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SOME RECURRENT PROPERTIES OF LP-SASAKIAN NANIFOLDS

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Abstract. The aim of the present paper is to study certain recurrent properties of LP-Sasakian manifolds. Here we first describe Ricci η-recurrent LP-Sasakian manifolds. Further we study semi-generalized recurrent and three dimensional locally generalized concircularly ϕ-recurrent LP-Sasakian manifolds and got interesting results.

1. Introduction

In 1989, Matsumoto [5] introduced the idea of LP-Sasakian manifolds. Later on, the authors Mihai and Rosca [7] developed independently on LP-Sasakian manifolds. During last two decades, the notion of LP-Sasakian manifolds has been studied by De et. al. [3], Shaikh et. al. [11] and many others such as [1, 6, 8] etc.,

On the other hand, the study of recurrent manifolds began with the work of Walker [13] and the idea of generelized recurrent manifold with non-zero associated 1-form was studied by De and Guha [2]. In [10], Ruse had shown that if the associated 1-form becomes zero, then the manifold reduces to a recurrent manifold. Next, Bhagwat Prasad [9] originated semi-generalized recurrent manifolds. Recently, Venkatesha et al. [12] introduced and studied hyper generalized ϕ-recurrent and quasi

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generalized ϕ-recurrent Sasakian manifold. During last five decades, the idea of recurrent manifolds has been weakened by many geometers with different structures.

Incited by the above studies, we made an attempt to study in this paper some recurrent properties of LP-Sasakian manifolds. In Section 2, we have given some basic concepts and preliminaries of LP-Sasakian manifolds which is needed throughout the paper. We have dealt Ricci η-recurrent LP-Sasakian manifolds in Section 3. Section 4 has concentrated to the study of semi-generalized recurrent LP-Sasakian manifolds. Here we have shown that the linear combination of 1-forms is always zero and the manifold becomes η-Einstein. Lastly we proved that three dimensional locally generalized concircularly ϕ-recurrent LP-Sasakian manifold is a space of constant curvature.

2. Preliminaries

An n-dimensional differentiable manifold M is said to be an LP-Sasakian manifold [7] if it admits a $(1,1)$ tensor field ϕ, a unit time like vector field ξ, a 1-form η and a lorentzian metric g which satisfies

$$\eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \;\; \phi^2 X = X + \eta(X)\xi, \tag{1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \;\; \nabla_X \xi = \phi X, \tag{2}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{3}$$

for any $X, Y \in T_pM$, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; rank\ \phi = n-1. \tag{4}$$

Again, if we put

$$\Omega(X,Y) \;\; = \;\; g(X, \phi Y),$$

for any $X, Y \in T_pM$, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field [5]. Also, as the vector field η is closed in an LP-Sasakian manifold, we have [5, 7]

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \;\; \Omega(X,\xi) = 0, \tag{5}$$

for any $X, Y \in T_pM$,

Let M be an n-dimensional LP-Sasakian manifold, then the following relations hold [5]:

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{6}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{7}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{8}$$

for any $X, Y, Z \in T_pM$, where R is the Riemannian curvature tensor and S is the Ricci tensor of the manifold.

Also in a three dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{9}$$

where Q is the Ricci operator i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.
Putting $Z = \xi$ in (9) and then by using (6), we have

$$\eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} - 1\right)[\eta(Y)X - \eta(X)Y]. \tag{10}$$

Again putting $Y = \xi$ in (10) and by virtue of (4) and (7), yields

$$\begin{aligned} QX &= \frac{1}{2}(r-2)X + \frac{1}{2}(r-6)\eta(X)\xi, \\ S(X,Y) &= \frac{1}{2}(r-2)g(X,Y) + \frac{1}{2}(r-6)\eta(X)\eta(Y). \end{aligned} \tag{11}$$

Substituting (11) in (9), we get

$$\begin{aligned} R(X,Y)Z &= \frac{(r-4)}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{(r-6)}{2}[g(Y,Z)\eta(X)\xi \\ &\quad -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{12}$$

Definition 2.1. An n-dimensional LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y),$$

for any vector fields X and Y, where α and β are constants. If $\beta = 0$, then the manifold M is an Einstein manifold and if $\alpha = 0$, then the manifold is a special type of η-Einstein.

3. Ricci η-recurrent LP-Sasakian manifolds

Definition 3.1. The Ricci tensor of an n-dimensional LP-Sasakian manifold is said to be η-recurrent if its Ricci tensor satisfies:

$$(\nabla_X S)(\phi(Y), \phi(Z)) = A(X)S(\phi(Y), \phi(Z)), \tag{13}$$

for any $X, Y, Z \in T_pM$, where $A(X) = g(X, \rho)$, ρ is the associated vector field of the 1-form A.

If the 1-form A vanishes, then the Ricci tensor is said to be η-parallel and this notion for Sasakian manifold was introduced by Yano [14].

Now it follows from (13) that

$$\begin{aligned}\nabla_Z S(\phi(X), \phi(Y)) - S(\nabla_Z \phi X, \phi(Y)) - S(\phi(X), \nabla_Z \phi Y)\\ = A(Z)S(\phi(X), \phi(Y)).\end{aligned} \tag{14}$$

By using (2), (3) and (8) in (14), we get

$$\begin{aligned}(\nabla_Z S)(X, Y) &= \eta(X)[S(Z, \phi Y) - (n-1)\Omega(Z, Y)] + \eta(Y)[S(\phi X, Z)\\ &\quad -(n-1)\Omega(Z, X)] + A(Z)[S(X, Y) + (n-1)\eta(X)\eta(Y)].\end{aligned} \tag{15}$$

Hence we can state the following:

Theorem 3.2. *In an n-dimensional LP-Sasakian manifold, the Ricci tensor is η-recurrent if and only if (15) holds.*

Putting $X = Y = e_i$ in (15), let e_i be an orthonormal basis of the tangent space at each point of the manifold, taking the summation over i, $1 \leq i \leq n$, we have

$$dr(Z) = [r - (n-1)]A(Z). \tag{16}$$

If the manifold has a constant scalar curvature r, then it follows from (16) that

$$A(Z) = 0, \forall Z.$$

Hence we can states the following:

Theorem 3.3. *If a n-dimensional Ricci η-recurrent LP-Sasakian manifold M is of constant scalar curvature, then the Ricci tensor of M is η-parallel.*

Again putting $X = Z = e_i$ in (15) and taking summation over i, $1 \leq i \leq n$, we get

$$dr(Y) = \kappa\eta(Y) + S(Y, \rho) + (n-1)\eta(Y)\eta(\rho), \tag{17}$$

where $\kappa = \sum S(\phi e_i, e_i)$. Using (16) in (17) yields

$$[r-(n-1)]A(Y) = \kappa\eta(Y) = S(Y,\rho) + (n-1)\eta(Y)\eta(\rho). \tag{18}$$

On plugging $Y = \xi$ in (18), yields

$$\kappa = [(n-1)-r]\eta(\rho). \tag{19}$$

Considering (19) in (18), we get

$$S(Y,\rho) = [r-(n-1)]g(Y,\rho) + [2(n-1)-r]\eta(Y)\eta(\rho). \tag{20}$$

This leads us to the following:

Theorem 3.4. *If the Ricci tensor in an n-dimensional LP-Sasakian manifold is η-recurrent, then its Ricci tensor along the associated vector field of the 1-form is given by (20).*

Replacing $Y = \phi Y$ in (20) and by virtue of (1), we get

$$S(Y,C) = Ng(Y,C), \tag{21}$$

where $N = (r-(n-1))$ and $C = \phi\rho$.
Hence we can state the following;

Theorem 3.5. *If the Ricci tensor in an LP-Sasakian manifold is η-recurrent, then $k = (r-(n-1))$ is an eigen value of Ricci tensor corresponding to the eigen vector $\phi\rho$.*

4. Semi-generalized recurrent LP-Sasakian manifolds

Definition 4.1. An n-dimensional Riemannian manifold is said to be semi-generalized recurrent manifold if its curvature tensor R satisfies the relation

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)g(Z,W)Y, \tag{22}$$

where A and B are two 1-forms, B is non-zero, ρ_1 and ρ_2 are two vector fields such that

$$g(X,\rho_1) = A(X), \quad g(X,\rho_2) = B(X), \tag{23}$$

for any vector field X and ∇ be the covariant differentiation operator with respect to the metric g.

Definition 4.2. A Riemannian Manifold M is said to be semi-generalized Ricci-recurrent LP-Sasakian manifold if

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + nB(X)g(Y,Z). \tag{24}$$

Taking cyclic sum of (22) with respect to X, Y, Z and then by using second Bianchi identity, we have

$$\begin{aligned} (25) \qquad & A(X)R(Y,Z)W + A(Y)R(Z,X)W + A(Z)R(X,Y)W \\ & +B(X)g(Z,W)Y + B(Y)g(X,W)Z + B(Z)g(Y,W)X = 0. \end{aligned}$$

Contracting (25) with respect to Y and U, we get

$$\begin{aligned} (26) \qquad & A(X)S(Z,W) - g(R(Z,X)\rho_1, W) - A(Z)S(X,W) \\ & +nB(X)g(Z,W) + g(X,W)g(Z,\rho_2) + B(Z)g(X,W) = 0, \end{aligned}$$

Again contracting above equation over Z and W, we obtain

$$(27) \qquad rA(X) + (n^2+2)B(X) - 2S(X,\rho_1) = 0.$$

Putting $X = \xi$ in (27) and by virtue of (7) and (23), we get

$$(28) \qquad r = \frac{1}{\eta(\rho_1)}\{2(n-1)\eta(\rho_1) - (n^2+2)\eta(\rho_2)\}.$$

For LP-Sasakian manifold $\eta(\rho_1) \neq 0$, and so we can state the following:

THEOREM 4.3. *Let M be an semi-generalized recurrent LP-Sasakian manifold, then the scalar curvature r is given by (28).*

On plugging $Z = \xi$ in (24), we get

$$(29) \qquad (\nabla_X S)(Y,\xi) = A(X)S(Y,\xi) + nB(X)g(Y,\xi).$$

By virtue of (5) and (7) in (29), yields

$$\begin{aligned} (30) \qquad & (n-1)g(X,\phi Y) - S(X,\phi Y) \\ & = (n-1)A(X)\eta(Y) + nB(X)\eta(Y), \end{aligned}$$

Again plugging $Y = \xi$ in (30), we obtain

$$(31) \qquad A(X) = \frac{n}{1-n}B(X).$$

This leads us to the following:

THEOREM 4.4. *In an n-dimensional semi-generalized Ricci recurrent LP-Sasakian manifold, the 1-forms A and B is always linearly dependent to each other.*

Replacing Y by ϕY in (30), we get

$$(32) \qquad S(X,Y) = (1-n)g(X,Y) + 2(1-n)\eta(X)\eta(Y).$$

Hence we can state the following:

THEOREM 4.5. *An n-dimensional semi-generalized Ricci recurrent LP-Sasakian manifold is η-Einstein.*

5. On three dimensional locally generalized concircularly ϕ-recurrent LP-Sasakian manifold

DEFINITION 5.1. A three dimensional LP-Sasakian manifold is said to be locally generalized concircularly ϕ-recurrent if its concircular curvature tensor $\tilde{C}$ defined by [14]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y], \tag{33}$$

satisfies the condition

$$\begin{aligned} &\phi^2((\nabla_W \tilde{C})(X,Y)Z) \\ &= A(W)\tilde{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{34}$$

for all vector fields X, Y, Z and W orthogonal to ξ.

Taking covariant derivative of (9) with respect to W, we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z \;=\; & \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ & + \frac{r-6}{2}[g(Y,Z)(\nabla_W \eta)(X)\xi - g(X,Z)(\nabla_W \eta)(Y)\xi \\ & + (\nabla_W \eta)(Y)\eta(Z)X - (\nabla_W \eta)(Z)\eta(Y)X \\ & - (\nabla_W \eta)(X)\eta(Z)Y - (\nabla_W \eta)(Z)\eta(X)Y] + \frac{dr(W)}{2}[g(Y,Z) \\ & \eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{35}$$

If we take X, Y, Z and W orthogonal to ξ, we obtain

$$\begin{aligned} (\nabla_W R)(X,Y)Z \;=\; & \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] \\ & + \frac{r-6}{2}[g(Y,Z)\Omega(W,X)\xi - g(X,Z)\Omega(W,Y)\xi]. \end{aligned} \tag{36}$$

Now equation (36) reduces to

$$\phi^2((\nabla_W R)(X,Y)Z) = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{37}$$

Taking covariant differentiation of (33) with respect to W, we get

$$(\nabla_W \tilde{C})(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{38}$$

From which it follows that

$$\phi^2((\nabla_W \tilde{C})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z) - \frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{39}$$

By virtue of (34), (37) in (39), we get

$$R(X,Y)Z = \left(\frac{r}{6} - \frac{B(W)}{A(W)} + \frac{dr(W)}{3A(W)}\right)[g(Y,Z)X - g(X,Z)Y]. \tag{40}$$

In a locally generalized concircularly ϕ-recrrrent LP-Sasakian manifold $A(W) \neq 0$. On contracting with respect to W, we obtain

$$R(X,Y)Z = \gamma[g(Y,Z)X - g(X,Z)Y], \tag{41}$$

where $\gamma = \{\frac{r}{6} - \frac{B(e_i)}{A(e_i)} + \frac{dr(e_i)}{3A(e_i)}\}$ is a scalar. Then by Schur's theorem [4] γ will be constant.
Hence we can state the following:

Theorem 5.2. *A three dimensional locally generalized concircularly ϕ-recurrent LP-Sasakian manifold is a space of constant curvature.*

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References

[1] C. S. Bagewadi, Venkatesha and N. S. Basavarajappa, *On LP-Sasakian manifolds,* Sci. Scr. A: Math. Sci. **16** (2008), 1–8.
[2] U. C. De and N. Guha, *On generalized recurrent manifolds,* J. Nat. Acad. Math. **9** (1991), 85–92.

[3] U. C. De, K. Matsumoto and A. A. Shaikh, *On Lorentzian Para-Sasakian manifolds,* Rendicontidel Seminaro Matemetico di Messina, Serien II, Supplemento al. **3** (1999), 149–156.

[4] S. Kobayashi and K. Nomizu, *Foundations of differential geometry,* Interscience Publishers, **1** (1963), 293–301.

[5] K. Matsumoto, *On Lorentzian paracontact manifolds,* Bull. Yamagata Univ. Natur. Sci. **12** (1989), 151–156.

[6] I. Mihai, U. C. De and A. A. Shaikh, *On Lorentzian Para-Sasakian manifolds,* Korean J. Math. Sci. **6** (1999), 1–13.

[7] I. Mihai and R. Rosca, *On Lorentzian Para-Sasakian manifolds,* Classical Analysis, World Scientific Publ., Singapore, (1992), 155–169.

[8] C. Murathan, A. Yildiz, K. Arsian and U. C. De, *On a class of Lorentzian Para-Sasakian manifolds,* Proc. Estonian Acad. Sci.Phys. Math. **55** (4) (2006), 210–219.

[9] B. Prasad, *On semi-generalized recurrent manifold,* Mathematica Balkanica, New series **14** (2000), 72–82.

[10] H. S. Ruse, *A classification of K^4 space,* London Mathematica Balkanical Society **53** (1951), 212–229.

[11] A. A. Shaikh and K. K. Baisha, *Some results on LP-Sasakian manifolds,* Bull. Math. Soc. Sci. Math., Roumanic Tome **49** (2) (2006), 193–205.

[12] V. Venkatesha, H.A. Kumara, D.M. Naik, *On a class of generalized φ-recurrent Sasakian manifold*, J. Egypt. Math. Soc. **27** (19) (2019), https://doi.org/10.1186/s42787-019-0022-0.

[13] A. G. Walker, *On Ruses spaces of recurrent curvature,* Proc. London Math. Soc. **52** (1976), 36–64.

[14] K. Yano and M. Kon, *Structures of manifolds,* World Scientific Publishing, Singapore 1984.

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