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LIFTING OF THE UNRAMIFIED IWASAWA MODULE

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Abstract. We give necessary and sufficient condition for the Greenberg's generalized conjecture conjecture on certain imaginary quadratic fields.

1. Introduction

Let p be a prime number, and k a number field. Write $[k:\mathbb{Q}] = r_1 + 2r_2$, where r_1 and r_2 is the number of real and complex embeddings of k, respectively. Suppose that K is a $\mathbb{Z}_p^d$-extension with $d \geq 1$ of k, so $K = \cup_{n\geq 0} k_n$ such that $k_0 = k, k_n \subset k_{n+1}$ and $Gal(k_n/k) \simeq (\mathbb{Z}/p^n\mathbb{Z})^d$. Denote by L_n the p-Hilbert class field of k_n. Then $Gal(L_n/k_n) \simeq A_n$ by Artin map, where A_n is the Sylow p-subgroup of the ideal class group of k_n. Let L_K be the maximal unramified abelian p-extension of K. Then $Y_K := Gal(L_K/K) = \varprojlim Gal(L_n/k_n) \simeq \varprojlim A_n$. It is known that the Iwasawa module Y_K is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$-module on which $\Gamma := Gal(K/k) \simeq \mathbb{Z}_p^d$ acts by inner automorphisms. For a $\mathbb{Z}_p$-basis $\{\gamma_1, \cdots, \gamma_d\}$ for Γ, the maps

$$\gamma_i \to 1 + T_i$$

extend to an isomorphism $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T_1, \cdots, T_d]]$. Note that $\Lambda_d := \mathbb{Z}_p[[T_1, \cdots, T_d]]$ is a unique factorization domain.

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A finitely generated torsion Λ_d-module M is called pseudo-null if M has two relatively prime annihilators in Λ_d. When $d = 1$ pseudo-nullity of M is equivalent to finiteness of M. A structure theorem of Iwasawa shows that there is a map from a finitely generated torsion Λ_d-module M to a unique module of the form $\oplus_i \Lambda_d/(f_i^{e_i})(f_i \in \Lambda_d$ irreducible) with pseudo-null kernel and cokernel. We call $f_M := \prod_i f_i^{e_i}$ to be the characteristic power series of M.

It is Greenberg's generalized conjecture that Y_K is pseudo-null. When k is a totally real abelian extension of $\mathbb{Q}$, $d = 1$ and K is the cyclotomic $\mathbb{Z}_p$-extension k_∞^c of k. In this case, it is called Greenberg's conjecture, so Greenberg's conjecture implies that Y_K is finite. Many authors proved that Greenberg's conjecture is true for certain real quadratic fields and for some p. However, not much has been done since Minardi in his thesis [2] proved Greenberg's generalized conjecture in some cases.

2. Proof of Theorems

Let K be the compositum of all $\mathbb{Z}_p$ of k. By class field theory, we see that $Gal(K/k) \simeq \mathbb{Z}_p^d$, with $r_2 + 1 \leq d$. Leopoldt's conjecture is that $d = r_2 + 1$. It is known that Leopoldt's conjecture is true for any prime p if k is an abelian extension of $\mathbb{Q}$.

From now on k is an imaginary quadratic extension of $\mathbb{Q}$. Hence $d = 2$. In this case, K is the compositum of the cyclotomic $\mathbb{Z}_p$-extension k_∞^c of k and the anti-cyclotomic $\mathbb{Z}_p$-extension k_∞^a of k. The anti-cyclotomic $\mathbb{Z}_p$-extension of k is the $\mathbb{Z}_p$-extension of k on which the complex conjugation acts inversely. If p is an odd prime, then $k_\infty^c \cap k_\infty^a = k$. Let γ_1, γ_2 be topological generators of $Gal(k_\infty^c/k), Gal(k_\infty^a/k)$ with $\gamma_1 = 1 + S, \gamma_2 = 1 + T$, respectively.

Minardi in his thesis [2] proved the Greenberg's generalized conjecture in some cases.

THEOREM 2.1. *Let k be an imaginary quadratic field. If p does not divide h_k, then Y_K is pseudo-null.*

Now we give some conditions under which Y_K is pseudo-null for certain imaginary quadratic fields. Define $Ker_T(Y_K)$ to be the submodule of Y_K killed by T. Since S and T commute each other, $Ker_T(Y_K)$ is a $\mathbb{Z}_p[[S]]$-module.

THEOREM 2.2. *Let p be an odd prime number and k an imaginary quadratic field with $A_k = \mathbb{Z}/p\mathbb{Z}$. Assume that only one prime $\mathfrak{p}$ of k lies above p and $\mathfrak{p}$ totally ramifies in K/k. Moreover, assume that the Iwasawa lambda invariant $\lambda_p(k^c_\infty/k) \geq 1$. Then we have*

$$Y_K \text{ is pseudo-null if and ony if } Ker_T(Y_K) \text{ is infinite.}$$

Proof. Since $A_k \simeq \mathbb{Z}/p$ and only one prime $\mathfrak{p}$ of k above p ramifies, $Y_{k^c_\infty}/SY_{k^c_\infty} \simeq \mathbb{Z}/p$. Hence we see that $Y_{k^c_\infty} = \mathbb{Z}_p[[S]]/\mathfrak{U}$ for some ideal $\mathfrak{U} \subset \mathbb{Z}_p[[S]]$ by Nakayama lemma. Moreover we see that $f(0) = p$ for some $f(S) \in \mathfrak{U}$, hence $\mathfrak{U}$ is generated by $f(S)$ because $f(S)$ is irreducible and $Y_{k^c_\infty}$ is not finite. Therefore $\mathfrak{U} = (f(S))$ for some irreducible $f(S) \in \mathbb{Z}_p[[S]]$. Again, by the condition,

$$Y_K/TY_K \simeq Y_{k^c_\infty} = \mathbb{Z}_p[[S]]/(f(S)) \tag{1}$$

By Nakayama lemma again, we see that

$$Y_K = \mathbb{Z}_p[[S,T]]/\mathfrak{I}. \tag{2}$$

for some ideal $\mathfrak{I} \in \mathbb{Z}_p[[S,T]]$. First suppose that Y_K is not pseudo-null. By (1) and (2), $\mathfrak{I} + T\mathbb{Z}_p[[S,T]] = (T, f(S))$. Thus there is an element $F(S,T)$ of $\mathfrak{I}$ such that $F(S,0) = f(S)$. Since $F(0,0) = f(0) = p$, $F(S,T)$ is irreducible. So every element of $\mathfrak{I}$ is divisible by $F(S,T)$ by the assumption of the non pseudo-nullity of Y_K. Hence $\mathfrak{I} = (F(S,T))$, where $F(S,T)$ is not a unit in $\mathbb{Z}_p[[S,T]]$. Note that $(f_{Y_K}(S,T)) = (F(S,T))$. By Perrin-Riou's formula, we have the following equation(see [2]).

$$f_{Y_K}(S,0)f_{Ker_T(Y_K)} = f(S)u(S),$$

where $u(S)$ is a unit in $\mathbb{Z}_p[[S]]$. Since $f(S)$ is irreducible and $f_{Y_K}(S,0)$ is not a unit, $f_{Ker_T(Y_K)}$ is a unit. Hence $Ker_T(Y_K)$ is finite.
Conversely, suppose that $Ker_T(Y_K)$ is finite. Then $f_{Ker_T(Y_K)}$ is a unit. Hence $f_{Y_K}(S,0)$ is irreducible, so $f_{Y_K}(S,T)$ is irreducible. Therefore Y_K is not pseudo-null. This completes the proof. □

REMARK 1. We give an example of an imaginary quadratic field k which satisfies the assumptions in Theorem 2.2. Let $k = \mathbb{Q}(\sqrt{-331})$. Then we see that the class number of k is three, $\lambda_3(k) = 1$(See [1]), and $p(= 3)$ stays prime in k. Note also that the class number of $\mathbb{Q}(\sqrt{993})$ is three. Hence it follows from the following Theorem 2.3 that the field $\mathbb{Q}(\sqrt{-331})$ satisfies all the conditions of Theorem 2.2.

THEOREM 2.3. [3, Theorem2]. *Let $d \not\equiv 3 mod 9$ be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field and K the compositum of all $\mathbb{Z}_3$-extension over k. Then*

$$H_k \cap K = k \Longleftrightarrow rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} = rank_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}$$

REMARK 2. Note that $KL_{k^c_\infty}$ is contained in L_K. So if $Gal(KL_{k^c_\infty}/K)$ is not a quotient of Y_K, but a subgroup of Y_K, then $Gal(KL_{k^c_\infty}/K) \subset Ker_T(Y_K)$, hence $Ker_T(Y_K)$ is infinite.

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