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A NOTE ON GENERALIZED PARAMETRIC MARCINKIEWICZ INTEGRALS

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Abstract. In the present paper, we establish certain L^p bounds for the generalized parametric Marcinkiewicz integral operators associated to surfaces generated by polynomial compound mappings with rough kernels in Grafakos-Stefanov class $\mathcal{F}_\beta(\mathrm{S}^{n-1})$. Our main results improve and generalize a result given by Al-Qassem, Cheng and Pan in 2012. As applications, the corresponding results for the generalized parametric Marcinkiewicz integral operators related to the Littlewood-Paley g_λ^*-functions and area integrals are also presented.

1. Introduction

During the last several years, a considerable amount of attention has been given to study the L^p bounds for the generalized parametric Marcinkiewicz integrals with various kinds of kernels (see for example, [1,2,6,10,14,27], among others). In this paper, we aim to establish some new results concerning this topic. To be precise, we will establish certain L^p bounds for the generalized parametric Marcinkiewicz integral operators associated to surfaces generated by polynomial compound mappings with rough kernels in Grafakos-Stefanov class. We point out that our main results greatly improve and generalize some known ones.

Throughout this paper, let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. For $y \in \mathbb{R}^n \setminus \{0\}$, we set $y' = y/|y|$. Let $\Gamma_{P,\varphi} = \{P(\varphi(|y|))y'; y \in \mathbb{R}^n\}$ be the surfaces generated by a continuous function $\varphi : [0, \infty) \to \mathbb{R}$ and a real polynomial P on $\mathbb{R}$ satisfying $P(0) = 0$. Assume that $\Omega \in L^1(\mathrm{S}^{n-1})$ is a homogeneous function of degree zero and satisfies

$$\int_{\mathrm{S}^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1}$$

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For $1 < q < \infty$ and $\rho = \varsigma + i\tau$ ($\varsigma, \tau \in \mathbb{R}$ with $\varsigma > 0$), we define the generalized parametric Marcinkiewicz integral operator $\mu^q_{\Omega,P,\varphi,\rho}$ along $\Gamma_{P,\varphi}$ by

$$(2) \qquad \mu^q_{\Omega,P,\varphi,\rho} f(x) = \Big(\int_0^\infty \Big| \frac{1}{t^\rho} \int_{|y|\le t} f(x - P(\varphi(|y|))y') \frac{\Omega(y')}{|y|^{n-\rho}} dy \Big|^q \frac{dt}{t} \Big)^{1/q},$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions. For the sake of simplicity, we denote $\mu^q_{\Omega,P,\varphi,\rho} = \mu^q_{\Omega,\rho}$ if $\varphi(t) = t$ and $P(t) = t$ and $\mu^q_{\Omega,P,\varphi,\rho} = \mu^q_{\Omega,P}$ if $\varphi(t) = t$ and $\rho = 1$. When $q = 2$ and $\rho = 1$, we write $\mu^q_{\Omega,\rho} = \mu_\Omega$. The operator μ_Ω is just the classical well-known Marcinkiewicz integral operator, which was first introduced and studied by Stein [28] who observed that μ_Ω is of type (p,p) $(1 < p \le 2)$ if $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for $0 < \alpha \le 1$. In 1960, Benedek, Calderón and Panzone [4] extended Stein's result to the case $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ and $1 < p < \infty$. Later on, the above results were improved greatly by many authors under much weaker conditions on Ω. For example, see [7] for the case $\Omega \in H^1(\mathrm{S}^{n-1})$ (the Hardy space on S^{n-1}), [3] for the case $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$, [9] for the case $\Omega \in B_r^{(0,-1/2)}(\mathrm{S}^{n-1})$ (the block space generated by r-blocks), [5] for the case $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ (the Grafakos-Stefanov class). For $q = 2$ and $\rho \neq 1$, the operator $\mu^q_{\Omega,\rho}$ is the classical parametric Marcinkiewicz integral operator $\mu_{\Omega,\rho}$. The L^p bounds for $\mu_{\Omega,\rho}$ with real (resp., complex) number ρ was first studied by Hörmander [13] (resp., Sakamoto and Yabuta [27]). Readers may consult [8, 15–20, 22–26, 30, 31] for their development and other extensions. For the Grafakos-Stefanov class and the operator $\mu^q_{\Omega,P}$, Chen, Fan and Pan [5] first proved that $\mu^2_{\Omega,P}$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (2\beta/(2\beta-1), 2\beta)$, provided that $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$. Recently, Wu [30] improved and extended the main result of [5] to the following.

Theorem A ([30])**.** *Let P be a real polynomial on $\mathbb{R}$ of $\deg(P) = N$ and satisfy $P(0) = 0$. Let $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1/2$ and satisfy* (1)*. Then*

$$\|\mu^2_{\Omega,P} f\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}$$

for all $p \in (1 + 1/(2\beta), 1 + 2\beta)$. The constant $C > 0$ is independent of the coefficients of P_N.

Recall that $\mathcal{F}_\beta(\mathrm{S}^{n-1})$ for $\beta > 0$, which is called the Grafakos-Stefanov class, is defined by

$$\mathcal{F}_\beta(\mathrm{S}^{n-1}) := \Big\{ \Omega \in L^1(\mathrm{S}^{n-1}) : \sup_{\xi \in \mathrm{S}^{n-1}} \int_{\mathrm{S}^{n-1}} |\Omega(y')| \log^\beta \frac{2}{|\xi \cdot y'|} d\sigma(y') < \infty \Big\},$$

which was introduced by Grafakos and Stefanov [12] in the study of L^p bounds for rough singular integrals. Clearly, it was pointed out in [12] that $\mathcal{F}_{\beta_1}(\mathrm{S}^{n-1}) \subset \mathcal{F}_{\beta_2}(\mathrm{S}^{n-1})$ for $0 < \beta_2 < \beta_1$, $\bigcup_{q>1} L^q(\mathrm{S}^{n-1}) \subset \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for $\beta > 0$, which are proper inclusions. Moreover, $\bigcap_{\beta>1} \mathcal{F}_\beta(\mathrm{S}^{n-1})$ is not included in $L(\log^+ L)(\mathrm{S}^{n-1})$.

When $\rho = 1$, we denote $\mu^q_{\Omega,\rho}$ by μ^q_Ω. In 2002, Chen, Fan and Ying [6] first introduced the operator μ^q_Ω and proved that μ^q_Ω is bounded from the homogeneous Triebel-Lizorkin space $\dot{F}^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p, q < \infty$, provided that $\Omega \in L^q(\mathrm{S}^{n-1})$ for some $q > 1$. Later on, the above result was improved greatly by many authors under much weaker conditions on Ω. For example, see [2, 14, 27] for the case $\Omega \in L(\log^+ L)(\mathrm{S}^{n-1})$, [2, 10] for the case $\Omega \in L(\log^+ L)^\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$, [2] for the case $\Omega \in B_r^{(0,1/q-1)}(\mathrm{S}^{n-1})$, [1] for the case $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$.

The main result of [1] is introduced as follows.

Theorem B ([1]). *Let $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$ and satisfy* (1). *Then*

$$\|\mu^q_\Omega f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $p \in (2\beta/(2\beta-1), 2\beta)$ and $q \in (2\beta/(2\beta-1), 2\beta)$.

By the fact that $\dot{F}^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$, Theorem A directly implies $\|\mu^q_{\Omega,P} f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$ for all $p \in (1+1/(2\beta), 1+2\beta)$ and $q = 2$ under the condition that $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1/2$. Compared with Theorems A and B, a natural question is the following.

Question 1.1. Is the operator $\mu^q_{\Omega,P}$ bounded from $\dot{F}^0_{p,q}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $p \in (1+1/(2\beta), 1+2\beta)$ and $q \in (1+1/(2\beta), 1+2\beta)$ under the condition that $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1/2$?

This is the main motivation of this paper. Our investigation will not only address this problem, but also deal with a more general class of operators. More precisely, we shall establish the following result.

Theorem 1.2. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0) = 0$ and $\varphi \in \mathfrak{F}$. Here $\mathfrak{F}$ is the set of all functions ϕ satisfying the following conditions*:

(a) *ϕ is a positive increasing $\mathcal{C}^1((0,\infty))$ function such that $t^\delta\phi'(t)$ is monotonic on $\mathbb{R}_+$ for some $\delta \in \mathbb{R}$;*

(b) *there exist C_ϕ, $c_\phi > 0$ such that $t\phi'(t) \ge C_\phi\phi(t)$ and $\phi(2t) \le c_\phi\phi(t)$ for all $t > 0$.*

Assume that $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1/2$ and satisfies (1). *Then*

$$\|\mu^q_{\Omega,P,\varphi,\rho} f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}$$

for $p \in (1+1/(2\beta), 1+2\beta)$ and $q \in (1+1/(2\beta), 1+2\beta)$. Here the constant $C_p > 0$ is independent of the coefficients of P, but may depend on $p, q, n, \varphi, \rho, N$.

Remark 1.3. (i) For the class $\mathfrak{F}$, there are some model examples such as t^α $(\alpha > 0)$, $t^\beta \ln(1+t)$ $(\beta \ge 1)$, $t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. Note that there exists $B_\varphi > 1$ such that $\varphi(2t) \ge B_\varphi\varphi(t)$ for any $\varphi \in \mathfrak{F}$ (see [15]).

(ii) Comparing Theorem B with Theorem 1.2, the range of β is extended to the case $\beta > 1/2$. Since $\mathcal{F}_{\beta_1}(\mathrm{S}^{n-1}) \subset \mathcal{F}_{\beta_2}(\mathrm{S}^{n-1})$ for $0 < \beta_2 < \beta_1$, which are proper inclusions. On the other hand, the range of (p,q) in Theorem 1.2

is larger than that of Theorem B. Thus, Theorem 1.2 improves and generalizes greatly the main result in [1], even in the special case $\rho = 1$, $\varphi(t) \equiv t$ and $P(t) \equiv t$.

(iii) Theorem 1.2 improves and generalizes greatly the main result in [6] since $\bigcup_{q>1} L^q(\mathrm{S}^{n-1}) \subset \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for $\beta > 0$, which are proper inclusions.

(iv) When $\rho = 1$, $\varphi(t) \equiv t$ and $q = 2$, Theorem 1.2 implies Theorem A by the fact that $\dot{F}^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Thus, Theorem 1.2 generalizes very much the main result in [30].

(v) Due to $\mathcal{F}_{\beta_1}(\mathrm{S}^{n-1}) \subset \mathcal{F}_{\beta_2}(\mathrm{S}^{n-1})$ for $0 < \beta_2 < \beta_1$ and $\dot{F}^0_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$, we see that Theorem 1.2 improves and generalizes greatly the result in [5].

As applications of Theorem 1.2, we consider the corresponding parametric Marcinkiewicz integral operators $\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho}$ and $\mathfrak{M}^{q}_{\Omega,P,\varphi,\rho,S}$ related to the Littlewood-Paley g^*_λ-function and the area integral S, respectively, which are defined by

$$\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho}f(x) := \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} \times \Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{\Omega(y')}{|y|^{n-\rho}}f(x-P(\varphi(|y|))y')dy\Big|^q\frac{dydt}{t^{n+1}}\Big)^{1/q},$$

where $\lambda > 0$ and $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$;

$$\mathfrak{M}^{q}_{\Omega,P,\varphi,\rho,S}f(x) := \Big(\iint_{\Gamma(x)}\Big|\frac{1}{t^\rho}\int_{|y|\le t}\frac{\Omega(y')}{|y|^{n-\rho}}f(x-P(\varphi(|y|))y')dy\Big|^q\frac{dydt}{t^{n+1}}\Big)^{1/q},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : \ |x-y| < t\}$ and Ω, P, φ, ρ are given as in (2).

As applications of Theorem 1.2, we obtain:

Theorem 1.4. *Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0) = 0$ and $\varphi \in \mathfrak{F}$. Let $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1/2$ and satisfy (1). Then for $q \in (1 + 1/(2\beta), 1 + 2\beta)$ and $p \in [q, 1 + 2\beta)$, there exists a constant $C > 0$ such that*

$$\|\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho}f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.$$

Here the constant $C > 0$ is independent of the coefficients of P, but may depend on $p, q, n, \lambda, \varphi, \rho, N$. The same result holds for $\mathfrak{M}^{q}_{\Omega,P,\varphi,\rho,S}$.

The paper is organized as follows. In Section 2 we recall the definition of the homogeneous Triebel-Lizorkin spaces and present a well-known characterization of homogeneous Triebel-Lizorkin spaces. In Section 3, we establish two vector-valued inequalities for some measures and a Littliewood-Paley type inequality, which play key roles in the proof of Theorem 1.2. The proofs of Theorems 1.2 and 1.4 will be given in Section 4. We would like to remark that the main method employed in this paper is a combination of ideas and arguments from [15], [21], [31], [32].

Throughout the paper, for any $p \in (1,\infty]$ we let p' denote the conjugate index of p which satisfies $1/p + 1/p' = 1$ (here we set $\infty' = 1$). The letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables.

2. Homogeneous Triebel-Lizorkin spaces

Let $\mathcal{S}'(\mathbb{R}^n)$ be the tempered distribution class on $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and $0 < p,\, q \le \infty\, (p \ne \infty)$, the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\},$$

where $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfies the conditions: $0 \le \phi(x) \le 1$; $\mathrm{supp}(\phi) \subset \{x \in \mathbb{R}^n : 1/2 \le |x| \le 2\}$; $\phi(x) \ge c > 0$ if $3/5 \le |x| \le 5/3$; $\sum_{j\in\mathbb{Z}} \phi(2^j\xi) = 1$ for $\xi \ne 0$. It is well-known that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and the following hold:

(1) $\dot{F}^{0}_{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$;

(2) $(\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n))^* = \dot{F}^{-\alpha}_{p',q'}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and $1 < p,\, q < \infty$;

(3) $\dot{F}^{\alpha}_{p,q_1}(\mathbb{R}^n) \subset \dot{F}^{\alpha}_{p,q_2}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $q_1 \le q_2$.

Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence such that $\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} \ge a > 1$. Let $\eta_0 \in \mathcal{C}^\infty(\mathbb{R})$ be an even function satisfying $0 \le \eta_0(t) \le 1$, $\eta_0(0) = 1$ and $\eta_0(t) = 0$ for $|t| \ge 1$. Set $\eta(\xi) = 1$ for $|\xi| \le 1$, $\eta(\xi) = \eta_0(\frac{|\xi|-1}{a-1})$, where $a > 1$. Then, η satisfies $\chi_{|\xi|\le 1}(\xi) \le \eta(\xi) \le \chi_{|\xi|\le a}(\xi)$ and $|\partial^\alpha \eta(\xi)| \le c_\alpha (a-1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, where c_α is independent of a. We define functions $\{\psi_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by $\psi_k(\xi) = \eta(a_{k+1}^{-1}\xi) - \eta(a_k^{-1}\xi)$. Then observe that

(i) $\mathrm{supp}(\psi_k) \subset \{a_k \le |\xi| \le a a_{k+1}\}$;

(ii) $\mathrm{supp}(\psi_k) \cap \mathrm{supp}(\psi_j) = \emptyset$ for $|j-k| \ge 2$;

(iii) $\sum_{k\in\mathbb{Z}} \psi_k(\xi) = 1$ for $\xi \in \mathbb{R}^n \backslash \{0\}$.

The following is a well-known characterization of homogeneous Triebel-Lizorkin spaces, which is one of the main ingredients of the proof of Theorem 1.2.

Lemma 2.1 ([32])**.** *Let Φ_k be defined on $\mathbb{R}^n$ by $\widehat{\Phi_k}(\xi) = \psi_k(\xi)$ and $\mathcal{A}_n$ denote the set of all polynomials on $\mathbb{R}^n$. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with $1 < a \le \frac{a_{k+1}}{a_k} \le b$ for all $k \in \mathbb{Z}$. For $\alpha \in \mathbb{R}$, $1 < p,\, q < \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)/\mathcal{A}_n$, we define the norm $\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)}$ by*

$$\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)} = \Big\| \Big(\sum_{k\in\mathbb{Z}} a_k^{\alpha q} |\Phi_k * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

Then $\|f\|_{\dot{F}^{\alpha}_{p,q}(\{\Phi_k\}_{k\in\mathbb{Z}},\mathbb{R}^n)}$ is equivalent to $\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}$ for $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$.

3. Some vector-valued inequalities

In this section we shall establish some vector-valued inequalities, which play key roles in the proof of Theorem 1.2. Let Ω, ρ be given as in (2) and $\Gamma : \mathbb{R}^n \to \mathbb{R}^d$ $(d \geq 1)$ be a suitable mapping. Define the family of measures $\{\sigma_{\Omega,\Gamma,k,t}\}$ on $\mathbb{R}^d$ by

$$\widehat{\sigma_{\Omega,\Gamma,k,t}}(\xi) = \frac{1}{(2^k t)^\rho} \int_{2^{k-1}t<|y|\leq 2^k t} e^{-2\pi i \xi\cdot\Gamma(y)} \frac{\Omega(y')}{|y|^{n-\rho}} dy. \tag{3}$$

The related maximal operator $\sigma^*_{\Omega,\Gamma}$ is defined by

$$\sigma^*_{\Omega,\Gamma}(f)(x) = \sup_{k\in\mathbb{Z}} \sup_{t>0} \big||\sigma_{\Omega,\Gamma,k,t}| * f(x)\big|,$$

where $|\sigma_{\Omega,\Gamma,k,t}|$ is defined in the same way as $\sigma_{\Omega,\Gamma,k,t}$, but with Ω replaced by $|\Omega|$.

Lemma 3.1. *Let $\Omega \in L^1(\mathrm{S}^{n-1})$ and $\Gamma(y) = \mathcal{P}(\varphi(|y|)y')$, where $\varphi \in \mathfrak{F}$ and $\mathcal{P} = (P_1, P_2, \ldots, P_d)$ with each P_j being a real-valued polynomial on $\mathbb{R}^n$. Then*

$$\|\sigma^*_{\Omega,\Gamma}(f)\|_{L^p(\mathbb{R}^d)} \leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}$$

for $1 < p < \infty$. The constant $C > 0$ is independent of Ω and the coefficients of $\{P_j\}_{j=1}^d$, but depends on φ and p.

Proof. By the change of variable and Hölder's inequality, one has

$$\begin{aligned}
&\big||\sigma_{\Omega,\Gamma,k,t}| * f(x)\big| \\
&\leq \int_{2^{k-1}t}^{2^k t} \int_{\mathrm{S}^{n-1}} |\Omega(y')||f(x - \mathcal{P}(\varphi(r)y'))| d\sigma(y') \frac{dr}{r} \\
&\leq \|\Omega\|^{1/\gamma}_{L^1(\mathrm{S}^{n-1})} \\
&\quad \times \Big(\int_{\mathrm{S}^{n-1}} |\Omega(y')| \int_{2^{k-1}t}^{2^k t} |f(x - \mathcal{P}(\varphi(r)y'))|^{\gamma'} \frac{dr}{r} d\sigma(y') \Big)^{1/\gamma'},
\end{aligned}$$

which yields

$$\begin{aligned}
&\sigma^*_{\Omega,\Gamma}(f)(x) \leq \|\Omega\|^{1/\gamma}_{L^1(\mathrm{S}^{n-1})} \\
&\quad \times \Big(\int_{\mathrm{S}^{n-1}} |\Omega(y')| \Big(\sup_{t>0} \int_{t/2}^{t} |f(x - \mathcal{P}(\varphi(r)y'))|^{\gamma'} \frac{dr}{r} \Big) d\sigma(y') \Big)^{1/\gamma'}.
\end{aligned} \tag{4}$$

Using a change of variable and the properties of φ, one can obtain

$$\begin{aligned}
&\int_{t/2}^{t} |f(x - \mathcal{P}(\varphi(r)y'))|^{\gamma'} \frac{dr}{r} \\
&= \int_{\varphi(t/2)}^{\varphi(t)} |f(x - \mathcal{P}(ry'))|^{\gamma'} \frac{dr}{\varphi^{-1}(r)\varphi'(\varphi^{-1}(r))} \\
&\leq \frac{1}{C_\varphi} \int_{\varphi(t/2)}^{\varphi(t)} |f(x - \mathcal{P}(ry'))|^{\gamma'} \frac{dr}{r}
\end{aligned}$$

$$\leq \frac{c_\varphi}{C_\varphi}\frac{1}{\varphi(t)}\int_0^{\varphi(t)} |f(x-\mathcal{P}(ry'))|^{\gamma'}dr.$$

It follows that

$$\sup_{t>0}\int_{t/2}^{t} |f(x-\mathcal{P}(\varphi(r)y'))|^{\gamma'}\frac{dr}{r} \leq C(\varphi)\sup_{t>0}\frac{1}{t}\int_0^t |f(x-\mathcal{P}(ry'))|^{\gamma'}dr. \tag{5}$$

By pp. 476–478 in [29], we obtain that there exists $C>0$ independent of the coefficients of $\{P_j\}_{j=1}^d$ such that

$$\Big\|\sup_{t>0}\frac{1}{t}\int_0^t |f(x-\mathcal{P}(ry'))|dr\Big\|_{L^p(\mathbb{R}^d)} \leq C_p\|f\|_{L^p(\mathbb{R}^d)} \tag{6}$$

for $1<p\leq\infty$. We get from (5) and (6) that

$$\Big\|\sup_{t>0}\int_{t/2}^{t} |f(x-\mathcal{P}(\varphi(r)y'))|^{\gamma'}\frac{dr}{r}\Big\|_{L^p(\mathbb{R}^d)} \leq C(\varphi,p)\|f\|_{L^p(\mathbb{R}^d)}$$

for $1<p\leq\infty$. The constant $C(\varphi,p)>0$ is independent of the coefficients of $\{P_j\}_{j=1}^d$. This together with (4) and Minkowski's inequality implies that

$$\|\sigma^*_{h,\Omega,\Gamma}(f)\|_{L^p(\mathbb{R}^d)} \leq C(\varphi,p)\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}$$

for $1<p<\infty$. The constant $C(\varphi,p)>0$ is independent of Ω and the coefficients of $\{P_j\}_{j=1}^d$. This proves Lemma 3.1. □

Applying Lemma 3.1, we have:

Lemma 3.2. *Let $\Omega\in L^1(\mathrm{S}^{n-1})$ and $\Gamma(y)=\mathcal{P}(\varphi(|y|)y')$, where $\varphi\in\mathfrak{F}$ and $\mathcal{P}=(P_1,P_2,\dots,P_d)$ with each P_j being a real-valued polynomial on $\mathbb{R}^n$. Then*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\sigma_{\Omega,\Gamma,k,t}*g_k|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq C_1\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}; \tag{7}$$

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,\Gamma,k,t}*g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq C_2\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{8}$$

hold for $1<p,q<\infty$ and any $t\in[1,2]$. The above constants C_1, $C_2>0$ are independent of Ω and the coefficients of $\{P_j\}_{j=1}^d$. Moreover, C_2 is also independent of t.

Proof. We first prove (7). Let $1<p<\infty$. By duality, there exists a nonnegative function $f\in L^{p'}(\mathbb{R}^d)$ with $\|f\|_{L^{p'}(\mathbb{R}^d)}=1$ such that

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}}\int_1^2 |\sigma_{\Omega,\Gamma,k,t}*g_k|dt\Big\|_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}\int_1^2 |\sigma_{\Omega,\Gamma,k,t}*g_k(x)|dtf(x)dx\\ &\leq \int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|g_k(x)|\sigma^*_{\Omega,\Gamma}(\tilde{f})(-x)dx.\end{aligned} \tag{9}$$

Applying Lemma 3.1 and Hölder's inequality, (9) leads to

$$\Big\|\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{\Omega,\Gamma,k,t}*g_k|dt\Big\|_{L^p(\mathbb{R}^d)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sum_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)}. \tag{10}$$

On the other hand, we get by Lemma 3.1 that

$$\begin{aligned}\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}|\sigma_{\Omega,\Gamma,k,t}*g_k|\Big\|_{L^p(\mathbb{R}^d)}&\leq\Big\|\sigma^*_{\Omega,\Gamma}\Big(\sup_{k\in\mathbb{Z}}|g_k|\Big)\Big\|_{L^p(\mathbb{R}^d)}\\&\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sup_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)}.\end{aligned} \tag{11}$$

Interpolation between (10) and (11) implies

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma_{\Omega,\Gamma,k,t}*g_k|^qdt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$

for $1<p,q<\infty$. This proves (7).

It remains to prove (8). Fix $t\in[1,2]$. Let $1<p<\infty$. By duality, there exists a nonnegative function $h\in L^{p'}(\mathbb{R}^d)$ with $\|h\|_{L^{p'}(\mathbb{R}^d)}=1$ such that

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,\Gamma,k,t}*g_k|\Big\|_{L^p(\mathbb{R}^d)}&=\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,\Gamma,k,t}*g_k(x)|h(x)dx\\&\leq\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}}|g_k(x)|\sigma^*_{\Omega,\Gamma}(\tilde{h})(-x)dx.\end{aligned} \tag{12}$$

Invoking Lemma 3.1, Hölder's inequality and (12), we obtain

$$\Big\|\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,\Gamma,k,t}*g_k|\Big\|_{L^p(\mathbb{R}^d)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sum_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)}. \tag{13}$$

On the other hand, by the argument similar to those used in deriving (11),

$$\Big\|\sup_{k\in\mathbb{Z}}|\sigma_{\Omega,\Gamma,k,t}*g_k|\Big\|_{L^p(\mathbb{R}^d)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\sup_{k\in\mathbb{Z}}|g_k|\Big\|_{L^p(\mathbb{R}^d)} \tag{14}$$

for $1<p<\infty$. Then (8) follows from the interpolation between (13) and (14). □

Lemma 3.3. *For each $k\in\mathbb{Z}$, define the multiplier operator S_k in $\mathbb{R}^n$ by $S_kf(x)=\Phi_k*f(x)$. Here Φ_k is defined as in Lemma 2.1. Let $1<q<\infty$.*

(i) *For $1<p<q$ and $1<r<p$, it holds that*

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^qdt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\leq C\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|g_{t,j,k}|^qdt\Big)^{1/q}\Big\|^r_{L^p(\mathbb{R}^n)}\Big)^{1/r}.\end{aligned} \tag{15}$$

(ii) *For $q<p<2$ and $1<r<p'$, it holds that*

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le C\Big(\sum_{j\in\mathbb{Z}}\Big(\int_1^2\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{t,j,k}|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)}dt\Big)^{r/q}\Big)^{1/r}.\end{aligned}\tag{16}$$

Proof. We shall prove this lemma by employing the idea in the proof of Proposition 3.1 in [31]. We notice that for each fixed $j\in\mathbb{Z}$ and any functions $\{h_{t,k}\}$,

$$\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}|S_{j-k}h_{t,k}|\Big\|_{L^p(\mathbb{R}^n)}\le\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]}|h_{t,k}|\Big\|_{L^p(\mathbb{R}^n)}\quad\text{for } 1<p<\infty;\tag{17}$$

$$\Big\|\sum_{k\in\mathbb{Z}}\int_1^2|S_{j-k}h_{t,k}|dt\Big\|_{L^1(\mathbb{R}^n)}\le C\Big\|\sum_{k\in\mathbb{Z}}\int_1^2|h_{t,k}|dt\Big\|_{L^1(\mathbb{R}^n)}.\tag{18}$$

Interpolation between (17) and (18) yields

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|S_{j-k}h_{t,k}|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|h_{t,k}|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\tag{19}$$

for all $1<p<q<\infty$. (19) and Minkowski's inequality imply

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|S_{j-k}g_{t,j,k}\Big|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le C\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|g_{t,j,k}|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\quad\text{for } 1<p<q<\infty.\end{aligned}\tag{20}$$

On the other hand, by the similar arguments as in getting (3.5) in [31], we can get

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^2 dt\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}\\&\le C\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|g_{t,j,k}|^2 dt\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}\Big)^{1/2}.\end{aligned}\tag{21}$$

Define the mapping $\hbar$ by

$$\hbar:\{g_{t,j,k}(x)\}_{j,k\in\mathbb{Z},t\in[1,2]}\to\Big\{\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}(x)\Big\}_{k\in\mathbb{Z},t\in[1,2]}.$$

It follows from (20) and (21) that $\hbar$ maps $\ell^1(L^p(\ell^q(L^q([1,2])),\mathbb{R}^n))$ into $L^p(\ell^q(L^q([1,2])),\mathbb{R}^n)$ for $1<p<q<\infty$, and maps $\ell^2(L^2(\ell^2(L^2([1,2])),\mathbb{R}^n))$ into $L^2(\ell^2(L^2([1,2])),\mathbb{R}^n)$. By interpolation we obtain that $\hbar$ is bounded from

$\ell^r(L^p(\ell^q(L^q([1,2])),\mathbb{R}^n))$ to $L^p(\ell^q(L^q([1,2])),\mathbb{R}^n)$ for $1<p<q$ and $1<r<p$. This gives (15).

It remains to prove (16). If follows from (21) that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^2dt\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}$$
$$\leq C\Big(\sum_{j\in\mathbb{Z}}\int_1^2\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{t,j,k}|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}dt\Big)^{1/2}.$$

This yields that $\hbar$ is bounded from $\ell^2(L^2(L^2(\ell^2,\mathbb{R}^n),[1,2]))$ to $L^2(\ell^2(L^2([1,2])),\mathbb{R}^n)$. On the other hand, by Minkowski's inequality and the Littlewood-Paley theory (see [11, Proposition 5.1.4]),

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k}g_{t,j,k}\Big|^qdt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\leq\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|S_{j-k}g_{t,j,k}\Big|^qdt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\leq\sum_{j\in\mathbb{Z}}\Big(\int_1^2\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j-k}g_{t,j,k}|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)}dt\Big)^{1/q}$$
$$\leq C\sum_{j\in\mathbb{Z}}\Big(\int_1^2\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{t,j,k}|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)}dt\Big)^{1/q}\quad\text{for } 1<q<p<\infty.$$

This yields that $\hbar$ is bounded from $\ell^1(L^q(L^p(\ell^q,\mathbb{R}^n),[1,2]))$ to $L^p(\ell^q(L^q([1,2])),\mathbb{R}^n)$ for $1<q<p<\infty$. Then (16) follows from the interpolation. □

4. Proofs of Theorems 1.2 and 1.4

This section is devoted to presenting the proofs of Theorems 1.2 and 1.4.

Proof of Theorem 1.2. Let P be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P(0)=0$. Without loss of generality we may assume that $P(t)=\sum_{i=1}^N b_it^i$ with each $b_i\neq 0$. Let $P_0(t)=0$ and $P_\lambda(t)=\sum_{i=1}^\lambda b_it^i$ for $\lambda\in\{1,2,\dots,N\}$. For $0\leq\lambda\leq N$, we define the family of measures $\{\sigma^\lambda_{k,t}\}$ by $\sigma^\lambda_{k,t}=\sigma_{\Omega,\Gamma,k,t}$ with $d=n$ and $\Gamma(y)=P_\lambda(\varphi(|y|))y'$. Here $\sigma_{\Omega,\Gamma,k,t}$ is defined as in (3). We first proved that for $1\leq\lambda\leq N$, the following are valid:

$$\widehat{\sigma^0_{k,t}}(\xi)=0; \tag{22}$$

$$|\widehat{\sigma^\lambda_{k,t}}(\xi)|\leq\|\Omega\|_{L^1(\mathrm{S}^{n-1})}; \tag{23}$$

$$|\widehat{\sigma^\lambda_{k,t}}(\xi)-\widehat{\sigma^{\lambda-1}_{k,t}}(\xi)|\leq\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\varphi(2^kt)^\lambda|b_\lambda\xi|; \tag{24}$$

$$|\widehat{\sigma^\lambda_{k,t}}(\xi)|\leq C(\log\varphi(2^kt)^\lambda|b_\lambda\xi|)^{-\beta}\quad\text{if } \varphi(2^kt)^\lambda|b_\lambda\xi|>1. \tag{25}$$

Here $C > 0$ is independent of the coefficients of P_N. By a change of variable,

$$\widehat{\sigma_{k,t}^{\lambda}}(\xi) = \frac{1}{(2^k t)^\rho} \int_{2^{k-1}t}^{2^k t} \int_{\mathrm{S}^{n-1}} e^{-2\pi i \xi \cdot P_\lambda(\varphi(r)) y'} \Omega(y') d\sigma(y') \frac{dr}{r^{1-\rho}}. \tag{26}$$

Then (22) follows easily from (26) and (1). (23) is obvious. We get easily from (26) and the property of φ that

$$|\widehat{\sigma_{k,t}^{\lambda}}(\xi) - \widehat{\sigma_{k,t}^{\lambda-1}}(\xi)| \le \|\Omega\|_{L^1(\mathrm{S}^{n-1})} \varphi(2^k t)^\lambda |b_\lambda \xi|.$$

This gives (24). On the other hand, by (23), Lemma 2.2 in [15] and the fact $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$,

$$\begin{aligned} |\widehat{\sigma_{k,t}^{\lambda}}(\xi)| &\le \int_{\mathrm{S}^{n-1}} |\Omega(y')| \Big| \frac{1}{(2^k t)^\rho} \int_{2^{k-1}t}^{2^k t} e^{-2\pi i P_\lambda(\varphi(r)) \xi \cdot y'} \frac{dr}{r^{1-\rho}} \Big| d\sigma(y') \\ &\le C \int_{\mathrm{S}^{n-1}} |\Omega(y')| \min\{1, |\varphi(2^k t)^\lambda b_\lambda \xi \cdot y'|^{-1/\lambda}\} d\sigma(y') \\ &\le C \int_{\mathrm{S}^{n-1}} |\Omega(y')| \frac{(\log e^{\beta\lambda} |\xi' \cdot y'|^{-1})^\beta}{(\log \varphi(2^k t)^\lambda |b_\lambda \xi|)^\beta} d\sigma(y') \\ &\le C (\log \varphi(2^k t)^\lambda |b_\lambda \xi|)^{-\beta}, \end{aligned} \tag{27}$$

whenever $\varphi(2^k t)^\lambda |b_\lambda \xi| > 1$. This proves (25). Here in the third inequality of (27) we have used the fact that $\frac{t}{(\log t)^\beta}$ is increasing in (e^β, ∞).

By Minkowski's inequality, we can write

$$\begin{aligned} \mu_{\Omega,P,\varphi,\rho}^q f(x) &= \Big(\int_0^\infty \Big| \sum_{k=-\infty}^{0} 2^{k\rho} \sigma_{k,t}^N * f(x) \Big|^q \frac{dt}{t} \Big)^{1/q} \\ &= \Big(\int_0^\infty \Big| \sum_{k=-\infty}^{0} 2^{k\rho} \sigma_{0,t}^N * f(x) \Big|^q \frac{dt}{t} \Big)^{1/q} \\ &\le \sum_{k=-\infty}^{0} 2^{k\varsigma} \Big(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{0,t}^N * f(x)|^q \frac{dt}{t} \Big)^{1/q} \\ &\le \frac{1}{1-2^{-\varsigma}} \Big(\sum_{k\in\mathbb{Z}} \int_1^2 |\sigma_{k,t}^N * f(x)|^q \frac{dt}{t} \Big)^{1/q} \\ &\le \frac{1}{1-2^{-\varsigma}} \Big(\sum_{k\in\mathbb{Z}} \int_1^2 |\sigma_{k,t}^N * f(x)|^q dt \Big)^{1/q}. \end{aligned} \tag{28}$$

Let ψ be a $\mathcal{C}_0^\infty(\mathbb{R})$ function such that $\psi(t) \equiv 1$ for $|t| \le 1/2$ and $\psi(t) \equiv 0$ for $|t| > 1$. For $1 \le \lambda \le N$ and $\xi \in \mathbb{R}^n$, we define the family of measures $\{\nu_{k,t}^\lambda\}$ by

$$\widehat{\nu_{k,t}^{\lambda}}(\xi) = \widehat{\sigma_{k,t}^{\lambda}}(\xi) \prod_{j=\lambda+1}^{N} \psi(\varphi(2^k t)^j |b_j \xi|) - \widehat{\sigma_{k,t}^{\lambda-1}}(\xi) \prod_{j=\lambda}^{N} \psi(\varphi(2^k t)^j |b_j \xi|). \tag{29}$$

From (22) we see that

$$\sigma_{k,t}^N=\sum_{\lambda=1}^N \nu_{k,t}^\lambda. \tag{30}$$

Here we use the convention $\Pi_{j\in\emptyset}a_j=1$. By (22)-(25) and (29), there exists $C>0$ independent of the coefficients of P_N such that for $1\le\lambda\le N$,

$$|\widehat{\nu_{k,t}^\lambda}(\xi)|\le C\min\{1,\varphi(2^kt)^\lambda|b_\lambda\xi|\}; \tag{31}$$

$$|\widehat{\nu_{k,t}^\lambda}(\xi)|\le C(\log\varphi(2^kt)^\lambda|b_\lambda\xi|)^{-\beta}\ \text{if}\ |\varphi(2^kt)^\lambda|b_\lambda\xi|>1. \tag{32}$$

We get from (28), (30) and Minkowski's inequality that

$$\mu_{\Omega,P,\varphi,\rho}^q f(x)\le C(\zeta)\sum_{\lambda=1}^N\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\nu_{k,t}^\lambda * f(x)|^q dt\Big)^{1/q}=:C(\zeta)\sum_{\lambda=1}^N\mathcal{D}_\lambda f(x), \tag{33}$$

where

$$\mathcal{D}_\lambda f(x)=\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\nu_{k,t}^\lambda * f(x)|^q dt\Big)^{1/q}.$$

Thus, to prove Theorem 1.2, it suffices to show that

$$\|\mathcal{D}_\lambda f\|_{L^p(\mathbb{R}^n)}\le C_p\|f\|_{\dot{F}_{p,q}^0(\mathbb{R}^n)} \tag{34}$$

for $p\in(1+1/(2\beta),1+2\beta)$ and $q\in(1+1/(2\beta),1+2\beta)$.

We now prove (34). For $1\le\lambda\le N$. Define the multiplier operator $S_{k,\lambda}$ in $\mathbb{R}^n$ by

$$S_{k,\lambda}f(x)=\Psi_{k,\lambda}*f(x),$$

where $\Psi_{k,\lambda}$ is defined by $\Psi_{k,\lambda}(\xi)=\Phi_k(\xi)$, where Φ_k is given as in Lemma 2.1 with $a_k=\varphi(2^{-k})^{-\lambda}|b_\lambda|^{-1}$. By the properties of φ we have

$$1<B_\varphi^\lambda\le\frac{a_{k+1}}{a_k}\le c_\varphi^\lambda\quad\forall k\in\mathbb{Z}.$$

This together with Lemma 2.1 yields that for $1\le\lambda\le N$ and $1<p,q<\infty$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_{k,\lambda}*f|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\sim\|f\|_{\dot{F}_{p,q}^0(\mathbb{R}^n)}. \tag{35}$$

By Minkowski's inequality and the definition of $\Psi_{k,\lambda}$, we can write

$$\mathcal{D}_\lambda f(x)=\Big(\sum_{k\in\mathbb{Z}}\int_1^2\Big|\sum_{j\in\mathbb{Z}}S_{j-k,\lambda}(\nu_{k,t}^\lambda*\Psi_{j-k,\lambda}*f)(x)\Big|^q\frac{dt}{t}\Big)^{1/q}. \tag{36}$$

We consider the following two cases:

Case 1. $q\in(1+1/(2\beta),1+2\beta)$ and $p\in(1+1/(2\beta),q)$. By (i) of Lemma 3.3 and (36), we can get

$$\|\mathcal{D}_\lambda f\|_{L^p(\mathbb{R}^n)}\le C\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\nu_{k,t}^\lambda*\Psi_{j-k,\lambda}*f(x)|^q dt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}^r\Big)^{1/r}$$

$$\leq C\Big(\sum_{j\in\mathbb{Z}}\|I_{\lambda,j,q}f\|^r_{L^p(\mathbb{R}^n)}\Big)^{1/r} \tag{37}$$

for $1<r<p$, where

$$I_{\lambda,j,q}f(x)=\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\nu^\lambda_{k,t}*\Psi_{j-k,\lambda}*f(x)|^q\frac{dt}{t}\Big)^{1/q}.$$

By (31)-(32) and Plancherel's theorem, we get that

$$\begin{aligned}\|I_{\lambda,j,q}f\|^2_{L^2(\mathbb{R}^n)}&=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\int_1^2|\nu^\lambda_{k,t}*\Psi_{j-k,\lambda}*f(x)|^2dtdx\\&\leq\sum_{k\in\mathbb{Z}}\int_{E_{j-k}}\int_1^2|\widehat{\nu^\lambda_{k,t}}(x)|^2dt|\hat{f}(x)|^2dx\\&\leq CB_j^2\sum_{k\in\mathbb{Z}}\int_{E_{j-k}}|\hat{f}(x)|^2dx\\&\leq CB_j^2\|f\|^2_{L^2(\mathbb{R}^n)},\end{aligned}\tag{38}$$

where $E_{j-k}=\{x\in\mathbb{R}^n:\varphi(2^{k-j+1})^{-\lambda}\leq|b_\lambda x|\leq\varphi(2^{k-j-1})^{-\lambda}\}$ and

$$B_j=|j|^{-\beta}\chi_{\{j\geq2\}}(j)+B_\varphi^{-|j|\lambda}\chi_{\{j\leq1\}}(j).$$

(38) together with the fact that $\dot{F}^0_{2,2}(\mathbb{R}^n)=L^2(\mathbb{R}^n)$ yields that

$$\|I_{\lambda,j,2}f\|_{L^2(\mathbb{R}^n)}\leq CB_j\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}. \tag{39}$$

On the other hand, by Lemma 3.2, we have that there exists $C>0$ independent of the coefficients of P_N such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\sigma^\lambda_{k,t}*g_k|^qdt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{40}$$

for $1<p<\infty$ and $1<q<\infty$. By (40) and the definition of $\nu_{\lambda,t}$, one can check that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\nu^\lambda_{k,t}*g_k|^qdt\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{41}$$

for $1<p<\infty$ and $1<q<\infty$. Combining (41) with (35) implies

$$\|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)}\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \tag{42}$$

for $1<p<\infty$ and $1<q<\infty$. By interpolation between (40) and (42), for $p\in(1+1/(2\beta),q)$, there exist $C>0$ and $\theta\in(2/(2\beta+1),1)$ such that

$$\|I_{\lambda,j,q}f\|_{L^p(\mathbb{R}^n)}\leq B_j^\theta\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}. \tag{43}$$

Fix $p \in (1+1/(2\beta), q)$, we can choose $1 < r < p$ such that $r\theta\beta > 1$. Thus, we get from (38) that for $p \in (1+1/(2\beta), q)$,

$$\|\mathcal{D}_\lambda f\|_{L^p(\mathbb{R}^n)} \le C\Big(\sum_{j\in\mathbb{Z}} B_j^{\theta r}\Big)^{1/r} \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.$$

This proves (34) for the case $q \in (1+1/(2\beta), 1+2\beta)$ and $p \in (1+1/(2\beta), q)$.

Case 2. $q \in (1+1/(2\beta), 1+2\beta)$ and $p \in (q, 1+2\beta)$. By (36) and (ii) of Lemma 3.3, we have

(44)

$$\|\mathcal{D}_\lambda f\|_{L^p(\mathbb{R}^n)} \le C\Big(\sum_{j\in\mathbb{Z}}\Big(\int_1^2 \Big\|\Big(\sum_{k\in\mathbb{Z}} |\nu_{k,t}^\lambda * \Psi_{j-k,\lambda} * f(x)|^q\Big)^{1/q}\Big\|^q_{L^p(\mathbb{R}^n)} dt\Big)^{r/q}\Big)^{1/r}$$

for any $r \in (1, p')$. For $t \in [1,2]$, let

$$J_{\lambda,j,q,t} f(x) = \Big(\sum_{k\in\mathbb{Z}} |\nu_{k,t}^\lambda * \Psi_{j-k,\lambda} * f(x)|^q\Big)^{1/q}.$$

We get from (44) that

$$\|\mathcal{D}_\lambda f\|_{L^p(\mathbb{R}^n)} \le C\Big(\sum_{j\in\mathbb{Z}}\Big(\int_1^2 \|J_{\lambda,j,q,t} f\|^q_{L^p(\mathbb{R}^n)} dt\Big)^{r/q}\Big)^{1/r}. \tag{45}$$

Fix $t \in [1,2]$. By Lemma 3.2 and the argument similar to those used to derive (41),

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |\nu_{k,t}^\lambda * g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_k|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \tag{46}$$

for $1 < p < \infty$ and $1 < q < \infty$. (46) together with (35) yields that

$$\|J_{\lambda,j,q,t} f\|_{L^p(\mathbb{R}^n)} \le C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \tag{47}$$

for $1 < p < \infty$ and $1 < q < \infty$. By the similar arguments as in getting (39),

$$\|J_{\lambda,j,2,t} f\|_{L^2(\mathbb{R}^n)} \le CB_j\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}. \tag{48}$$

Interpolation between (47) and (48) yields that for fixed $p \in (q, 1+2\beta)$, we can choose $r \in (1, p')$ and $\delta \in (2/(2\beta+1), 1)$ such that $r\delta\beta > 1$ and

$$\|J_{\lambda,j,q,t} f\|_{L^p(\mathbb{R}^n)} \le CB_j^\delta\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}. \tag{49}$$

Here $C > 0$ is independent of t and the coefficients of P. (49) together with (45) yields (34) for the case $q \in (1+1/(2\beta), 1+2\beta)$ and $p \in (q, 1+2\beta)$.

The case $p = q$ and $q \in (1+1/(2\beta), 1+2\beta)$ can be obtained by the interpolation between Case 1 and Case 2. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.4. By the argument similar to those used in deriving Lemma 4.2 in [21], we can obtain that for $\lambda > 1$ and $1 < q < \infty$, there exists a constant $C(n,\lambda) > 0$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} (\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho} f(x))^q g(x)dx \le C(n,\lambda) \int_{\mathbb{R}^n} (\mu^q_{\Omega,P,\varphi,\rho} f(x))^q Mg(x)dx, \tag{50}$$

where M is the usual Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Fix $1 < q \le p < \infty$, by duality, L^p bounds for M, Hölder's inequality and (50), we have

$$\begin{aligned}\|\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho} f\|^q_{L^p(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{(p/q)'}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} (\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho} f(x))^q g(x)dx \\ &\le C(n,\lambda) \sup_{\|g\|_{L^{(p/q)'}(\mathbb{R}^n)} \le 1} \int_{\mathbb{R}^n} (\mu^q_{\Omega,P,\varphi,\rho} f(x))^q Mg(x)dx \\ &\le C(n,\lambda,p,q) \|\mu^q_{\Omega,P,\varphi,\rho} f\|^q_{L^p(\mathbb{R}^n)},\end{aligned}$$

which together with Theorem 1.2 yields Theorem 1.4 for $\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho}$. On the other hand, one can easily check that

$$\mathfrak{M}^q_{\Omega,P,\varphi,\rho,S} f(x) \le 2^{n\lambda/q} \mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho} f(x).$$

This together with the bounds for $\mathfrak{M}^{\lambda,q,*}_{\Omega,P,\varphi,\rho}$ implies the bounds for $\mathfrak{M}^q_{\Omega,P,\varphi,\rho,S}$. □

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References

[1] H. Al-Qassem, L. Cheng, and Y. Pan, *Boundedness of rough integral operators on Triebel-Lizorkin spaces*, Publ. Mat. **56** (2012), no. 2, 261–277. https://doi.org/10.5565/PUBLMAT_56212_01

[2] ———, *On rough generalized parametric Marcinkiewicz integrals*, J. Math. Inequal. **11** (2017), no. 3, 763–780. https://doi.org/10.7153/jmi-2017-11-60

[3] A. Al-Salman, H. Al-Qassem, L. C. Cheng, and Y. Pan, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett. **9** (2002), no. 5-6, 697–700. https://doi.org/10.4310/MRL.2002.v9.n5.a11

[4] A. Benedek, A.-P. Calderón, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365. https://doi.org/10.1073/pnas.48.3.356

[5] J. Chen, D. Fan, and Y. Pan, *A note on a Marcinkiewicz integral operator*, Math. Nachr. **227** (2001), 33–42.

[6] ———, *Singular integral operators on function spaces*, J. Math. Anal. Appl. **276** (2002), no. 2, 691–708. https://doi.org/10.1016/S0022-247X(02)00419-5

[7] Y. Ding, D. Fan, and Y. Pan, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernels*, Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 4, 593–600. https://doi.org/10.1007/s101140000015

[8] ______, *On the L^p boundedness of Marcinkiewicz integrals*, Michigan Math. J. **50** (2002), no. 1, 17–26. https://doi.org/10.1307/mmj/1022636747

[9] Y. Ding, Q. Xue, and K. Yabuta, *Boundedness of the Marcinkiewicz integrals with rough kernel associated to surfaces*, Tohoku Math. J. (2) **62** (2010), no. 2, 233–262. https://doi.org/10.2748/tmj/1277298647

[10] D. Fan and H. Wu, *On the generalized Marcinkiewicz integral operators with rough kernels*, Canad. Math. Bull. **54** (2011), no. 1, 100–112. https://doi.org/10.4153/CMB-2010-085-3

[11] L. Grafakos, *Classical Fourier Analysis*, second edition, Graduate Texts in Mathematics, **249**, Springer, New York, 2008.

[12] L. Grafakos and A. Stefanov, *L^p bounds for singular integrals and maximal singular integrals with rough kernels*, Indiana Univ. Math. J. **47** (1998), no. 2, 455–470. https://doi.org/10.1512/iumj.1998.47.1521

[13] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140. https://doi.org/10.1007/BF02547187

[14] H. V. Le, *Singular integrals with mixed homogeneity in Triebel-Lizorkin spaces*, J. Math. Anal. Appl. **345** (2008), no. 2, 903–916. https://doi.org/10.1016/j.jmaa.2008.05.018

[15] F. Liu, *Integral operators of Marcinkiewicz type on Triebel-Lizorkin spaces*, Math. Nachr. **290** (2017), no. 1, 75–96. https://doi.org/10.1002/mana.201500374

[16] ______, *On the Triebel-Lizorkin space boundedness of Marcinkiewicz integrals along compound surfaces*, Math. Inequal. Appl. **20** (2017), no. 2, 515–535. https://doi.org/10.7153/mia-20-35

[17] ______, *A note of Littlewood-Paley functions on Triebel-Lizorkin spaces*, Bull. Korean Math. Soc. **55** (2018), no. 2, 659–672. https://doi.org/10.4134/BKMS.b170212

[18] ______, *A note on Marcinkiewicz integral associated to surfaces of revolution*, J. Aust. Math. Soc. **104** (2018), no. 3, 380–402. https://doi.org/10.1017/S1446788717000143

[19] ______, *Boundedness and continuity of maximal operators associated to polynomial compound curves on Triebel-Lizorkin spaces*, Math. Inequal. Appl. **22** (2019), no. 1, 25–44. https://doi.org/10.7153/mia-2019-22-02

[20] F. Liu, Z. Fu, and S. T. Jhang, *Boundedness and continuity of Marcinkiewicz integrals associated to homogeneous mappings on Triebel-Lizorkin spaces*, Front. Math. China **14** (2019), no. 1, 95–122. https://doi.org/10.1007/s11464-019-0742-3

[21] F. Liu, Z. Fu, Y. Zheng, and Q. Yuan, *A rough Marcinkiewicz integral along smooth curves*, J. Nonlinear Sci. Appl. **9** (2016), no. 6, 4450–4464. https://doi.org/10.22436/jnsa.009.06.84

[22] F. Liu and S. Mao, *L^p bounds for nonisotropic Marcinkiewicz integrals associated to surfaces*, J. Aust. Math. Soc. **99** (2015), no. 3, 380–398. https://doi.org/10.1017/S1446788715000191

[23] F. Liu and H. Wu, *L^p bounds for Marcinkiewicz integrals associated to homogeneous mappings*, Monatsh. Math. **181** (2016), no. 4, 875–906. https://doi.org/10.1007/s00605-016-0968-z

[24] F. Liu, H. Wu, and D. Zhang, *L^p bounds for parametric Marcinkiewicz integrals with mixed homogeneity*, Math. Inequal. Appl. **18** (2015), no. 2, 453–469. https://doi.org/10.7153/mia-18-34

[25] F. Liu, Q. Xue, and K. Yabuta, *Boundedness and continuity of maximal singular integrals and maximal functions on Triebel-Lizorkin spaces*, Sci. China Math. (to appear). doi:10.1007/s11425-017-9416-5.

[26] F. Liu and D. Zhang, *Parabolic Marcinkiewicz integrals associated to polynomials compound curves and extrapolation*, Bull. Korean Math. Soc. **52** (2015), no. 3, 771–788. https://doi.org/10.4134/BKMS.2015.52.3.771

[27] M. Sakamoto and K. Yabuta, *Boundedness of Marcinkiewicz functions*, Studia Math. **135** (1999), no. 2, 103–142.

[28] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466. https://doi.org/10.2307/1993226

[29] ______, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, **43**, Princeton University Press, Princeton, NJ, 1993.

[30] H. Wu, *On Marcinkiewicz integral operators with rough kernels*, Integral Equations Operator Theory **52** (2005), no. 2, 285–298. https://doi.org/10.1007/s00020-004-1339-z

[31] ______, *L^p bounds for Marcinkiewicz integrals associated to surfaces of revolution*, J. Math. Anal. Appl. **321** (2006), no. 2, 811–827. https://doi.org/10.1016/j.jmaa.2005.08.087

[32] K. Yabuta, *Triebel-Lizorkin space boundedness of Marcinkiewicz integrals associated to surfaces*, Appl. Math. J. Chinese Univ. Ser. B **30** (2015), no. 4, 418–446. https://doi.org/10.1007/s11766-015-3358-8

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