$ is continuous. Therefore, by the Lévy theorem, $\widetilde{W}_t$ is a Brownian motion under $\mathbb{Q}$. The new dynamic (3.4) follows directly by inserting $\widetilde{W}_t$ to (2.2). □

4. Markov property and the infinitesimal generator

Our model (2.1) has a delay effect in the underlying dynamic. This causes mathematical difficulties in pricing and hedging. The major reason is the non-Markov property of model. In other word, model (2.1) itself is not Markov. Consequently, it is hard to find a pricing PDE of financial derivatives under model (2.1). However, we can construct a vector process including S_t which is Markov under Assumption 2.1. For the simplicity, we consider the case, $\tilde{m} = 2$. General cases, $\tilde{m} \geq 3$, can be derived by adding vector processes and derivative terms. This is an often used technique in the Asian option pricing. Using this vector process instead, we can derive the pricing equation

Let us consider the vector process (S_t, A_t, X_t) where $A_t = \int_{-\tilde{s}}^t S_u\, f(t,u) du$ and $X_t := \int_{-\tilde{s}}^t S_u \frac{\partial f}{\partial t}(t,u) du$. We next show that (S_t, A_t, X_t) is indeed Markov. We start this with a Lemma.

Lemma 4.1. *For all $t > -\tilde{s}$, the next equation holds.*

$$A_t = A_0 + \int_0^t S_a f(a,a)da + \int_0^t \int_{-\tilde{s}}^a S_u \frac{\partial f}{\partial a}(a,u)duda \ \ a.s.. \tag{4.1}$$

Proof. For each fixed $\omega \in \Omega$, we obtain an SDE,

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{A_{t+\Delta t} - A_t}{\Delta t} &= \lim_{\Delta t \to 0} \Big(\frac{1}{\Delta t} \int_t^{t+\Delta t} S_u f(t,u)du \\ &\qquad + \int_{-\tilde{s}}^t S_u \frac{(f(t+\Delta t, u) - f(t,u))}{\Delta t} du \Big) \\ &= S_t f(t,t) + \int_{-\tilde{s}}^t S_u \frac{\partial f}{\partial t}(t,u)du. \end{aligned} \tag{4.2}$$

Therefore, we have

$$A_t = A_0 + \int_0^t S_a f(a,a)da + \int_0^t \int_{-\tilde{s}}^a S_u \frac{\partial f}{\partial a}(a,u)duda \ \ a.s.. \tag{4.3}$$

□

Recall that a vector process, under the risk-neutral measure, (S_t, A_t, X_t) follows the dynamic

$$\begin{aligned} dS_t &= \sigma\beta(A_t)d\tilde{W}_t, \\ dA_t &= S_t f(t,t)dt + X_t dt, \\ X_t &= \int_{-\tilde{s}}^t S_u \frac{\partial f}{\partial t}(t,u)du. \end{aligned} \tag{4.4}$$

Since the density function f satisfies PDE which is introduced by (2.4), $\frac{\partial f}{\partial t} = -\mathcal{L}_{\tau_t} f$, we have

$$\begin{aligned} dX_t &= S_t\Big(\gamma t \frac{\partial f}{\partial u}(t,t) - \frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)dt + \Big(\int_{-\tilde{s}}^t S_u \frac{\partial^2 f}{\partial t^2}(t,u)du\Big)dt \\ &= S_t\Big(\gamma t \frac{\partial f}{\partial u}(t,t) - \frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)dt - \Big(\int_{-\tilde{s}}^t S_u \frac{\partial(\mathcal{L}_{\tau_t} f)}{\partial t}(t,u)du\Big)dt \\ &= S_t\Big(\gamma t \frac{\partial f}{\partial u}(t,t) - \frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)dt - \Big(\int_{-\tilde{s}}^t S_u (\mathcal{L}_{\tau_t} \frac{\partial f}{\partial t}(t,u))du\Big)dt \\ &= S_t\Big(\gamma t \frac{\partial f}{\partial u}(t,t) - \frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)dt + \Big(\int_{-\tilde{s}}^t S_u (\mathcal{L}_{\tau_t}\mathcal{L}_{\tau_t} f(t,u))du\Big)dt \\ &= S_t\Big(\gamma t \frac{\partial f}{\partial u}(t,t) - \frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)dt + \Big(\int_{-\tilde{s}}^t S_u (\mathcal{L}^2_{\tau_t} f(t,u))du\Big)dt, \end{aligned} \tag{4.5}$$

where $\mathcal{L}^2_{\tau_t} f := \mathcal{L}_{\tau_t}(\mathcal{L}_{\tau_t} f)$. Under the condition $\frac{\partial^2 f}{\partial t^2} = 0$, the second term of the above equation becomes zero. In this case, our problem becomes a time-inhomogeneous SDE problem for (S_t, A_t, X_t). Also all parameters and functions do not require historical values but need only present values at time t.

In fact, the assumption $\frac{\partial^2 f}{\partial t^2} = 0$ is not strong at all, since the delay effect does not need to depend on time t in a very complicated way.

Under this assumption, the next theorem gives the Markov property.

Theorem 4.1. *Under the Assumption 2.1 with $\tilde{m} = 2$, the vector process (S_t, A_t, X_t) is Markov.*

Proof. By Lemma 4.1, A_t is generated by W_u for $0 < u < t$. Therefore (S_t, A_t, X_t) is adapted to the filtration $\mathcal{F}_t$. Moreover the pair (S_t, A_t, X_t) is the unique solution of SDEs (4.4) and (4.5) by Theorem 2.1. Now, we have

$$
\begin{aligned}
|X_{t_1} - X_{t_0}| &= |\int_{t_0}^{t_1} S_u \frac{\partial f}{\partial t}(t,u)du| \\
&\le \int_{t_0}^{t_1} |S_u \frac{\partial f}{\partial t}(t,u)|du \\
&\le C \int_{t_0}^{t_1} |S_u| du \\
&\le C S_u^\star |t_1 - t_0|,
\end{aligned} \tag{4.6}
$$

where

$$S_u^\star = \max_{[t_0, t_1]} S_u.$$

By the Doob's maximal inequality, X_t is Lipschitz function in $L^2(\mathbb{Q})$. Then, (S_t, A_t, X_t) is Markov by Theorem 32 in Chapter 5 of [15]. □

For a general case $\tilde{m}$, we take a vector process $X_t = (X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(\tilde{m}-1)})$ where $X_t^{(i)} := \int_{-\tilde{s}}^t S_u \frac{\partial^i f}{\partial t^i}(t,u)du = (-1)^i \int_{-\tilde{s}}^t S_u(\mathcal{L}_{\tau_t}^{(i)} f(t,u))du$ for $1 \le i \le (\tilde{m} - 1)$. Then $(S_t, A_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(\tilde{m}-1)})$ is also Markov and $dX_t^{(i)}$ is represented by $\mathcal{L}_{\tau_t}^{(i)} f(t,u)$ and $X_t^{(i+1)}$.

5. Option pricing and hedging

In the previous section, we studied the Markov property of the model. We next find the infinitesimal generator of it.

Recall that S_t satisfies the following SDDE on $\mathbb{Q}$:

$$
\begin{aligned}
dS_t &= \sigma\beta(\int_{-\tilde{s}}^t S_u \, p_t(du))d\tilde{W}_t, \quad 0 < t \le T < \infty, \\
S_t &= \phi(t), \quad -\tilde{s} \le t \le 0,
\end{aligned} \tag{5.1}
$$

where $\tilde{W}_t$ is a standard Brownian motion on $\mathbb{Q}$ and ϕ is a real valued continuous function on $[-\tilde{s}, t]$. Without loss of generality, we set $\tilde{s} = 1$ and $\tau_0 = 0$.

Now, we consider the case $(S_t, A_t, X_t) = (s, a, x)$ with path function $S_u = \phi(u)$ for $-1 \le u < t$, i.e., we already have observed historical data of S_s for $s < t$. Let P_e be the European option price $E^{\mathbb{Q}}[e^{-rT} h(S_T)|\mathcal{F}_t]$, where $h(\cdot)$ is the payoff function. By the Markov property of the vector process (S_t, A_t, X_t), we have $E^{\mathbb{Q}}[e^{-rT}h(S_T)|\mathcal{F}_t] = E^{\mathbb{Q}}[e^{-rT}h(S_T)|(S_t, A_t, X_t)]$. The next theorem gives us the infinitesimal generator $\mathcal{L}_{(S_t,A_t,X_t)}$ of (S_t, A_t, X_t). Precisely, $\mathcal{L}_{(S_t,A_t,X_t)}$ depends on t, however we abbreviate the notation $\mathcal{L}_{(S_t,A_t,X_t)}$ to $\mathcal{L}$ for convenience.

Theorem 5.1. *Recall that the infinitesimal generator* $\mathcal{L}_{\tau_t}$ *of* τ_t *is* $-\gamma\tau_t\frac{\partial}{\partial\tau} + \frac{1}{2}D(\tau_t)^2\frac{\partial^2}{\partial\tau^2}$. *We assume* $\tilde{P}_e(s,a,x) \in C_0^2(\mathbb{R}^3)$. *Under the Assumption 2.1 with* $\tilde{m} = 2$, *we can derive the infinitesimal generator* $\mathcal{L}$ *for* $(S_t, A_t, X_t) = (s, a, x)$,

$$\begin{aligned}\mathcal{L}\tilde{P}_e = &\frac{1}{2}\sigma^2\beta^2(a)\frac{\partial^2\tilde{P}_e}{\partial s^2} + sf(t,t)\frac{\partial\tilde{P}_e}{\partial a} + x\frac{\partial\tilde{P}_e}{\partial a}\\ &+ \Big(s\gamma t\frac{\partial f}{\partial u}(t,t) - s\frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)\frac{\partial\tilde{P}_e}{\partial x}.\end{aligned} \tag{5.2}$$

Proof. The semigroup $\mathcal{P}_T$ with respect to the vector precess (S_t, A_t, X_t) satisfies

$$\mathcal{P}_T\tilde{P}_e(s,a,x) = E^{\mathbb{Q}}[\tilde{P}_e(S_T, A_T, X_T) \mid S_t = s, A_t = a, X_t = x, (\phi(\rho),\ -1 \le \rho < t)]. \tag{5.3}$$

Recall that the generator $\mathcal{L}$ satisfies

$$\mathcal{L}\tilde{P}_e(s,a,x) = \lim_{T\to t}\frac{\mathcal{P}_T\tilde{P}_e(s,a,x) - \tilde{P}_e(s,a,x)}{T-t}. \tag{5.4}$$

Since (4.5) and $dA_tdS_t = dA_tdX_t = dX_tdS_t = 0$, it follows that

$$\begin{aligned}&E^{\mathbb{Q}}[d\tilde{P}_e(S_t, A_t, X_t)]\\ &= E^{\mathbb{Q}}[\frac{\partial\tilde{P}_e}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2\tilde{P}_e}{\partial s^2}(dS_t)^2 + \frac{\partial\tilde{P}_e}{\partial a}dA_t + \frac{\partial\tilde{P}_e}{\partial x}dX_t]\\ &= E^{\mathbb{Q}}[\frac{1}{2}\sigma^2\beta^2(\int_{-1}^t S_u\, f(t,u)du)\frac{\partial^2\tilde{P}_e}{\partial s^2}dt + S_tf(t,t)\frac{\partial\tilde{P}_e}{\partial a}dt\\ &\quad + \int_{-1}^t S_u\frac{\partial f}{\partial t}(t,u)du\frac{\partial\tilde{P}_e}{\partial a}dt + S_t\Big(\gamma t\frac{\partial f}{\partial u}(t,t) - \frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)\frac{\partial\tilde{P}_e}{\partial x}dt\\ &\quad + \int_{-1}^t \phi(u)(\mathcal{L}_{\tau_t}^2 f(t,u))du\frac{\partial\tilde{P}_e}{\partial x}dt].\end{aligned} \tag{5.5}$$

Therefore, we have

$$\begin{aligned}\mathcal{L}\tilde{P}_e(s,a,x) = &\lim_{T\to t}\frac{\mathcal{P}_T\tilde{P}_e - \tilde{P}_e}{T-t}\\ &= \frac{1}{2}\sigma^2\beta^2(a)\frac{\partial^2\tilde{P}_e}{\partial s^2} + sf(t,t)\frac{\partial\tilde{P}_e}{\partial a} + x\frac{\partial\tilde{P}_e}{\partial a}\\ &\quad + \Big(s\gamma t\frac{\partial f}{\partial u}(t,t) - s\frac{1}{2}D^2(t)\frac{\partial^2 f}{\partial u^2}(t,t)\Big)\frac{\partial\tilde{P}_e}{\partial x}.\end{aligned} \tag{5.6}$$

□

By Theorem 5.1 and the martingale representation theorem in [5], [6] and [7], we derive the partial differential equation (PDE) for the European option pricing at t in the following.

Theorem 5.2. *Let us denote the European option price* $P_e(t,s,a,x;\phi(\rho),\ -1\leq\rho<t)\in C_0^2(\mathbb{R}^+\times\mathbb{R}^3)$. *Then* $P_e(t,s,a,x)$ *satisfies that*

$$\begin{aligned}&\frac{\partial P_e}{\partial t}+\mathcal{L}P_e=0,\\&P_e(T,s,a,x)=h(s),\end{aligned}\tag{5.7}$$

where $f(t,u)$ *is a solution of the followings,*

$$\begin{aligned}&\frac{\partial f}{\partial t}(t,u)+\mathcal{L}_{\tau_t}f(t,u)=0,\\&f(0,u)=\delta_{\tau_0}(u).\end{aligned}\tag{5.8}$$

Proof. By Theorem 5.1 and the martingale representation theorem, the European option price is decomposed into three terms, dt, dB_t and dW_t. Since the European option price is a martingale, dt term should be zero. Therefore, it follows that $P_e(t,s,a,x)$ with zero spot rate of interest satisfies the above PDE. Refer to [14]. Moreover, probability density function $p_t(u)$ satisfies the Kolmogorov backward equation of the process τ_t. □

Let us consider a European style contingent claim $H:=h(S_T)\in L^2(\mathbb{Q})$ and define $V_t=E^{\mathbb{Q}}[H|\mathcal{F}_t]$. The price of H is given by $V_t=P_e(t,S_t,A_t,X_t)$ where P_e is the solution in Theorem 5.2. By the self-financing condition of a hedging strategy ξ_s^H, we have

$$V_t=V_0+\int_0^t\xi_s^H dS_s,\tag{5.9}$$

and

$$\xi^H=\frac{d\langle V,S\rangle}{d\langle S,S\rangle}.\tag{5.10}$$

Now, we assume that $P_e\in C^{1,2,2,2}([0,T]\times R^3)$. The next theorem shows that our hedging strategy is in fact a delta hedging.

Theorem 5.3. ξ^H *is represented by*

$$\xi_t^H=\frac{\partial P_e}{\partial S_t}.\tag{5.11}$$

Proof. By (5.10), we need to calculate $d\langle P_e,S\rangle_t$. By the functional martingale representation theorem, $P_e(t,S_t,A_t,X_t)$ satisfies that

$$\begin{aligned}&P_e(t,S_t,A_t,X_t)-P_e(0,S_0,A_0,X_0)\\&=\int_0^t\mathcal{L}P_e(0,S_0,A_0,X_0)du+\int_0^t\sigma\frac{\partial P_e}{\partial s}\beta(A_0)d\tilde{W}_s.\end{aligned}\tag{5.12}$$

This leads to

$$d\langle P_e, S\rangle_t = \sigma^2 \frac{\partial P_e}{\partial s}\beta^2(A_0)dt. \tag{5.13}$$

By (5.10) and (5.1), we finally get

$$\xi_t^H = \frac{\partial P_e}{\partial S_t}. \tag{5.14}$$

□

6. Conclusion

We have studied option pricing and hedging in the presence of the short term model dependency through a stochastic delay effect. The short term dependency is captured either through the stopping time or the backward time integration. We studied the case when the dependency is captured by the backward time integration. Our model is not a risk neutral one for generality. Therefore, we found the equivalent martingale measure, and the corresponding pricing and hedging formulae.

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