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ON THE GENERALIZED BANACH SPACES

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Abstract. For any non-negative real number ϵ_0, we shall introduce a concept of the ϵ_0-Cauchy sequence in a normed linear space V and also introduce a concept of the ϵ_0-completeness in those spaces. Finally we introduce a concept of the generalized Banach spaces with these concepts.

1. Introduction

In this section, we briefly introduce the concept of the generalized limits of the multi-valued sequences and functions on the normed spaces which we need later. Let's denote by $B(x, \epsilon)$ (resp. $\overline{B}(x, \epsilon)$) the open (resp. closed) ball in the normed linear space V with radius ϵ and center at x.

Definition 1.1. Let $\{x_n\}$ be a multi-valued infinite sequence of elements of the normed linear space $(V, \|\cdot\|)$. And let $\epsilon_0 \geq 0$ be a fixed non-negative real number. If a subset S of V satisfies the following condition, we call that the ϵ_0 generalized limit (or ϵ_0-limit) of $\{x_n\}$ as n goes to ∞ is S, and we denote it by $\underset{n \longrightarrow \infty}{\boxed{\epsilon_0 - \lim}} x_n = S$: S is the set of all the vectors $\alpha \in V$ satisfying the condition

$$\forall \epsilon > \epsilon_0, \exists K \in N \ s.t. (\forall n \in N) n \geq K, (\forall x_n) \Rightarrow \|x_n - \alpha\| < \epsilon.$$

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If the set S in the definition above is not empty we say that $\{x_n\}$ is an ϵ_0-convergent sequence or ϵ_0-converges to S. We also define that any member $\alpha \in S$ is an approximate value of the generalized limit of $\{x_n\}$ with the limit of the error ϵ_0. Then we can regard $\alpha \in S$ as the approximate value of the limit of $\{x_n\}$ whether $\{x_n\}$ converges in the usual sense or not. From now on, $V \neq \{0\}$ denotes a normed linear space.

Definition 1.2. Let $\{x_n\}$ be a multi-valued infinite sequence in V. We define that $\{x_n\}$ is ultimately bounded if and only if there exist real numbers K and M such that $(\forall n \in N) n \geq K, \forall x_n \Rightarrow \|x_n\| \leq M$.

Lemma 1.3. *(Representation) Let $\{x_n\}$ be a multi-valued infinite sequence in the normed linear space $V \neq \{0\}$ which satisfies the Heine-Borel property. And let $\epsilon_0 \geq 0$ be a non-negative real number. Suppose that $\{x_n\}$ is ultimately bounded. If $\boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n = S$ then S is a convex and compact subset of V such that $S = \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. Here*

$$SSL = SSL(\{x_n\}) = \{\alpha \in V | \exists \{x_{n_k}\} \leq \{x_n\} \ s.t. \lim_{k \to \infty} x_{n_k} = \alpha\}$$

and $\{x_{n_k}\} \leq \{x_n\}$ means that $\{x_{n_k}\}$ is a single-valued subsequence of $\{x_n\}$.

Proof. $(\subseteq)$ Let any element $\beta \in S \neq \emptyset$ be given. Then

$$\forall \epsilon > \epsilon_0, \exists K_1 \in N \ s.t. (\forall n \in N) n \geq K_1, (\forall x_n) \Rightarrow \|x_n - \beta\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}.$$

If $\alpha \in SSL$ is any element, then there exists a single-valued and convergent subsequence $\{x_{n_k}\}$ such that $\lim_{k \to \infty} x_{n_k} = \alpha$. Thus we have

$$\forall \epsilon > \epsilon_0, \exists K_2 \in N \ s.t. (\forall k \in N) k \geq K_2 \Rightarrow \|x_{n_k} - \alpha\| < \frac{\epsilon - \epsilon_0}{2}.$$

Choosing a natural number $K = \max\{K_1, K_2\}$, we have

$$\begin{aligned} \|\beta - \alpha\| &= \|\beta - x_{n_K} + x_{n_K} - \alpha\| \\ &\leq \|\beta - x_{n_K}\| + \|x_{n_K} - \alpha\| \\ &< \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} + \frac{\epsilon - \epsilon_0}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > \epsilon_0$ was arbitrary, we have $\|\beta - \alpha\| \leq \epsilon_0$. That is, $\beta \in \overline{B}(\alpha, \epsilon_0)$. Since $\alpha \in SSL$ was arbitrary, we have $\beta \in \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. Since $\beta \in S$

was also arbitrary, we have $S \subseteq \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. ($\supseteq$) Since $V \neq \{0\}$, $S \neq V$ since $\{x_n\}$ is ultimately bounded. In order to show that the opposite inclusion is also satisfied, let $\beta \notin S$ be any element of $V - S \neq \emptyset$. Then we have

$$\exists \epsilon_1 > \epsilon_0 \; s.t. (\forall k \in N, \exists n_k \in N, \exists x_{n_k} \;\; s.t. \; \|x_{n_k} - \beta\| \geq \epsilon_1).$$

Since $\{x_n\}$ is ultimately bounded, $\{x_{n_k}\}$ is a bounded sequence in V. Thus $\{x_{n_k} : k \in N\}$ is a subset of some closed bounded ball $\bar{B}(x, r)$ for some $x \in V$ and $r > 0$. Since V satisfies the Heine-Borel property, the closed ball $\bar{B}(x, r)$ is a compact subset of V. Since $\{x_{n_k}\}$ is a sequence in the compact set $\bar{B}(x, r)$, there is a convergent subsequence $\{x_{n_{k_p}}\}$ of $\{x_{n_k}\}$. Hence we may assume that $\lim_{p \to \infty} x_{n_{k_p}} = \alpha_0$ for some $\alpha_0 \in V$. Then we have, for such an $\epsilon_1 > \epsilon_0$,

$$\exists K \in N \; s.t. \; p \geq K \Rightarrow \|x_{n_{k_p}} - \alpha_0\| < \frac{\epsilon_1 - \epsilon_0}{2}.$$

Therefore, we have

$$\begin{aligned} \|\beta - \alpha_0\| &= \|\beta - x_{n_{k_K}} + x_{n_{k_K}} - \alpha_0\| \\ &\geq \|\beta - x_{n_{k_K}}\| - \|x_{n_{k_K}} - \alpha_0\| \\ &> \epsilon_1 - \frac{\epsilon_1 - \epsilon_0}{2} = \frac{\epsilon_1 + \epsilon_0}{2}. \end{aligned}$$

Since the last quantity satisfies the relation $\frac{\epsilon_1 + \epsilon_0}{2} > \epsilon_0$, this implies that $\beta \notin \overline{B}(\alpha_0, \epsilon_0)$. Since $\alpha_0 \in SSL$, this also implies that $\beta \notin \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. Hence $\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0) \subseteq S$. Consequently, we have $S = \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. On the other hand, since S is the intersection of the closed balls $\overline{B}(\alpha, \epsilon_0)$ which are bounded, closed and convex, S is convex and compact in V. Finally, if $S = \emptyset$ then S is clearly convex and compact, and $\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0) \subseteq S = \emptyset$. $\square$

Note in the lemma above that if $SSL = \{a\}$ for some $a \in V$ then we have $\boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n = \overline{B}(a, \epsilon_0)$ for all $\epsilon_0 \geq 0$.

LEMMA 1.4. *Let $\{x_n\}$ be a multi-valued infinite sequence in the normed linear space V which satisfies the Heine-Borel property and $\epsilon_0 \geq 0$. Suppose that $\{x_n\}$ is ultimately bounded. Then the set SSL*

of all the single-valued subsequential limits of $\{x_n\}$ *is a non-empty and compact subset of* V.

Proof. The ultimate boundedness of the sequence $\{x_n\}$ implies that the set SSL is non-empty and bounded since V satisfies the Heine-Borel property. In order to verify that SSL is a closed subset of V, let any member $\alpha \in \overline{SSL}$ be given. If α is an element of SSL then we are done. Suppose that $\alpha \notin SSL$. Then α must be an accumulation point of the set SSL. By means of choosing the open balls $B(\alpha, \frac{1}{k})$ for all natural numbers $k \in N$, we have a single-valued sequence $\{\alpha_k\} \subseteq SSL$ such that $\lim_{k\to\infty} \alpha_k = \alpha$. Since the first term α_1 of the sequence $\{\alpha_k\}$ is an element of SSL, there is one value, say x_{n_1}, of the multi-valued term x_{n_1} in $\{x_n\}$ such that $\|x_{n_1} - \alpha_1\| < 1$. Similarly, since $\alpha_2 \in SSL$, there is one value, say x_{n_2}, of the multi-valued term x_{n_2} in $\{x_n\}$ such that $\|x_{n_2} - \alpha_2\| < \frac{1}{2}$ and $n_2 > n_1$. By applying those methods, we can inductively choose a single-valued subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - \alpha_k\| < \frac{1}{k}$ for all natural number $k \in N$. Since $\|x_{n_k} - \alpha\| \leq \|x_{n_k} - \alpha_k\| + \|\alpha_k - \alpha\|$, if we take the limit on both sides we have $\lim_{k\to\infty} x_{n_k} = \alpha$. Thus we have $\alpha \in SSL$ which completes the proof. $\square$

Definition 1.5. Let D be a subset of a normed space V and $f : D \to W$ be a multi-valued function into the normed space W. We define that f is ϵ_0-uniformly continuous on D if and only if we have

$$\begin{aligned} \forall \epsilon > \epsilon_0, \exists \delta > 0 \quad \text{s.t.} \quad & (\forall x, y \in D)\|x - y\| < \delta, \forall f(x), \forall f(y) \\ \Rightarrow \quad & \|f(x) - f(y)\| < \epsilon. \end{aligned}$$

2. The generalized Banach space

In this section, we define the concept of the ϵ_0 generalized completeness of a set and the concept of the ϵ_0 generalized Banach space. In this section, V denotes a normed linear space and ϵ_0 denotes a fixed non-negative real number.

Definition 2.1. Let $\{x_n\}$ be a multi-valued sequence in V. We define that $\{x_n\}$ is an ϵ_0-Cauchy sequence if and only if

$$\forall \epsilon > \epsilon_0, \exists K \in N s.t. (\forall m, n) m, n \geq K, \forall x_m, \forall x_n \Rightarrow \|x_m - x_n\| < \epsilon.$$

Note that it is easy to prove that any ϵ_0-Cauchy sequence is ultimately bounded.

DEFINITION 2.2. Let S be any non-empty subset of V. Then we define that S is ϵ_0-complete in V if and only if $\underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n \cap S \neq \emptyset$ for any ϵ_0-Cauchy sequence $\{x_n\}$ in S.

LEMMA 2.3. *Let V be a normed linear space which satisfies the Heine-Borel property, and let $\{x_n\}$ be an ϵ_0-Cauchy sequence in V. Then we have*

$$SSL \subseteq \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n.$$

Proof. Let $\{x_n\}$ be the given ϵ_0-Cauchy sequence in V. Then we have

$$\begin{aligned} &\forall \epsilon > \epsilon_0, \exists K \in N \quad s.t. \quad (\forall m, n) m, n \geq K, \forall x_m, \forall x_n \\ &\qquad\qquad \Longrightarrow \quad \|x_m - x_n\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} \end{aligned}$$

since $\epsilon_0 + \frac{\epsilon-\epsilon_0}{2} > \epsilon_0$. Since V satisfies the Heine-Borel property, we have $SSL \neq \emptyset$. Suppose that $\alpha \in SSL$. Then there is a single-valued and convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \alpha$. Since $n_k \geq k$, we have, by replacing x_n to x_{n_k},

$$\begin{aligned} &\forall \epsilon > \epsilon_0, \exists K \in N \quad s.t. \quad (\forall m, k) m, k \geq K, \forall x_m \\ &\qquad\qquad \Longrightarrow \quad \|x_m - x_{n_k}\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}. \end{aligned}$$

For each fixed term number m and each value of x_m, by taking the limit as k goes to ∞, we have

$$\begin{aligned} &\forall \epsilon > \epsilon_0, \exists K \in N \quad s.t. \quad (\forall m) m \geq K, \forall x_m \\ &\qquad\qquad \Longrightarrow \quad \|x_m - \alpha\| \leq \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} = \frac{\epsilon + \epsilon_0}{2} < \epsilon. \end{aligned}$$

Thus we have $\alpha \in \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n$. Consequently, $SSL \subseteq \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n$. □

COROLLARY 2.4. *Let $\{x_n\}$ be an ϵ_0-Cauchy sequence in a normed linear space V which satisfies the Heine-Borel property. If we denote by $hull(SSL)$ the convex hull of SSL then $hull(SSL) \neq \emptyset$ and*

$$hull(SSL) \subseteq \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n = \bigcap_{\alpha\in SSL} \overline{B}(\alpha, \epsilon_0).$$

Proof. Since the convex hull of SSL is the smallest convex subset of V which contains the set SSL, this corollary follows from lemmas 1.3, 2.3 and the convex property of the ϵ_0-limit. □

Lemma 2.5. *Let $\{x_n\}$ be an ϵ_0-Cauchy sequence in a normed linear space V. If $\alpha, \beta \in SSL$ then $\|\alpha - \beta\| \le \epsilon_0$. Hence the diameter of SSL is less than or equal to ϵ_0.*

Proof. Since $\{x_n\}$ is an ϵ_0-Cauchy sequence in V, we have

$$\forall \epsilon > \epsilon_0, \exists K \in N \quad s.t. \quad (\forall m, n) m, n \ge K, \forall x_m, \forall x_n$$
$$\Longrightarrow \quad \|x_m - x_n\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}$$

since $\epsilon_0 + \frac{\epsilon - \epsilon_0}{2} > \epsilon_0$. And since $\alpha, \beta \in SSL$, there are two single-valued and convergent subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim\limits_{k \to \infty} x_{m_k} = \alpha$ and $\lim\limits_{k \to \infty} x_{n_k} = \beta$. Since $m_k, n_k \ge k$, we have

$$\forall \epsilon > \epsilon_0, \exists K \in N s.t. (\forall k) k \ge K \Longrightarrow \|x_{m_k} - x_{n_k}\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}.$$

If we take the limit as k goes to ∞, we have

$$\|\alpha - \beta\| \le \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} = \frac{\epsilon + \epsilon_0}{2} < \epsilon.$$

Since $\epsilon > \epsilon_0$ was arbitrary, this implies that $\|\alpha - \beta\| \le \epsilon_0$. Hence the diameter of SSL is less than or equal to ϵ_0. $\square$

Theorem 2.6. *Let $\{x_n\}$ be an ϵ_0-Cauchy sequence in a normed linear space V which satisfies the Heine-Borel property. If $\epsilon_0 > 0$ and $diam(SSL(\{x_n\})) = d$ then there exists an open convex subset G of V such that*

$$hull(SSL) \cap G \neq \emptyset \ \text{ and } \overline{G} \subseteq \boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n.$$

Proof. Since $\{x_n\}$ is ultimately bounded, SSL is non-empty and compact by lemma 1.4. Hence there is a point $\alpha \in SSL$. If $SSL = \{\alpha\}$ is a singleton then we choose the open set G as $G = B(\alpha, \epsilon_0)$. Then we have $hull(SSL) \cap G = \{\alpha\} \neq \emptyset$ and $\overline{G} = \overline{B}(\alpha, \epsilon_0) = \boxed{\epsilon_0 - \lim\limits_{n \longrightarrow \infty}} x_n$. Suppose that SSL is not a singleton. Then $hull(SSL)$ is not a singleton, too, and has the same diameter. Hence there are two points $\alpha, \beta \in hull(SSL)$ such that $\|\alpha - \beta\| = d > 0$ since $hull(SSL)$ is also compact and $diam(hull(SSL)) = d > 0$. For each element $x \in T = \overline{B}(\alpha, d) \cap \overline{B}(\beta, d)$, the quantity $sup\{\|y - x\| : y \in hull(SSL)\}$ is a non-negative real number since $hull(SSL)$ is compact. Hence the infimum $r = inf\{sup\{\|y - x\| : y \in hull(SSL)\} : x \in T\}$ exists. At the first step, we will prove that this infimum r is less than the diameter d of $hull(SSL)$. Assume that $r \geq d$. Then we have $sup\{\|y-x\| : y \in hull(SSL)\} \geq d$ for all $x \in T$. In particular, we have $sup\{\|y-\gamma\| : y \in hull(SSL)\} \geq d$. Here $\gamma = \frac{\alpha+\beta}{2}$. Since γ is the center point of the line segment $\overline{\alpha\beta} \subseteq hull(SSL)$, we must have $sup\{\|y-\gamma\| : y \in hull(SSL)\} = d$. Since $hull(SSL)$ is compact, there is a point $y_\gamma \in hull(SSL)$ such that $\|y_\gamma - \gamma\| = sup\{\|y-\gamma\| : y \in hull(SSL)\} = d$. Thus $y_\gamma \in \partial(\overline{B}(\gamma, d))$. Now consider the midpoint $\eta = \frac{\gamma+y_\gamma}{2}$. Since η is a point of the set T, we also have $sup\{\|y-\eta\| : y \in hull(SSL)\} \geq r \geq d$ by the assumption $r \geq d$. And there is an element $y_\eta \in hull(SSL)$ such that $\|y_\eta - \eta\| = sup\{\|y-\eta\| : y \in hull(SSL)\} \geq r \geq d$ since $hull(SSL)$ is compact. But we have $y_\eta \in [\overline{B}(\gamma, d) - B(\eta, d)]$ and this set $\overline{B}(\gamma, d) - B(\eta, d)$ is disjoint from the closed ball $\overline{B}(y_\gamma, d)$. For if $z \in \overline{B}(\gamma, d) - B(\eta, d) \cap \overline{B}(y_\gamma, d)$, then we have $\|z-\gamma\| \leq d$, $\|z-\eta\| > d$ and $\|z-y_\gamma\| \leq d$ which is a contradiction since $\eta = \frac{\gamma+y_\gamma}{2}$. Thus we have $\|y_\eta - y_\gamma\| > d$ which is a contradiction with the fact that $diam(hull(SSL)) = d$. Therefore, the infimum r must satisfy the relation $r < d$. And this infimum is in fact the minimum of that set since $hull(SSL)$ and T are compact. Hence there is a point $x_0 \in T$ and is the minimum real number r_0 such that $0 < r_0 < d$ and $hull(SSL) \subseteq \overline{B}(x_0, r_0)$. At the next step, since the number r_0 is the minimal number such that $r_0 = inf\{sup\{\|y-x_0\| : y \in hull(SSL)\} : x_0 \in T\}$, it is obvious that x_0 can be chosen so that $x_0 \in hull(SSL)$. Then we have $hull(SSL) \cap B(x_0, r_0) \neq \emptyset$ and $SSL \subseteq \overline{B}(x_0, r_0)$. Moreover, by taking $G = B(x_0, \epsilon_0 - r_0)$, we have

$$\begin{aligned} \overline{G} = \overline{B}(x_0, \epsilon_0 - r_0) &= \bigcap_{\alpha \in \overline{B}(x_0, r_0)} \overline{B}(\alpha, \epsilon_0) \\ &\subseteq \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0) \\ &= \boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n \end{aligned}$$

which completes the proof. □

Corollary 2.7. *If $D \subseteq R^m$ satisfies $\bigcup_{b \in D} \overline{B}(b, \left\{1 - \frac{\sqrt{3}}{2}\right\} \epsilon_0) = R^m$ then D is ϵ_0-complete.*

Proof. At first, assume that $\epsilon_0 = 0$ and let any 0-Cauchy sequence $\{x_n\}$ be given. Then any single-valued subsequence of $\{x_n\}$ is a Cauchy sequence in the usual sense. Since R^m is complete in the usual sense and $\{x_n\}$ is a 0-Cauchy sequence, the set of all the subsequential limits $SSL(\{x_n\})$ must be a singleton. Thus $\{x_n\}$ is a 0-convergent sequence. Now suppose that $\epsilon_0 > 0$ and any ϵ_0-Cauchy sequence $\{x_n\}$ in D be given. If $hull(SSL) = \{\alpha\}$ is a singleton, then the ϵ_0-limit of $\{x_n\}$ is $\overline{B}(\alpha, \epsilon_0)$ which implies that the sequence $\{x_n\}$ is ϵ_0-convergent. Suppose that $hull(SSL)$ is not a singleton. At the first step, we will show that the minimum r_0 in the theorem just above satisfies the inequality $r_0 \leq \frac{\sqrt{3}}{2}d$ if the diameter of $hull(SSL(x_n))$ is d for an ϵ_0-Cauchy sequence $\{x_n\}$ in D. Since $hull(SSL)$ is not a singleton, there are two distinct elements $x_0, y_0 \in hull(SSL)$ such that $\|x_0 - y_0\| = d$ since $hull(SSL)$ is compact. By an appropriate rotation and translation of the axes and the origin in the usual Euclidean coordinate system of R^m, we may assume that $x_0 = (-\frac{d}{2}, 0, \cdots, 0)$, $y_0 = (\frac{d}{2}, 0, \cdots, 0)$ and $\frac{x_0+y_0}{2} = (0, 0, \cdots, 0)$. Then we must have

$$hull(SSL) \subseteq \overline{B}(x_0, d) \cap \overline{B}(y_0, d)$$

since $diam(hull(SSL)) = d$. But the equation of the most far boundary from the origin of the intersection of the boundaries $\partial\overline{B}(x_0, d)$ and $\partial\overline{B}(y_0, d)$ is given by

$$(x_1 - \frac{d}{2})^2 + x_2^2 + \cdots + x_m^2 = d^2 = (x_1 + \frac{d}{2})^2 + x_2^2 + \cdots + x_m^2.$$

That is, we have

$$x_1 = 0,\ x_2^2 + \cdots + x_m^2 = \frac{3}{4}d^2.$$

Thus the distance between the origin and the boundary of the intersection $\overline{B}(x_0, d) \cap \overline{B}(y_0, d)$ satisfies the inequality

$$dist(0, \partial\left\{\overline{B}(x_0, d) \cap \overline{B}(y_0, d)\right\}) \leq \frac{\sqrt{3}}{2}d.$$

Hence $hull(SSL)$ is contained in the closed ball with the radius $\frac{\sqrt{3}}{2}d$. Then, by the theorem just above, there is a point $x \in hull(SSL)$ and exists a real number $r_0 \leq \frac{\sqrt{3}}{2}d$ such that $\overline{B}(x, \epsilon_0 - r_0) \subseteq \boxed{\epsilon_0 - \lim_{n\longrightarrow\infty}} x_n$. But we have

$$\epsilon_0 - r_0 \geq \epsilon_0 - \frac{\sqrt{3}}{2}d \geq \epsilon_0 - \frac{\sqrt{3}}{2}\epsilon_0 = (1 - \frac{\sqrt{3}}{2})\epsilon_0.$$

Since this inequality implies that $\overline{B}(x,(1-\frac{\sqrt{3}}{2})\epsilon_0) \subseteq \overline{B}(x,\epsilon_0 - r_0)$, we have $D \cap \overline{B}(x,\epsilon_0 - r_0) \neq \emptyset$ which implies that $\{x_n\}$ is an ϵ_0-convergent sequence. Therefore, D is ϵ_0-complete. □

Note that if V is a normed linear space which satisfies the Heine-Borel property and $\epsilon_0 > 0$, then any dense subset D of V in the usual sense is ϵ_0-complete since $D \cap \overline{B}(x,r) \neq \emptyset$ for all $x \in V$ and all $r > 0$.

THEOREM 2.8. *Let V be a normed linear space which satisfies the Heine-Borel property. Then any closed subset D of V is ϵ_0-complete for all $\epsilon_0 \geq 0$.*

Proof. Suppose that D is a closed subset of V and let any ϵ_0-Cauchy sequence $\{x_n\} \subseteq D$ be given. By corollary 2.4, we have

$$SSL \subseteq \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n.$$

But the set $SSL(\{x_n\}) \neq \emptyset$ since $\{x_n\}$ is ultimately bounded. Since $SSL \subseteq \overline{D}$, this implies that

$$\emptyset \neq SSL \subseteq \overline{D} \cap \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n.$$

But we have $\overline{D} = D$ since D is closed. Thus D is ϵ_0-complete for all $\epsilon_0 \geq 0$. □

COROLLARY 2.9. *Let V be a normed linear space which satisfies the Heine-Borel property. Let $D \neq \emptyset$ be a subset of V and a real number $\epsilon_0 \geq 0$ be given. If D is ϵ_0-complete then $\overline{D}$ is ϵ_0-complete. But the converse is not true in general.*

Proof. By the theorem just above, it is clear that $\overline{D}$ is ϵ_0-complete. Now consider the subset D of R given by

$$D = \{-\frac{1}{n}, 1 + \frac{1}{n} : n \in N\}.$$

Then $\overline{D} = D \cup \{0,1\}$ is 1-complete since it is closed. But if we choose a sequence $\{x_n\}$ such that $x_{2n} = -\frac{1}{2n}$ and $x_{2n-1} = 1 + \frac{1}{2n-1}$ for each $n \in N$ then $SSL(\{x_n\}) = \{0,1\}$. Hence we have

$$\underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} x_n = \bigcap_{\alpha \in \{0,1\}} \overline{B}(\alpha, 1) = [0,1].$$

Since $D \cap [0,1] = \emptyset$, D is not 1-complete. □

Theorem 2.10. *Let V be a normed linear space which satisfies the Heine-Borel property. Then any convex subset D of V is ϵ_0-complete for all $\epsilon_0 > 0$.*

Proof. Suppose that D is a convex subset of V. Since $\emptyset$ is ϵ_0-complete, we may assume that $D \neq \emptyset$. And let any ϵ_0-Cauchy sequence $\{x_n\} \subseteq D$ be given. Since $\{x_n\}$ is also an ϵ_0-Cauchy sequence in $\overline{D}$ which is ϵ_0-complete by theorem 2.8, we have

$$\emptyset \neq hull(SSL) \subseteq \overline{D} \cap \boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n = \overline{D} \cap \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$$

since $\overline{D}$ is also convex. If $D \cap hull(SSL) \neq \emptyset$ then we are done since the intersection of D and the ϵ_0-limit of $\{x_n\}$ is not an empty set. Now suppose that $D \cap hull(SSL) = \emptyset$. Then $hull(SSL)$ is a subset of the derived set D', the set of all the accumulation points of D. That is, it is a subset of the set $D' - D$. By the theorem 2.6, there is an open convex subset G of V such that

$$hull(SSL) \cap G \neq \emptyset \quad \text{and } \overline{G} \subseteq \boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n.$$

Choose a point $\alpha \in hull(SSL) \cap G$. Then $\alpha \in D' - D$ and $\alpha \in G$. Since G is an open set containing the accumulation point α of D, there is a point $\beta \in D$ such that $\beta \in G$ and $\beta \neq \alpha$. Then

$$\beta \in D \cap G \subseteq D \cap \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0).$$

Thus $D \cap \boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n \neq \emptyset$ which completes the proof. □

Note that the convex subset of V is not 0-complete in general.

Proposition 2.11. *(1) The union of the ϵ_0-complete subsets does not need to be ϵ_0-complete. (2) The intersection of the ϵ_0-complete subsets does not need to be ϵ_0-complete.*

Proof. (1) Let $D_1 = \{-\frac{1}{n} : n \in N\}$ and $D_2 = \{1+\frac{1}{n} : n \in N\}$. In order to prove that D_1 is 1-complete, let any 1-Cauchy sequence $\{x_n\} \subseteq D_1$ be given. Then $SSL(\{x_n\}) \neq \emptyset$ and $SSL \subseteq D_1 \cup \{0\}$. Hence we have

$$[-1, 0] \subseteq \bigcap_{\alpha \in D_1 \cup \{0\}} \overline{B}(\alpha, 1) \subseteq \bigcap_{\alpha \in SSL} \overline{B}(\alpha, 1) = \boxed{1 - \lim_{n \longrightarrow \infty}} x_n.$$

Thus the intersection of D_1 and the 1-limit of $\{x_n\}$ is not an empty set. Hence D_1 is 1-complete. Since the diameter of D_2 is 1, we can prove by the same method that D_2 is also 1-complete. But the union

$$D_1 \cup D_2 = \{-\frac{1}{n}, 1 + \frac{1}{n} : n \in N\}$$

is not 1-complete as in the proof of corollary 2.9. (2) Let $D_1 = \{-\frac{1}{n}, 0, 1+\frac{1}{n} : n \in N\}$ and $D_2 = \{-\frac{1}{n}, 1, 1 + \frac{1}{n} : n \in N\}$. In order to prove that D_1 is 1-complete, let any 1-Cauchy sequence $\{x_n\} \subseteq D_1$ be given. Since the diameter of SSL satisfies the inequality $Diam(SSL) \leq 1$, the following three cases occur.

$$\begin{aligned} &(i) \quad \emptyset \neq SSL = \{0, 1\}, \\ &(ii) \quad \emptyset \neq SSL \subseteq \{-\frac{1}{n}, 0 : n \in N\}, \\ &(iii) \quad \emptyset \neq SSL \subseteq \{1 + \frac{1}{n}, 1 : n \in N\}. \end{aligned}$$

(i) If $SSL = \{0, 1\}$ then $D_1 \cap \boxed{1 - \lim_{n \longrightarrow \infty}} x_n = D_1 \cap [0, 1] = \{0\} \neq \emptyset$. (ii) If $SSL \subseteq \{-\frac{1}{n}, 0 : n \in N\}$ then $D_1 \cap \boxed{1 - \lim_{n \longrightarrow \infty}} x_n \supseteq \{-\frac{1}{n}, 0 : n \in N\} \neq \emptyset$. (iii) If $SSL \subseteq \{1 + \frac{1}{n}, 1 : n \in N\}$ then $D_1 \cap \boxed{1 - \lim_{n \longrightarrow \infty}} x_n \supseteq \{1 + \frac{1}{n} : n \in N\} \neq \emptyset$. Therefore, D_1 is 1-complete. On the other hand, we can prove by the same method that D_2 is also 1-complete. But the intersection

$$D_1 \cap D_2 = \{-\frac{1}{n}, 1 + \frac{1}{n} : n \in N\}$$

is not 1-complete as in the proof of (1). □

PROPOSITION 2.12. *Let V be a normed linear space which satisfies the Heine-Borel property and let $\epsilon_0 > 0$ be a positive real number. If a subset D of V is not ϵ_0-complete then there is an ϵ_0-Cauchy sequence $\{x_p\}$ such that $hull(SSL) \cap B(\gamma, r) \neq \emptyset$, $SSL \cap B(\gamma, r) = \emptyset$ and $diam(SSL) = \epsilon_0$ for some $\gamma \in V$ and some positive real number $r > 0$. Moreover, SSL satisfies the following condition.*

$$\forall \alpha \in SSL, \exists \beta \in SSL \ s.t. \ \|\alpha - \beta\| = \epsilon_0.$$

Proof. Suppose that D is not ϵ_0-complete. Then there is an ϵ_0-Cauchy sequence $\{x_p\}$ in D such that $D \cap \boxed{\epsilon_0 - \lim_{p \longrightarrow \infty}} x_p = \emptyset$. If $hull(SSL) \cap D \neq \emptyset$

then we have

$$\emptyset \neq D \cap hull(SSL) \subseteq D \cap \{\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)\} \subseteq D \cap \underset{p \longrightarrow \infty}{\boxed{\epsilon_0 - \lim}} x_p.$$

This is a contradiction. Hence $hull(SSL) \cap D = \emptyset$ and $SSL \subseteq D' - D$ since $SSL \subseteq \overline{D}$. On the other hand, there is an element γ and is a real number $r > 0$ by theorem 2.6 such that

$$hull(SSL) \cap B(\gamma, r) \neq \emptyset \text{ and } \overline{B}(\gamma, r) \subseteq \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0).$$

It is obvious that $D \cap B(\gamma, r) = \emptyset$. And if $SSL \cap B(\gamma, r) \neq \emptyset$ then there exists an element $\alpha_0 \in SSL \subseteq D' - D$ such that $\alpha_0 \in B(\gamma, r)$. Since α_0 is an accumulation point of D and $B(\gamma, r)$ is an open set, there exists an element $x \in D$ such that $x \in B(\gamma, r)$. Hence we have $D \cap \{\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)\} \neq \emptyset$ which is a contradiction. Hence we have $SSL \cap B(\gamma, r) = \emptyset$. Now suppose that there is an element $\alpha_0 \in SSL$ such that $\|\alpha_0 - \beta\| < \epsilon_0$ for all elements $\beta \in SSL$. Then we have

$$\max\{\|\alpha_0 - \beta\| : \beta \in SSL\} = r_0 < \epsilon_0$$

since SSL is compact. Then we have

$$\alpha_0 \in B(\alpha_0, \epsilon_0 - r_0) \subseteq \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0).$$

Since $\alpha_0 \in D' - D$ and $B(\alpha_0, \epsilon_0 - r_0)$ is an open set containing α_0, we have $D \cap B(\alpha_0, \epsilon_0 - r_0) \neq \emptyset$. This is a contradiction as the above. Since the diameter of SSL is not greater than ϵ_0, this contradiction implies that

$$\forall \alpha \in SSL, \exists \beta \in SSL \text{ s.t. } \|\alpha - \beta\| = \epsilon_0$$

and $diam(SSL) = \epsilon_0$. □

THEOREM 2.13. *Let D be a non-empty subset of a normed linear space V which satisfies the Heine-Borel property and let $\epsilon_0 > 0$. Then D is not ϵ_0-complete if and only if there is a compact subset S of $D' - D$ such that $diam(S) = \epsilon_0$ and $D \cap \{\bigcap_{\alpha \in S} \overline{B}(\alpha, \epsilon_0)\} = \emptyset$.*

Proof. (⇒) Suppose that D is not ϵ_0-complete. Then we have an ϵ_0-Cauchy sequence $\{x_p\}$ such that $D \cap \underset{p \longrightarrow \infty}{\boxed{\epsilon_0 - \lim}} x_p = \emptyset$. As in the proof of the proposition just above, we have $SSL(\{x_p\}) \subseteq D' - D$ and $diam[SSL] = \epsilon_0$. Now put $S = SSL(\{x_p\})$. Then S is compact by lemma 1.4. And $diam(S) = \epsilon_0$ and $S \subseteq D' - D$ as in the proof of the proposition just above. Moreover,

$$D \cap \{\bigcap_{\alpha \in S} \overline{B}(\alpha, \epsilon_0)\} = D \cap \{\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)\} = \emptyset$$

since $\bigcap_{\alpha \in SSL(\{x_p\})} \overline{B}(\alpha, \epsilon_0) = \boxed{\epsilon_0 - \lim}_{p \longrightarrow \infty} x_p$. ($\Leftarrow$) Suppose that there exists a compact subset S of $D' - D$ such that $D \cap \{\bigcap_{\alpha \in S} \overline{B}(\alpha, \epsilon_0)\} = \emptyset$ and $diam(S) = \epsilon_0$. Since $S \subseteq D' - D$, for each $\alpha \in S$, there is a single-valued sequence $\{x_{\alpha_p}\}$ in D such that $\|x_{\alpha_p} - \alpha\| < \frac{1}{p}$ for each $p \in N$. In order to verify that D is not ϵ_0-complete, let's choose a multi-valued sequence $\{x_p\}$ so that $x_p = \{x_{\alpha_p} : \alpha \in S\}$ for each $p \in N$. In order to show that $\{x_p\}$ is an ϵ_0-Cauchy sequence, let any positive number $\epsilon > \epsilon_0$ be given. Choosing a natural number $K \in N$ so large that $K > \frac{2}{\epsilon - \epsilon_0}$, we have, since $\|\alpha - \beta\| \leq \epsilon_0$ for all $\alpha, \beta \in S$,

$$\begin{aligned}
\forall \epsilon > \epsilon_0, \quad \exists \quad & K \in N \text{ s.t. } (\forall p, q) p, q \geq K, \forall x_{\alpha_p} \in x_p, \forall x_{\beta_q} \in x_q \\
\Rightarrow \quad & \|x_{\alpha_p} - x_{\beta_q}\| \leq \|x_{\alpha_p} - \alpha\| + \|\alpha - \beta\| + \|\beta - x_{\beta_q}\| \\
\leq \quad & \frac{1}{p} + \epsilon_0 + \frac{1}{q} \leq \frac{2}{K} + \epsilon_0 \\
< \quad & \epsilon - \epsilon_0 + \epsilon_0 = \epsilon.
\end{aligned}$$

Thus the sequence $\{x_p\}$ is an ϵ_0-Cauchy sequence in D. Since the limit of the subsequential limits is also a subsequential limit, we have $SSL(\{x_p\}) = \overline{S}$. But $\overline{S} = S$ since S is closed. Thus $SSL(\{x_p\}) = S$. Finally, by the assumption, we have

$$D \cap \{\bigcap_{\alpha \in SSL(\{x_p\})} \overline{B}(\alpha, \epsilon_0)\} = D \cap \{\bigcap_{\alpha \in S} \overline{B}(\alpha, \epsilon_0)\} = \emptyset$$

Consequently, D is not ϵ_0-complete. □

Proposition 2.14. *(Criterion) Let V, W be two normed linear spaces such that both V and W satisfy the Heine-Borel property. Let $f : D \to W$ be a multi-valued function defined on a bounded subset D of V. Then f is ϵ_0-uniformly continuous on D if and only if $\{f(x_p)\}$ is an ϵ_0-Cauchy sequence in W for every 0-Cauchy sequence $\{x_p\}$ on D.*

Proof. ($\Rightarrow$) Suppose that f is ϵ_0-uniformly continuous on D and any 0-Cauchy sequence $\{x_n\}$ on D be given. Then we have

$$\begin{aligned}
\forall \epsilon > \epsilon_0, \exists \delta > 0 \quad \text{s.t.} \quad & (\forall x, y \in D)\|x - y\| < \delta, \forall f(x), \forall f(y) \\
\Rightarrow \quad & \|f(x) - f(y)\| < \epsilon.
\end{aligned}$$

Since $\{x_n\}$ is a 0-Cauchy sequence, we have

$$\exists K \in N, \text{s.t.}(\forall p, q \in N) p, q \geq K, \forall x_p, \forall x_q \Rightarrow \|x_p - x_q\| < \delta.$$

Thus we have

$$\begin{aligned} \forall \epsilon > \epsilon_0, \exists K \in N \quad & \text{s.t.} \quad (\forall p, q \in N) p, q \geq K, \forall f(x_p), \forall f(x_q) \\ & \Rightarrow \quad \|f(x_p) - f(x_q)\| < \epsilon. \end{aligned}$$

Therefore, $\{f(x_p)\}$ is an ϵ_0-Cauchy sequence in W. ($\Leftarrow$) Suppose that f is not ϵ_0-uniformly continuous on D. Then we have

$$\begin{aligned} \exists \epsilon_1 > \epsilon_0 \quad \text{s.t.} \quad & \{\forall \delta > 0, \exists x_\delta, y_\delta \in D, \exists f(x_\delta), f(y_\delta) \in W \\ \text{s.t.} \quad & \|x_\delta - y_\delta\| < \delta, \|f(x_\delta) - f(y_\delta)\| \geq \epsilon_1\}. \end{aligned}$$

Choosing $\delta = \frac{1}{p}$ for each natural number $p \in N$, we have

$$\begin{aligned} \exists \{x_p\}, \{y_p\} \subseteq D \quad \wedge \quad & \exists \{f(x_p)\}, \{f(y_p)\} \subseteq W \\ & \text{s.t.} \quad \|x_p - y_p\| < \frac{1}{p} \wedge \|f(x_p) - f(y_p)\| \geq \epsilon_1. \end{aligned}$$

Since $\{x_p\}$ and $\{y_p\}$ are bounded sequences in a bounded subset D and the closure $\overline{D}$ is compact, we may assume that $\lim_{p\to\infty} x_p = \lim_{p\to\infty} y_p = \alpha$ for some $\alpha \in \overline{D}$ by choosing single-valued and convergent subsequences. Let's define a sequence $\{z_p\}$ by $z_{2p-1} = x_p$ and $z_{2p} = y_p$ for each natural number $p \in N$. Then $\lim_{p\to\infty} z_p = \alpha$ and $\{z_p\}$ is a 0-Cauchy sequence in D. But we have

$$\|f(z_{2p-1}) - f(z_{2p})\| = \|f(x_p) - f(y_p)\| \geq \epsilon_1$$

for all $p \in N$. Hence $\{f(z_p)\}$ is not an ϵ_0-Cauchy sequence. This is a contradiction which completes the proof. □

THEOREM 2.15. *Let V, W be two normed linear spaces such that both V and W satisfy the Heine-Borel property. And let $f : D \to W$ be a multi-valued function defined on a 0- complete subset D of V. If f is ϵ_0-uniformly continuous on D then, for every 0-Cauchy sequence $\{x_p\}$ on D, there is an element $\alpha \in D$ such that $\{f(x_p)\}$ ϵ_0-converges to $f(\alpha) \in f(D)$.*

Proof. Let any 0-Cauchy sequence $\{x_p\}$ on D be given. Since $f(x)$ is ϵ_0-uniformly continuous on D, we have

$$\begin{aligned} \forall \epsilon > \epsilon_0, \exists \delta > 0 \quad & \text{s.t.} \quad (\forall x, y \in D) \|x - y\| < \delta, \forall f(x), \forall f(y) \\ & \Rightarrow \quad \|f(x) - f(y)\| < \epsilon. \end{aligned}$$

But we have $\underset{p\longrightarrow\infty}{\boxed{0-\lim}}x_p = \{\alpha\}$ for some $\alpha \in D$ since D is 0-complete. Hence we have

$$\exists K \in N \text{ s.t. } \forall p \geq K, \forall x_p \Rightarrow \|x_p - \alpha\| < \delta.$$

Hence we have

$$\begin{aligned} \forall \epsilon > \epsilon_0, \exists K \in N \quad &\text{s.t.} \quad \forall p \geq K, \forall f(x_p), \forall f(\alpha) \\ &\Rightarrow \quad \|f(x_p) - f(\alpha)\| < \epsilon. \end{aligned}$$

Thus we have $f(\alpha) \in \underset{p\longrightarrow\infty}{\boxed{0-\lim}} f(x_p)$ for all values of $f(\alpha)$. Since $f(\alpha) \in f(D)$ for all values of $f(\alpha)$, the sequence $\{f(x_p)\}$ is an ϵ_0-convergent sequence of $f(D)$. $\square$

Now we introduce a concept of the generalized Banach spaces.

Definition 2.16. Let $\epsilon_0 \geq 0$ be a non-negative real number. A linear space V on a field F is called the ϵ_0-Banach space if and only if V is an ϵ_0-complete normed linear space.

Proposition 2.17. *Let V be a real normed linear space which satisfies the Heine-Borel property. Then V is the ϵ_0-Banach space for all real number $\epsilon_0 \geq 0$.*

Proof. Let any ϵ_0-Cauchy sequence $\{x_n\}$ in V be given. Then we have

$$\forall \epsilon > \epsilon_0, \exists K \in N \text{ such that } \forall m, n \geq K, \forall x_m, x_n \Rightarrow \|x_m - x_n\| < \epsilon.$$

Since $\{x_n\}$ is ultimately bounded, the set SSL of all the subsequential limits of $\{x_n\}$ is not empty and compact. Hence, by lemma 2.3,

$$\emptyset \neq SSL \subseteq \underset{n\longrightarrow\infty}{\boxed{\epsilon_0-\lim}} x_n.$$

Hence V is ϵ_0-complete which completes the proof. $\square$

Theorem 2.18. *Let V be a real normed linear space which satisfies the Heine-Borel property. Then any linear subspace W of V is the ϵ_0-Banach space for all real number $\epsilon_0 > 0$.*

Proof. Any linear subspace W is a convex subset of V. By the theorem 2.10, W is ϵ_0-complete. Hence W is also an ϵ_0-Banach space for all real number $\epsilon_0 > 0$. $\square$

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