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ESTIMATES FOR RIESZ TRANSFORMS ASSOCIATED WITH SCHRÖDINGER TYPE OPERATORS

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Abstract. Let $\mathcal{L}_2 = (-\Delta)^2 + V^2$ be the Schrödinger type operator, where nonnegative potential V belongs to the reverse Hölder class RH_s, $s > n/2$. In this paper, we consider the operator $T_{\alpha,\beta} = V^{2\alpha}\mathcal{L}_2^{-\beta}$ and its conjugate $T^*_{\alpha,\beta}$, where $0 < \alpha \leq \beta \leq 1$. We establish the (L^p, L^q)-boundedness of operator $T_{\alpha,\beta}$ and $T^*_{\alpha,\beta}$, respectively, we also show that $T_{\alpha,\beta}$ is bounded from Hardy type space $H^1_{\mathcal{L}_2}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$ and $T^*_{\alpha,\beta}$ is bounded from $L^{p_1}(\mathbb{R}^n)$ into BMO type space $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$, where $p_1 = \frac{n}{4(\beta-\alpha)}$, $p_2 = \frac{n}{n-4(\beta-\alpha)}$.

1. Introduction

For $1 < s < \infty$, a nonnegative locally L^s-integrable function V is said to belong to RH_s if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^s dy\right)^{1/s} \leq \frac{C}{|B|}\int_B V(y)dy$$

holds for every ball $B \subset \mathbb{R}^n$.

Consider the Schrödinger type operators

$$\mathcal{L}_j = (-\Delta)^j + V^j \ (j = 1, 2) \ \text{ on } \ \mathbb{R}^n, \ n \geq 2j + 1,$$

where Δ is the Laplacian on $\mathbb{R}^n$ and the potential V belongs to the reverse Hölder class RH_s for $s > n/2$. Note that $\mathcal{L}_2$ is biharmonic operator when $V = 0$. The boundedness of the operator $V^\alpha\mathcal{L}_1^{-\beta}$ has been well studied in some spaces (see [7], [8], [9], [13]). In this paper, we concentrate on the boundedness of the operators $T_{\alpha,\beta} = V^{2\alpha}\mathcal{L}_2^{-\beta}$ and its conjugate $T^*_{\alpha,\beta} = \mathcal{L}_2^{-\beta}V^{2\alpha}$ for $0 < \alpha \leq \beta \leq 1$.

The operator $T_{\alpha,\beta}$ has been studied under the condition $V \in RH_s$ for $s > n/2$. Zhong [14] and Sugano [13] showed the L^p-boundedness of $T_{1,1}$, Chen et

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al. [2] established the L^p-boundedness of $T_{1/2,1/2}$. In this paper, we establish the following (L^p, L^q)-boundedness.

Theorem 1.1. *Suppose $V \in RH_s$ for some $s > n/2$, $0 < \alpha \le \beta \le 1$.*

(i) *If $\left(\frac{s}{2\alpha}\right)' < p < \frac{n}{4(\beta-\alpha)}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)}{n}$, then*

$$\|T^*_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *If $1 < p < \frac{1}{\frac{2\alpha}{s} + \frac{4(\beta-\alpha)}{n}}$ and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)}{n}$, then*

$$\|T_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}.$$

Let us recall the concept of Hardy space related to Schrödinger type operator. The Schrödinger type operators $\mathcal{L}_j$ $(j = 1, 2)$ generate C_0 semigroups $\{e^{-t\mathcal{L}_j}\}_{t>0}$. The maximal functions with respect to the semigroups $\{e^{-t\mathcal{L}_j}\}_{t>0}$ are given by

$$M^{\mathcal{L}_j} f(x) = \sup_{t>0} |e^{-t\mathcal{L}_j} f(x)|.$$

By [1, 4], a function $f \in L^1(\mathbb{R}^n)$ is said to be in $H^1_{\mathcal{L}_j}(\mathbb{R}^n)$ if the semigroup maximal function $M^{\mathcal{L}_j} f$ belongs to $L^1(\mathbb{R}^n)$. The norms of such a function are defined by

$$\|f\|_{H^1_{\mathcal{L}_j}(\mathbb{R}^n)} = \|M^{\mathcal{L}_j} f\|_{L^1(\mathbb{R}^n)}, \quad j = 1, 2.$$

Theorem 1.1 in [1] showed that $H^1_{\mathcal{L}_2}(\mathbb{R}^n) = H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ with equivalent norms.

We consider the boundedness of $T_{\alpha,\beta}$ at the endpoint $p = 1$, and get the following result.

Theorem 1.2. *Suppose $V \in RH_s$ for some $s > n/2$, $0 < \alpha < \beta \le 1$. Then,*

$$\|T_{\alpha,\beta}(f)\|_{L^{p_2}(\mathbb{R}^n)} \le C\|f\|_{H^1_{\mathcal{L}_2}(\mathbb{R}^n)},$$

where $p_2 = \frac{n}{n-4(\beta-\alpha)}$.

The dual space of $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ is the BMO type space $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ (see [3]). Let f be a locally function on $\mathbb{R}^n$ and $B = B(x, r)$. Set $f_B = \frac{1}{|B|}\int_B f(y)dy$ and $f(B, V) = f_B$ if $r < \rho(x)$; $f(B, V) = 0$ if $r \ge \rho(x)$. We say $f \in BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ if

$$\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f(y) - f(B, V)|dy < \infty.$$

It follows from [3] that $\|f\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)}$ is actually a norm which makes $BMO_{\mathcal{L}_1}(\mathbb{R}^n)$ a Banach space. Due to $H^1(\mathbb{R}^n) \subset H^1_{\mathcal{L}_1}(\mathbb{R}^n)$, we conclude by duality that $BMO_{\mathcal{L}_1}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

By Theorem 1.2 and duality, we get:

Corollary 1.3. *Suppose $V \in RH_s$ for $s > \frac{n}{2}$. Let $0 < \alpha < \beta \le 1$. Then*

$$\|T^*_{\alpha,\beta}(f)\|_{BMO_{\mathcal{L}_1}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)},$$

where $p_1 = \frac{n}{4(\beta-\alpha)}$.

2. Some preliminaries

As in [12], for a given potential $V \in RH_s$ with $s \geq n/2$, we define the auxiliary function

$$\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

We recall some important properties concerning the auxiliary function which will play an important role to obtain the main results. Throughout this section we always assume $V \in RH_s$ with $s > n/2$.

Lemma 2.1 ([12]). *For $0 < r < R < \infty$, we have*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left(\frac{R}{r} \right)^{n/s-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy.$$

Lemma 2.2 ([5]). *There exists a constant $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C \left(1 + \frac{r}{\rho(x)} \right)^{l_0}.$$

Lemma 2.3 ([12]). *There exists $k_0 \geq 1$ such that*

$$C^{-1}\rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq C\rho(x) \left(1 + \frac{|x-y|}{\rho(x)} \right)^{\frac{k_0}{1+k_0}}$$

for all $x, y \in \mathbb{R}^n$.

It is easy to get the following result from Lemma 2.3.

Lemma 2.4. *Let $k \in \mathbb{N}$ and $x \in 2^{k+1}B(x_0, r) \setminus 2^k B(x_0, r)$. Then we have*

$$\frac{1}{\left(1 + \frac{2^k r}{\rho(x)} \right)^N} \leq \frac{C}{\left(1 + \frac{2^k r}{\rho(x_0)} \right)^{N/(k_0+1)}}.$$

A ball $B(x_0, \rho(x_0))$ is called critical. Assume that $Q = B(x_0, \rho(x_0))$, for $x \in Q$, Lemma 2.3 tell us that $\rho(x) \sim \rho(y)$ if $|x-y| < C\rho(x)$.

We give the estimates of fundamental solutions. Denote $\Gamma_{\mathcal{L}_2}(x, y, \lambda)$ by the fundamental solution of $\mathcal{L}_2 + \lambda$, where $\lambda \in [0, \infty)$. When $\lambda = 0$ it follows from Theorem 2 in [13] that for any positive integer N there exists a positive constant C_N such that

$$0 \leq \Gamma_{\mathcal{L}_2}(x, y, 0) \doteq \Gamma_{\mathcal{L}_2}(x, y) \leq \frac{C_N}{\left(1 + \frac{|x-y|}{\rho(x)} \right)^N} \frac{1}{|x-y|^{n-4}}.$$

When $\lambda \neq 0$, we have the following result [10].

Lemma 2.5. *Let $0 < \lambda < \infty$. For any positive integer N, there exists a positive constant C_N such that*

$$0 \le \Gamma_{\mathcal{L}_2}(x,y,\lambda) \le \frac{C_N}{\left(1+\lambda^{\frac{1}{2}}|x-y|^2\right)^N \left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-4}}.$$

Lemma 2.6 ([11])**.** *Assume that $(\mathcal{L}_2+\lambda)u = 0$ in $B(x_0, 2R)$ for some $x_0 \in \mathbb{R}^n$. Then there exists a k_0' such that*

$$\left(\int_{B(x_0,R)} |\nabla u|^t dx\right)^{1/t} \le CR^{2n/s-4}\left(1+\frac{R}{\rho(x_0)}\right)^{k_0'} \sup_{B(x_0,2R)} |u|,$$

where $1/t = 2/s - 3/n$.

Let K_β be the kernel of operator $\mathcal{L}_2^{-\beta}$. We have the following estimates for $K_\beta(x,y)$.

Lemma 2.7. *Suppose $V \in RH_s$ for $s > \frac{n}{2}, 0 < \beta \le 1$.*

(i) *For every $N > 0$, there exists a constant C such that*

$$|K_\beta(x,y)| \le \frac{C}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-4\beta}};$$

(ii) *For every $N > 0$, and any $0 < h < |x-y|/16$, we have*

$$|K_\beta(x,y+h) - K_\beta(x,y)| \le \frac{C}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{h^\delta}{|x-y|^{n-4\beta+\delta}},$$

where $\delta = 4 - 2n/s$.

Proof. (i) When $\beta = 1$, we have

$$K_\beta(x,y) = \Gamma_{\mathcal{L}_2}(x,y,0) \le \frac{C_N}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-4}}.$$

For $0 < \beta < 1$, by the functional calculus, we may write

$$\mathcal{L}_2^{-\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{-\beta}(\mathcal{L}_2+\lambda)^{-1} d\lambda.$$

Let $f \in C_0^\infty$. From $(\mathcal{L}_2+\lambda)^{-1} f(x) = \int_{\mathbb{R}^n} \Gamma_{\mathcal{L}_2}(x,y,\lambda) f(y) dy$, it follows that

$$\mathcal{L}_2^{-\beta}(f)(x) = \int_{\mathbb{R}^n} K_\beta(x,y) f(y) dy,$$

where

$$K_\beta(x,y) = \frac{\sin \pi\beta}{\pi} \int_0^\infty \lambda^{-\beta} \Gamma_{\mathcal{L}_2}(x,y,\lambda) d\lambda.$$

Note that

$$\int_0^\infty \lambda^{-\beta} \frac{1}{\left(1+\lambda^{\frac{1}{2}}|x-y|^2\right)^N} d\lambda \leq C|x-y|^{4\beta-4}$$

for $0<\beta<1$. So, by Lemma 2.5 we get

$$|K_\beta(x,y)| \leq \frac{C}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-4\beta}}.$$

(ii) For $\beta=1$, by Lemma 3.3 in [2] we get (ii). When $0<\beta<1$, by the functional calculus, we have

$$K_\beta(x,y+h)-K_\beta(x,y) = \frac{\sin\beta\pi}{\pi} \int_0^\infty \lambda^{-\beta}\big(\Gamma_{\mathcal{L}_2}(x,y+h,\lambda)-\Gamma_{\mathcal{L}_2}(x,y,\lambda)\big)d\lambda.$$

We need to estimate $|\Gamma_{\mathcal{L}_2}(x,y+h,\lambda)-\Gamma_{\mathcal{L}_2}(x,y,\lambda)|$ in advance.

Fix $x,y\in\mathbb{R}^n$ and let $R=|x-y|/8, 1/t=1/s-3/n$. It follows from the Morrey embedding theorem (see [6]) and Lemma 2.6 that

$$\begin{aligned}
&|\Gamma_{\mathcal{L}_2}(x,y+h,\lambda)-\Gamma_{\mathcal{L}_2}(x,y,\lambda)| \\
&\leq Ch^{1-n/t}\left(\int_{B(y,R)} |\nabla_y \Gamma_{\mathcal{L}_2}(x,z,\lambda)|^t du\right)^{1/t} \\
&\leq C\left(\frac{h}{R}\right)^{1-n/t}\left(1+\frac{R}{\rho(y)}\right)^{k_0'} \sup_{z\in B(y,2R)} \Gamma_{\mathcal{L}_2}(x,z,\lambda) \\
&\leq C\left(\frac{h}{R}\right)^{1-n/t}\left(1+\frac{R}{\rho(x)}\right)^{k_0''} \sup_{z\in B(y,2R)} \Gamma_{\mathcal{L}_2}(x,z,\lambda).
\end{aligned}$$

Note $z\in B(y,2R), |x-y|=8R$, then $6R\leq|z-x|\leq 10R$. By Lemma 2.5 we get

$$\begin{aligned}
\Gamma_{\mathcal{L}_2}(x,z,\lambda) &\leq \frac{C}{\left(1+\lambda^{\frac{1}{2}}|z-x|^2\right)^N\left(1+\frac{|z-x|}{\rho(x)}\right)^N} \frac{1}{|z-x|^{n-4}} \\
&\leq \frac{C}{\left(1+\lambda^{\frac{1}{2}}R^2\right)^N\left(1+\frac{R}{\rho(x)}\right)^N} \frac{1}{R^{n-4}}.
\end{aligned}$$

Then

$$\begin{aligned}
&|\Gamma_{\mathcal{L}_2}(x,y+h,\lambda)-\Gamma_{\mathcal{L}_2}(x,y,\lambda)| \\
&\leq \frac{C}{\left(1+\lambda^{\frac{1}{2}}R^2\right)^N\left(1+\frac{R}{\rho(x)}\right)^N} \frac{h^\delta}{R^{n-4+\delta}} \\
&\leq \frac{Ch^\delta}{\left(1+\lambda^{\frac{1}{2}}|x-y|^2\right)^N\left(1+\frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-4+\delta}}.
\end{aligned}$$

So

$$\begin{aligned} &|K_\beta(x,y+h)-K_\beta(x,y)| \\ &\le C\int_0^\infty \lambda^{-\beta}\big|\Gamma_{\mathcal{L}_2}(x,y,\lambda)-\Gamma_{\mathcal{L}_2}(x,x_0,\lambda)\big|d\lambda \\ &\le \frac{C}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\frac{h^\delta}{|x-y|^{n-4+\delta}}\int_0^\infty \lambda^{-\beta}\frac{1}{\left(1+\lambda^{\frac12}|x-y|^2\right)^N}d\lambda \\ &\le \frac{C}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\frac{h^\delta}{|x-y|^{n-4\beta+\delta}}. \end{aligned}$$

□

Let $f\in L^q_{loc}(\mathbb{R}^n)$. Denote $|B|$ by the Lebesgue measure of the ball $B\subset\mathbb{R}^n$. The fractional Hardy-Littlewood maximal function $M_{\sigma,\gamma}$ is defined by

$$M_{\sigma,\gamma}(f)(x)=\sup\left(\frac{1}{|B|^{1-\frac{\sigma\gamma}{n}}}\int_B|f(y)|^\gamma dy\right)^{1/\gamma}.$$

Lemma 2.8. *Suppose* $1<\gamma<p<\frac{n}{\sigma}$, *and* $\frac1q=\frac1p-\frac{\sigma}{n}$. *Then*

$$\|M_{\sigma,\gamma}f\|_{L^q(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}.$$

3. (L^p,L^q)-estimates

In this section, we prove Theorem 1.1. By Lemma 2.5,

$$T^*_{\alpha,\beta}f(x)=\int_{\mathbb{R}^n}K_\beta(y,x)V(y)^{2\alpha}f(y)dy,$$

where

$$|K_\beta(y,x)|\le\frac{C_N}{\left(1+\frac{|x-y|}{\rho(y)}\right)^N}\frac{1}{|x-y|^{n-4\beta}}.$$

We will prove part (i) and part (ii) follows by duality.

Let $r=\rho(x)$. Then by Lemma 2.4 we have

$$\begin{aligned} |T^*_{\alpha,\beta}f(x)|=&\int_{\mathbb{R}^n}|K_\beta(y,x)|V(y)^{2\alpha}|f(y)|dy \\ &\le C\int_{\mathbb{R}^n}\frac{V(y)^{2\alpha}}{(1+\frac{|x-y|}{\rho(y)})^N}\frac{|f(y)|}{|x-y|^{n-4\beta}}dy \\ &\le C\sum_{k=-\infty}^{\infty}\frac{(2^kr)^{4\beta}}{(1+2^k)^{N/(k_0+1)}}\frac{1}{(2^kr)^n}\int_{|x-y|\le 2^kr}V(y)^{2\alpha}|f(y)|dy. \end{aligned}$$

By Hölder inequality and $V\in RH_s$, $s>n/2$, we have

$$\frac{1}{(2^kr)^n}\int_{|x-y|\le 2^kr}V(y)^{2\alpha}|f(y)|dy$$

$$\leq C\left(\frac{1}{(2^k r)^n}\int_{|x-y|\leq 2^k r} V(y)dy\right)^{2\alpha}$$

$$\times\left(\frac{1}{(2^k r)^n}\int_{|x-y|\leq 2^k r} |f(y)|^{(\frac{s}{2\alpha})'}dy\right)^{1/(\frac{s}{2\alpha})'}.$$

For $k \geq 1$, by Lemma 2.2, there exists $l_0 > 0$ such that

$$\left(\frac{1}{(2^k r)^n}\int_{|x-y|\leq 2^k r} V(y)dy\right)^{2\alpha}$$

$$\leq C(2^k r)^{-4\alpha}\left(\frac{1}{(2^k r)^{n-2}}\int_{|x-y|\leq 2^k r} V(y)dy\right)^{2\alpha}$$

$$\leq C(2^k r)^{-4\alpha}(1+2^k)^{2l_0\alpha} \leq C(2^k r)^{-4\alpha}2^{2l_0\alpha k}.$$

For $k \leq 0$, by Lemma 2.1, we have

$$\left(\frac{1}{(2^k r)^n}\int_{|x-y|\leq 2^k r} V(y)dy\right)^{2\alpha}$$

$$\leq C(2^k r)^{-4\alpha}\left(\frac{r}{2^k r}\right)^{(n/s-2)2\alpha}\left(\frac{1}{r^{n-2}}\int_{|x-y|\leq r} V(y)dy\right)^{2\alpha}$$

$$\leq C(2^k r)^{-4\alpha}2^{2\alpha k(2-n/s)}.$$

Taking $N > 2(k_0+1)l_0\alpha$, then

$$|T^*_{\alpha,\beta}f(x)| \leq C\left(\frac{1}{(2^k r)^{n-4(\beta-\alpha)(\frac{s}{2\alpha})'}}\int_{|x-y|\leq 2^k r} |f(y)|^{(\frac{s}{2\alpha})'}dy\right)^{1/(\frac{s}{2\alpha})'}$$

$$\times\left(\sum_{k=1}^{\infty}\frac{1}{2^{k(\frac{N}{k_0+1}-2l_0\alpha)}} + \sum_{k=-\infty}^{0} 2^{2\alpha k(2-n/s)}\right)$$

$$\leq CM_{4(\beta-\alpha),(\frac{s}{2\alpha})'}(f)(x).$$

By Lemma 2.6, we get

$$\|T^*_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

where $\left(\frac{s}{2\alpha}\right)' < p < \frac{n}{4(\beta-\alpha)}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)}{n}$. This finishes the proof of (i).

By duality, we get

$$\|T_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $\frac{n}{n-4(\beta-\alpha)} < q < \frac{s}{2\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)}{n}$. These conditions are equivalent to

$$1 < p < \frac{1}{\frac{2\alpha}{s}+\frac{4(\beta-\alpha)}{n}} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{4(\beta-\alpha)}{n}.$$

This completes the proof of Theorem 1.1.

4. $(H^1_{\mathcal{L}_2}, L^{p_2})$-estimates

Because of $H^1_{\mathcal{L}_2}(\mathbb{R}^n) = H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ and their norms are equivalent, we only need to show that the operator $T_{\alpha,\beta}$ maps the Hardy space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ continuously into $L^{p_2}(\mathbb{R}^n)$, where $p_2 = \frac{n}{n-4(\beta-\alpha)}$. Firstly, we review the concept of $(1,q)_\rho$-atom.

Let $1 < q \le \infty$. A measurable function a is called a $(1,q)_\rho$-atom associated with the ball $B(x,r)$ if $r < \rho(x)$ and the following conditions hold:

(i) supp $a \subset B(x,r)$ for some $x \in \mathbb{R}^n$ and $r > 0$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(x,r)|^{1/q-1}$;

(iii) if $r < \rho(x)/4$, then $\int_{B(x,r)} a(x)dx = 0$.

By [4], the Hardy space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ admits the following atomic decomposition:

Lemma 4.1. *$f \in H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $(1,q)_\rho$-atoms and $\sum_j |\lambda_j| < \infty$. Moreover*

$$\|f\|_{H^1_{\mathcal{L}_1}} \sim \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum is taken over all atomic decompositions of f into $H^1_{\mathcal{L}_1}$-atoms.

We choose q_1 and q_2 such that

$$1 < q_1 < \frac{1}{\frac{2\alpha}{s} + \frac{4(\beta-\alpha)}{n}}$$

and

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{4(\beta-\alpha)}{n}.$$

By Lemma 4.1, we only need to prove

$$\|T_{\alpha,\beta}a\|_{L^{p_2}(\mathbb{R}^n)} \le C$$

holds for any $(1,q_1)_\rho$-atom a, where $C > 0$ is independent of a.

Assume that supp$a \subset B(x_0,r), r < \rho(x_0)$. Then

$$\|T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \le \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} + \|\chi_{(16B)^c}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} = I_1 + I_2.$$

By Hölder inequality, Theorem 1.1 and $\frac{1}{p_2} = 1 - \frac{4(\beta-\alpha)}{n}$, we have

$$\begin{aligned} I_1 &= \|\chi_{16B}T_{\alpha,\beta}(a)\|_{L^{p_2}(\mathbb{R}^n)} \\ &\le C|16B|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_{\mathbb{R}^n}|T_{\alpha,\beta}a(x)|^{q_2}dx\right)^{1/q_2} \\ &\le C|16B|^{\frac{1}{p_2}-\frac{1}{q_2}}\left(\int_B|a(x)|^{q_1}dx\right)^{1/q_1} \\ &\le C|16B|^{\frac{1}{p_2}-\frac{1}{q_2}}|B|^{\frac{1}{q_1}-1} \le C. \end{aligned}$$

We divided into two case for the estimate of I_2: $r \geq \rho(x_0)/4$ and $r < \rho(x_0)/4$.

Case I: $r \geq \rho(x_0)/4$. In this case, we have $r \sim \rho(x_0)$. Let $E_k = 2^k B \setminus 2^{k-1}B$. By Lemma 2.4 and Lemma 2.5, we get

$$\begin{aligned}
I_2 = & \left(\int_{(16B)^c} |T_{\alpha,\beta} a(x)|^{p_2} dx \right)^{1/p_2} \\
\leq & C \left(\sum_{k=5}^{\infty} \int_{E_k} V(x)^{2\alpha p_2} \left(\int_B |K_\beta(x,y) a(y)| dy \right)^{p_2} dx \right)^{1/p_2} \\
\leq & C \left(\sum_{k=5}^{\infty} \int_{E_k} V(x)^{2\alpha p_2} \left(\int_B \frac{|a(y)| dy}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N |x-y|^{n-4\alpha}} \right)^{p_2} dx \right)^{1/p_2} \\
\leq & C \left(\sum_{k=5}^{\infty} \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{Np_2/(k_0+1)}} \frac{(2^k r)^{4\beta p_2}}{(2^k r)^{np_2}} \int_{2^k B} V(x)^{2\alpha p_2} dx \right)^{1/p_2} \\
& \times \int_B |a(y)| dy.
\end{aligned}$$

Notice

$$p_2 = \frac{n}{n - 4(\beta - \alpha)} < \frac{n}{4\alpha} < \frac{s}{2\alpha}.$$

By Hölder inequality and $V \in RH_s$, we have

$$\begin{aligned}
& \frac{1}{|2^k B|} \int_{2^k B} V(x)^{2\alpha p_2} dx \\
\leq & C \left(\frac{1}{|2^k B|} \int_{2^k B} V(x)^s dx \right)^{2\alpha p_2/s} \\
\leq & C \left(\frac{1}{|2^k B|} \int_{2^k B} V(x) dx \right)^{2\alpha p_2}.
\end{aligned}$$

Then, by Lemma 2.2 we get

$$\frac{1}{|2^k B|} \int_{2^k B} V(x)^{2\alpha p_2} dx \leq C (2^k r)^{-4\alpha p_2} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{2l_0 \alpha p_2}.$$

Due to a is a $(1, q_1)_\rho$-atom, by Hölder inequality we get

$$\int_B |a(y)| dy \leq C |B|^{1 - \frac{1}{q_1}} \left(\int_B |a(y)|^{q_1} dy \right)^{1/q_1} \leq C.$$

Then, we have

$$I_2 \leq C \left(\sum_{k=5}^{\infty} \frac{(2^k r)^{n - p_2(n - 4(\beta - \alpha))}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\frac{Np_2}{k_0+1} - 2l_0 \alpha p_2}} \right)^{1/p_2}.$$

Noticing $r \sim \rho(x_0)$, and $p_2 = \frac{n}{n-4(\beta-\alpha)}$, we get

$$I_2 \le C\left(\sum_{k=5}^{\infty} \frac{1}{(2^k)^{\frac{Np_2}{k_0+1}-2l_0\alpha p_2}}\right)^{1/p_2}.$$

Taking $\frac{N}{k_0+1} - 2\alpha l_0 > 0$, we obtain $I_2 \le C$.

Case II: $r < \rho(x_0)/4$. By the vanishing condition of a, Lemma 4.1 and Lemma 2.4 we have

$$\begin{aligned} I_2 &\le C\left(\int_{(16B)^c} V(x)^{2\alpha p_2}\left(\int_B |(K_\beta(x,y) - K_\beta(x,x_0))a(y)|dy\right)^{p_2} dx\right)^{1/p_2} \\ &\le C\left(\sum_{k=5}^{\infty}\int_{E_k} V(x)^{2\alpha p_2}\left(\int_B \frac{|a(y)|}{\left(1+\frac{|x-y|}{\rho(x)}\right)^N}\frac{|y-x_0|^\delta}{|x-y|^{n+\delta-4\beta}}dy\right)^{p_2} dx\right)^{1/p_2} \\ &\le C\left(\sum_{k=5}^{\infty}\frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{\frac{Np_2}{k_0+1}}}\frac{r^{\delta p_2}}{(2^k r)^{(n+\delta-4\beta)p_2}}\int_{2^kB} V(x)^{2\alpha p_2}dx\right)^{1/p_2} \\ &\quad \times \int_B |a(y)|dy. \end{aligned}$$

Note that

$$\frac{1}{|2^kB|}\int_{2^kB} V(x)^{2\alpha p_2}dx \le C(2^k r)^{-4\alpha p_2}\left(1+\frac{2^k r}{\rho(x_0)}\right)^{2l_0\alpha p_2},$$

$$\int_B |a(y)|dy \le C,$$

$p_2 = \frac{n}{n-4(\beta-\alpha)}$, and $\delta > 0$. Then, taking $N \ge 2l_0\alpha(k_0+1)$, we get

$$\begin{aligned} I_2 &\le C\left(\sum_{k=5}^{\infty}\frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{\frac{Np_2}{k_0+1}-2l_0\alpha p_2}}\frac{1}{(2^k r)^{(n-4(\beta-\alpha))p_2-n}}\frac{r^{\delta p_2}}{(2^k r)^{\delta p_2}}\right)^{1/p_2} \\ &\le C\left(\sum_{k=5}^{\infty}\frac{1}{2^{k\delta p_2}}\right)^{1/p_2} \le C. \end{aligned}$$

This completes the proof of Theorem 1.2.

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