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ON THE PRODUCT OF QUASI-PARTIAL METRIC SPACES

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Abstract. This paper is mainly concerned with the existence and uniqueness of fixed points of $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, where X is a quasi- partial metric space and mapping f satisfies appropriate conditions. Results are also supported with relevant examples.

1. Introduction

Banach fixed point theorem [3] is a powerful tool which can be applied in the study of nonlinear phenomena. After presenting this principle, many authors have generalized this theorem in different directions, for example see [4, 6-10, 18-19] In 1965, S. Presic [19], extended Banach fixed point theorem to operators defined on product of metric spaces. In recent paper [18], Pacurar proved the convergence of a Presic type k-step iterative method for a new class of operators $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, satisfying a general Presic type contraction condition on metric spaces. Another interesting generalization is due to Matthews' extension of the Banach contraction principle from metric spaces to partial metric spaces [17]. Since then, several authors have studied fixed point theorems in partial metric spaces. See [2, 5, 11-17, 21] and the references there in. Huang et al. [13] defined the concept of expanding mapping in the setting of partial metric spaces and obtained some results for two mappings in

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partial metric spaces. The concept of a quasi-partial-metric space was introduced by Karapinar et al. [14]. Shahzad and Valero [20], presented a Nemytskii-Edelstein type fixed point theorem for self mappings in partial metric spaces.

THEOREM 1.1. [20] *Let (X, p) be a compact partial metric space. If f is a mapping from (X, p) into itself which is conjugate continuous and satisfies*

$$p(f(x), f(y)) < p(x, y)$$

for all $x, y \in X$ with $x \neq y$, then f has a unique fixed point.

Clearly, by the same method of the proof of Theorem 1.1, one can show that this theorem also holds for mapping $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, whenever (X, p) is a partial metric space.

Remember that a point x in a nonempty set X is a fixed point of function $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, if and only if it is a fixed point of $F : X \longrightarrow X$ defined by

$$F(x) = f(x, x, \dots, x)$$

for all $x \in X$.

In this paper, inspired and motivated by Shahzad and Valero [20], Huang et al. [13] and Karapinar et al. [14], we consider appropriate conditions for a class of mappings on product of quasi-partial metric spaces and stablish some fixed point results. In Section 2, some basic definitions and properties which will be used later in the paper are provided. In Section 3, main results on fixed point of mappings in the setting of partial metric spaces and product of quasi-partial metric spaces follows with a detailed proof. In order to certify the validity of the main results, we shall also includes some examples.

2. Preliminaries

We start by recalling some basic definitions and properties which will be used in this paper.

Let X be a nonempty set. A quasi metric on X is a function $q : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:,

(*i*) $q(x,y) = q(y,x) = 0 \Leftrightarrow x = y$.

(*ii*) $q(x,y) \leq q(x,z) + q(z,y)$.

Each quasi-metric q on X generates a T_0-topology $\tau(q)$ on X which has as a base, the family of open q-balls $\{B_q(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_q(x,\varepsilon) = \{y \in X : q(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that, the function $q^{-1} : X \times X \to \mathbb{R}^+$ defined by $q^{-1}(x,y) = q(y,x)$, known as conjugate quasi-metric of q, is a quasi-metric and function q^s on $X \times X$ defined by $q^s(x,y) = \max\{q(y,x), q(x,y)\}$ is a metric on X.

Example 2.1. [15] (a) The function $d_u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ given by $d_u(x,y) = max\{y - x, 0\}$, for all $x, y \in \mathbb{R}$, is a quasi-metric on $\mathbb{R}$. The topology, $\tau(d_u)$, is called upper topology on $\mathbb{R}$ and $(\mathbb{R}, d_u)$ is called upper quasi-metric space.

(b) The quasi-metric space $(\mathbb{R}, d_u^{-1})$ with $d_u^{-1}(x,y) = max\{x - y, 0\}$, for all $x, y \in \mathbb{R}$ is called the lower quasi-metric space.

Note that for any $x \in \mathbb{R}$ and $\varepsilon > 0$, $B_{d_u}(x,\varepsilon) = (-\infty, x + \varepsilon)$ and $B_{d_u^{-1}}(x,\varepsilon) = (x - \varepsilon, \infty)$. Therefore $\tau(d_u) \neq \tau(d_u^{-1})$.

Definition 2.2. [17] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in$ X;
(*i*) $p(x,x) = p(x,y) = p(y,y) \Leftrightarrow x = y$.
(*ii*) $p(x,x) \leq p(x,y)$.
(*iii*) $p(x,y) = p(y,x)$.
(*iv*) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. Clearly, a metric p on a set X is a partial metric such that $p(x,x) = 0$ for all $x \in X$. Each partial metric p on X generates a T_0-topology τ_p on X which has as a base, the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. The topological space (X, τ_p) is first countable. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X,p) converges to a point

$x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n,x_m)$, and a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Every partial metric p on X, induces the metric $p^s : X \times X \longrightarrow \mathbb{R}^+$ defined by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x,y \in X$ and the quasi-metric $d_p : X \times X \longrightarrow \mathbb{R}^+$ defined by $d_p(x,y) = p(x,y) - p(x,x)$ for all $x,y \in X$ such that $\tau(p)$ is finer than $\tau(p^s)$ and $\tau(p) = \tau(d_p)$ [17].

Example 2.3. Let $X = \mathbb{R}$. Consider the function $p : X \times X \to \mathbb{R}^+$ given by

$$p(x,y) = \frac{1}{2}(|x-y| + |x| + |y|) \qquad (x,y \in \mathbb{R}).$$

It is easy to see that (X,p) is a partial metric space and $p^s(x,y) = |x-y|$, for all $x,y \in \mathbb{R}$. For $x \in \mathbb{R}$ and $\varepsilon > 0$, the open balls are as follows,

$$B_p(x,\varepsilon) = (-\varepsilon, x+\varepsilon) \subset (x-\varepsilon, x+\varepsilon) = B_{p^s}(x,\varepsilon) \textit{ whenever } x > 0,$$
$$B_p(x,\varepsilon) = (x-\varepsilon, \varepsilon) \subset B_{p^s}(x,\varepsilon) \textit{ whenever } x < 0,$$
$$B_p(0,\varepsilon) = (-\varepsilon, \varepsilon) = B_{p^s}(0,\varepsilon).$$

Definition 2.4. [14] A quasi-partial metric on a nonempty set X is a function $qp : X \times X \to \mathbb{R}^+$ satisfying
(i) If $qp(x,x) = qp(x,y) = qp(y,y)$, then $x = y$.
(ii) $qp(x,x) \le qp(x,y)$.
(iii) $qp(x,x) \le qp(y,x)$.
(iv) $qp(x,y) + qp(z,z) \le qp(x,z) + qp(z,y)$ *for all* $x,\ y,\ z \in X$.

A quasi-partial metric space is a pair (X,qp) such that X is a nonempty set and qp is a quasi-partial metric on X. Every quasi-partial metric qp on X induces the metric $qp^s : X \times X \to \mathbb{R}^+$ defined by $qp^s(x,y) = qp(x,y) + qp(y,x) - qp(x,x) - qp(y,y)$ for all $x,y \in X$. If $qp(x,y) = qp(y,x)$ for all $x,y \in X$, then qp is a partial metric on X.

For the quasi-partial metric qp on a nonempty set X, the following functions are quasi-metrics on X [16],

$$q_{qp}(x,y) = qp(x,y) - qp(x,x),$$
$$q_{qp}^{-1}(x,y) = q_{qp}(y,x) = qp(y,x) - qp(y,y),$$
$$\overline{q_{qp}}(x,y) = qp^{-1}(x,y) - qp^{-1}(x,x) = qp(y,x) - qp(x,x),$$
$$\overline{q_{qp}}^{-1}(x,y) = \overline{q_{qp}}(y,x) = qp(x,y) - qp(y,y).$$

Similarly, as in the case of partial metrics we can introduce ε-balls of points to define topologies on X. So for $\varepsilon > 0$, we obtain the following ε-balls at $x \in X$.

$$\begin{aligned}
B_{q_{qp}}(x,\varepsilon) &= \{y \in X : qp(x,y) - qp(x,x) < \varepsilon\},\\
B_{q_{qp}^{-1}}(x,\varepsilon) &= \{y \in X : qp(y,x) - qp(y,y) < \varepsilon\},\\
B_{\overline{q_{qp}}}(x,\varepsilon) &= \{y \in X : qp(y,x) - qp(x,x) < \varepsilon\},\\
B_{\overline{q_{qp}}^{-1}}(x,\varepsilon) &= \{y \in X : qp(x,y) - qp(y,y) < \varepsilon\}.
\end{aligned}$$

In each of the above cases, the collection of all these balls yields a base for a T_0-topology on X, which as usual, we shall denote by $\tau(q_{qp})$, $\tau(q_{qp}^{-1})$, $\tau(\overline{q_{qp}})$ and $\tau(\overline{q_{qp}}^{-1})$, respectively.

Example 2.5. [12] Let $X = \mathbb{R}$. Define $qp(x,y) = |x-y| + |x|$. Then qp is a quasi-partial metric on $\mathbb{R}$. We have

$$\overline{q_{qp}}(x,y) = |x-y| + |y| - |x|, \qquad B_{\overline{q_{qp}}}(x,\varepsilon) = \begin{cases} (-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}), & x > 0\\ (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), & x = 0\\ (x-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}), & x < 0, \end{cases}$$

$$\overline{q_{qp}}^{-1}(x,y) = |x-y| + |x| - |y|, \quad B_{\overline{q_{qp}}^{-1}}(x,\varepsilon) = \begin{cases} (x-\frac{\varepsilon}{2}, \infty), & x > 0\\ (-\infty, \infty), & x = 0\\ (-\infty, x+\frac{\varepsilon}{2}), & x < 0, \end{cases}$$

$$\begin{aligned}
q_{qp}^{-1}(x,y) = |x-y| = q_{qp}(x,y), \quad B_{q_{qp}^{-1}}(x,\varepsilon) &\\
&= (x-\varepsilon, x+\varepsilon)\\
&= B_{q_{qp}}(x,\varepsilon).
\end{aligned}$$

We can see that q_{qp}, q_{qp}^{-1}, $\overline{q_{qp}}$ and $\overline{q_{qp}}^{-1}$ are quasi-metrics and $B_{q_{qp}}(x,\varepsilon) = (x-\varepsilon, x+\epsilon) = B_{q_{qp}^{-1}}(x,\varepsilon)$. So the open balls in $\tau(q_{qp})$, $\tau(q_{qp}^{-1})$ and $\tau(|.|)$ are equal.

Lemma 2.6. [14] *For a quasi-partial metric qp on X,*

$$p_{qp}(x,y) = \frac{1}{2}[qp(x,y) + qp(y,x)] \qquad\qquad (x,y \in X),$$

is a partial metric on X.

Lemma 2.7. [14] *Let (X, qp) be a quasi-partial metric space, let (X, p_{qp}) be the corresponding partial metric space, and let $(X, d_{p_{qp}})$ be the corresponding metric space. Then the following statements are*

equivalent:
(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (X, qp) and (X, qp) is complete.
(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (X, p_{qp}) and (X, p_{qp}) is complete.
(iii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in (X, d_{qp}) and $(X, d_{p_{qp}})$ is complete. Also,

$$\begin{aligned}\lim_{n\to\infty} d_{p_{qp}}(x, x_n) = 0 \Leftrightarrow p_{qp}(x, x) &= \lim_{n\to\infty} p_{qp}(x, x_n) = \lim_{n,m\to\infty} p_{qp}(x_n, x_m)\\ \Leftrightarrow \ qp(x, x) &= \lim_{n\to\infty} qp(x, x_n) = \lim_{n,m\to\infty} qp(x_n, x_m)\\ &= \lim_{n\to\infty} qp(x_n, x) = \lim_{n,m\to\infty} qp(x_m, x_n).\end{aligned}$$

We use the following lemma in the proof of main theorems.

Lemma 2.8. [12] *Let (X, qp) be a quasi-partial metric space. Then the following hold:*
(A) If $qp(x, y) = 0$, then x = y.
(B) If $x \neq y$, then $qp(x, y) > 0$ and $qp(y, x) > 0$.

3. Main results

Remember that a function f from a topological space (X, τ) into $(\mathbb{R}, \tau(|.|))$ is upper semicontinuous on (X, τ) if and only if f is continuous from (X, τ) to $(\mathbb{R}, \tau(d_u))$ where d_u is upper quasi-metric.
In order to prove our main theorem, we need the following proposition.

Proposition 3.1. *If qp is a quasi-partial metric on a nonempty set X, then $qp : (X, \tau(\overline{q_{qp}})) \times (X, \tau(q_{qp})) \longrightarrow (\mathbb{R}^+, \tau(|.|))$ is upper semicontinuous.*

Proof. It is enough to show that $qp : (X, \tau(\overline{q_{qp}})) \times (X, \tau(q_{qp})) \longrightarrow (\mathbb{R}^+, \tau(d_u))$ is continuous. Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq X$ and $x, y \in X$ be such that $\lim_{n\to\infty} \overline{q_{qp}}(x, x_n) = \lim_{n\to\infty} q_{qp}(y, y_n) = 0$. So for a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\overline{q_{qp}}(x, x_n) < \frac{\varepsilon}{2}$ and $q_{qp}(y, y_n) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. Then, for $n \geq n_0$ we have

$$\begin{aligned} qp(x_n, y_n) - qp(x, y) &\leq qp(x_n, x) + qp(x, y_n) - qp(x, x) - qp(x, y) \\ &= \overline{q_{qp}}(x, x_n) + qp(x, y_n) - qp(x, y) \\ &< \frac{\varepsilon}{2} + qp(x, y_n) - qp(x, y) \\ &\leq \frac{\varepsilon}{2} + qp(x, y) + qp(y, y_n) - qp(y, y) - qp(x, y) \\ &= \frac{\varepsilon}{2} + q_{qp}(y, y_n) \\ &< \varepsilon. \end{aligned}$$

Hence $d_u(qp(x_n, y_n), qp(x, y)) = max\{qp(x_n, y_n) - qp(x, y), 0\} < \varepsilon$ for all $n \geq n_0$. Therefore, according to the argument of the beginning of this section $qp : (X, \tau(\overline{q_{qp}})) \times (X, \tau(q_{qp})) \longrightarrow (\mathbb{R}^+, \tau(|.|))$ is upper semicontinuous. □

Clearly, by Proposition 3.1, every partial metric p on a nonempty set X with the topology arising of the quasi-metric d_p is upper semicontinuous.

Next example supports Proposition 3.1 and shows that for

$$qp : (X, \tau(q_{qp})) \times (X, \tau(\overline{q_{qp}})) \longrightarrow (\mathbb{R}^+, \tau(|.|)) \tag{1}$$

Proposition 3.1 does not hold.

Example 3.2. Let $X = \mathbb{R}$, and qp be the quasi-partial metric which is introduced in example 2.5. Let $\{x_n\}_{n\in\mathbb{N}} = \{2, 1, 2, 1, \cdots\}$ and $\{y_n\}_{n\in\mathbb{N}} = \{3 + \frac{1}{n}\}_{n\in\mathbb{N}}$ be two sequences in X. Clearly

$$\lim_{n\to\infty} \overline{q_{qp}}(3, x_n) = \lim_{n\to\infty} q_{qp}(3, y_n) = 0 \tag{2}$$

and $\lim_{n\to\infty} d_u(qp(x, y), qp(x_n, y_n)) = 0$, but

$$d_u(qp(y, x), qp(y_n, x_n)) = \begin{cases} \frac{2}{n} + 2, & n = 2k \\ \frac{2}{n} + 1, & n = 2k + 1. \end{cases}$$

is not convergent.

Definition 3.3. Let X be a non-empty set and τ_1, τ_2 be two topologies on X. The function $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, is sequentially (τ_1, τ_2)-continuous if $\{f(x_n, x_n, \ldots, x_n)\}_{n\in\mathbb{N}}$ converges to $f(x, x, \ldots, x)$ in (X, τ_2) whenever $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converges to x in (X, τ_1).

Recall that, every upper semicontinuous function on a compact topological space attains a maximum value [1].

THEOREM 3.4. *Let qp be a quasi-partial metric on a nonempty set X. Suppose that $(X, \tau(\overline{q_{qp}}))$ is compact and $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, is sequentially $(\tau(\overline{q_{qp}}), \tau(q_{qp}))$-continuous and satisfying*

$$qp(f(x, \cdots, x), f(y, \cdots, y)) > qp(x, y) \tag{3}$$

for all $x, y \in X$ with $f(x, \cdots, x) \neq f(y, \cdots, y)$, then f has a unique fixed point.

Proof. Define $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = qp(x, f(x, \cdots, x))$. Using the compactness of $(X, \tau(\overline{q_{qp}}))$, Proposition 3.1 and sequentially $(\tau(\overline{q_{qp}}), \tau(q_{qp}))$-continuity of f implies that the function φ attains a maximum value at some $x_0 \in X$, i.e, $\varphi(x_0) \geq \varphi(x)$ for all $x \in X$. If $f(x_0, \cdots, x_0) = x_0$, then x_0 is a fixed point of f, otherwise, we claim that $y_0 = f(x_0, \cdots, x_0)$ is a fixed point of f. Suppose that $y_0 \neq f(y_0, \cdots, y_0)$. Then by Lemma 2.8 we have $qp(y_0, f(y_0, \cdots, y_0)) \neq 0$ and by (2)

$$\begin{aligned} \varphi(y_0) &= qp(y_0, f(y_0, \cdots, y_0)) \\ &= qp(f(x_0, \cdots, x_0), f(y_0, \cdots, y_0)) \\ &> qp(x_0, y_0) \\ &= qp(x_0, f(x_0, \cdots, x_0)), \end{aligned}$$

a contradiction. Then y_0 is a fixed point of f. To prove uniqueness, if w and z are two distinct fixed points of f, then by Lemma 2.8, $qp(z, w) \neq 0$ and by (2)

$$qp(z, w) = qp(f(z, \cdots, z), f(w, \cdots, w)) > qp(z, w).$$

So the fixed point of f is unique. □

The following example illustrates Theorem 3.4.

EXAMPLE 3.5. Suppose that $X = \{0, \frac{1}{3}, 1\}$. Define $qp : X \times X \longrightarrow \mathbb{R}^+$ by

$$qp(x, y) = \begin{cases} 2x + y + 2, & x \neq y \\ 1, & x = y. \end{cases}$$

Clearly, qp is a quasi-partial metric on X. Let $f : X^2 \longrightarrow X$ be a function defined by

$$f(x, y) = \begin{cases} \frac{1}{3}, & x = 0,\ y \in X \\ 1, & x \in \{\frac{1}{3}, 1\},\ y \in X. \end{cases}$$

Since X is a finite set, $(X^2, \tau(q_{qp}))$ and $(X^2, \tau(\overline{q_{qp}}))$ are compact spaces and f is sequentially $(\tau(\overline{q_{qp}}), \tau(q_{qp}))$-continuous. It easy to check that

$$qp(f(x,x), f(y,y)) > qp(x,y)$$

for all $x, y \in X$ with $f(x,x) \neq f(y,y)$. Therefore all the conditions of Theorem 3.4 are satisfied and $x = 1$ is the unique fixed point of the function f, i.e., $f(1,1) = 1$.

Next example shows that the compactness of the domain of the function f can not be deleted in Theorem 3.4, in order to guarantee the existence of the fixed point.

EXAMPLE 3.6. The function $p_{\max} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by $p_{\max}(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$ is a partial metric and so is a quasi-partial metric on $\mathbb{R}^+$ [17]. We have also $p^s_{\max}(x,y) = |y-x|$, $\overline{q_{qp}}(x,y) = q_{qp}(x,y) = d_{p_{\max}}(x,y)$ for all $x, y \in \mathbb{R}^+$. Let $X = (0,1)$. Define $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, by $f(x_1, \cdots, x_k) = \sqrt{x_1}$, for all $x_1, \cdots, x_k \in X$. Then

$$p(f(x, \cdots, x), f(y, \cdots, y)) > p(x,y)$$

for all $x, y \in X$ with $f(x, \cdots, x) \neq f(y, \cdots, y)$. Also, $f : X^k \longrightarrow X$ is a sequentially $(\tau(d_p), \tau(d_p))$-continuous map. Clearly, $(X, \tau(d_p))$ is not compact and f is fixed point free.

Next, we show that Theorem 3.4 does not yield the uniqueness of fixed point in general when the contraction condition (2) in the statement of Theorem 3.4 is omitted.

EXAMPLE 3.7. Let $X = [0,1]$. We can see that

$$p_{\max}(f(1, \ldots, 1), f(x, \ldots, x)) = 1 = p_{\max}(1, x),$$

for all $x \in X$ and $f(x, \ldots, x) = x$ for $x = 0, 1$ where $p_{\max}$ and f are as defined in Example 3.6.

In Theorem 3.4, if we arrange the two topology $\tau(q_{qp})$ and $\tau(\overline{q_{qp}})$, we obtain the following result.

THEOREM 3.8. *Let qp be a quasi-partial metric on a nonempty set X, k be a positive integer and $(X, \tau(q_{qp}))$ be a compact space. If $f : X^k \longrightarrow X$ is a sequentially $(\tau(q_{qp}), \tau(\overline{q_{qp}}))-$continuous map satisfying*

$$qp(f(x, \cdots, x), f(y, \cdots, y)) > qp(x,y)$$

for all $x, y \in X$ with $f(x, \cdots, x) \neq f(y, \cdots, y)$, then f has a unique fixed point.

Proof. The procedure of the proof is the same as the proof of Theorem 3.4, when we define the function $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = qp(f(x, \cdots, x), x)$ for all $x \in X$. $\square$

Example 3.5 is also illustrate Theorem 3.8.

COROLLARY 3.9. *Let (X, p) be a partial metric space and suppose that $(X, \tau(d_p))$ is compact. If $f : X^k \longrightarrow X$, $k \in \mathbb{N}$, is a sequentially $(\tau(d_p), \tau(d_p))$-continuous map satisfying*

$$p(f(x, \cdots, x), f(y, \cdots, y)) > p(x, y)$$

for all $x, y \in X$ with $f(x, \cdots, x) \neq f(y, \cdots, y)$, then f has a unique fixed point.

Proof. The partial metric p is a quasi-partial metric and $\overline{q_p} = q_p = d_p$. Then f satisfies conditions of Theorem 3.4 and Theorem 3.8. Therefore the desired result is obtained. $\square$

By considering $k = 1$ in Theorem 3.4, we obtain the following result.

COROLLARY 3.10. *Let p be a partial metric on a set X and $(X, \tau(d_p))$ be a compact space. If f is a continuous self map on $(X, \tau(d_p))$ and there exists $n \in \mathbb{N}$ such that*

$$p(f^n(x), f^n(y)) > p(x, y)$$

for all $x, y \in X$ with $x \neq y$, then f has a unique fixed point.

Proof. First of all, note that any partial metric p is a quasi-partial metric, $\overline{q_p} = q_p = d_p$ is a quasi-metric, and by continuity of f, the self mapping f^n on $(X, \tau(d_p))$ is continuous. Then by Theorem 3.4, f^n has a unique fixed point, x_0, in X. We have also

$$f^n(f(x_0)) = f(f^n(x_0)) = f(x_0).$$

Hence by uniqueness of the fixed point of f^n, we have $f(x_0) = x_0$. Then f has a unique fixed point. $\square$

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