J. Appl. Math. & Informatics Vol. **37**(2019), No. 5 - 6, pp. 481 - 489
https://doi.org/10.14317/jami.2019.481

EXISTENCE OF PICARD-JUNGCK OPERATOR USING C_G-SIMULATION FUNCTIONS IN GENERALIZED METRIC SPACES[†]

SUMIT CHANDOK

Abstract. *In this manuscript, we provide some new results with short proofs for the existence of Picard-Jungck operators in the framework of generalized metric spaces using C_G-simulation functions. An example is also provided to illustrate the usability of the results.*

AMS Mathematics Subject Classification : 47H10, 54H25, 46Nxx.
Key words and phrases : Point of coincidence; Common fixed point; Simulation function; C-class function; Property C_G.

1. INTRODUCTION AND PRELIMINARIES

The study of the Picard operators is similar to the study of contractive-type mappings in the setting of metric spaces. It is easy to see that almost all contractive-type mappings on a complete metric space are Picard operators. In the present paper, we propose a class of Picard-Jungck operators for a pair of mappings on generalized metric spaces in the sense of Branciari [5] by taking into account of the C_G-simulation functions. Also some new results for the existence of such operators for a pair of self mappings in the setting of metric spaces are obtained. A nontrivial example is also provided to show the usability of the results. The result proved here are short and generalize many known results existing in the literature. For different variants of simulation function we can obtain very interesting results.

To begin with, we have the following definitions, notations and results which will be used in the sequel.

Definition 1.1. Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ be a map such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them is different from x and y:

Received April 16, 2019. Revised September 5, 2019. Accepted September 6, 2019.
[†]This work was supported by the AISTDF/DST research grant CRD/2018/000017.

(d1) $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.
Then d is called a generalized metric on X and (X,d) is called a generalized metric space.

Let (X,d) be a generalized metric space, let $\{x_n\} \subset X$ be a sequence, and $x \in X$. Then we say that

(a) $\{x_n\}$ is convergent to x (denoted by $\lim_{n\to\infty} x_n = x$) if and only if $\lim_{n\to\infty} d(x_n, x) = 0$;
(b) $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;
(c) (X,d) is complete if and only if every Cauchy sequence in X is convergent to some point in X.
(d) a mapping $T : X \to X$ is continuous at $x \in X$ if, for any $V \in \tau$ containing Tx, there exists $U \in \tau$ containing x such that $TU \subset V$, where τ is the topology on X induced by the generalized metric d. That is,

$$\tau = \{U \subset X : \forall x \in U, \exists B \in \beta, x \in B \subset U\},$$
$$\beta = \{B(x,r) : x \in X, \forall r > 0\},$$
$$B(x,r) = \{y \in X : d(x,y) < r\}.$$

If T is continuous at each point $x \in X$, then it is called continuous.

Note that T is continuous if and only if it is sequentially continuous, i.e., $\lim_{n\to\infty} d(Tx_n, Tx) = 0$ for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.

If d is a generalized metric on X, then it is not continuous in each coordinate.

Now, here we define the C-class function as follows:

Definition 1.2. A mapping $G : [0,+\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies $G(s,t) \leq s$ for all $s,t \in [0,+\infty)$.

Definition 1.3. A mapping $G : [0,+\infty)^2 \to \mathbb{R}$ has the property C_G, if there exists an $C_G \geq 0$ such that

(C_G1) $G(s,t) > C_G$ implies $s > t$;
(C_G2) $G(t,t) \leq C_G$, for all $t \in [0,+\infty)$.

Some examples of C-class functions that have property C_G are as follows:
a) $G(s,t) = s - t$, $C_G = r$, $r \in [0,+\infty)$;
b) $G(s,t) = s - \frac{(2+t)t}{1+t}$, $C_G = 0$;
c) $G(s,t) = \frac{s}{1+kt}$, $k \geq 1$, $C_G = \frac{r}{1+k}$, $r \geq 2$.
For more examples of C-class functions that have property C_G see [3, 13] .

Definition 1.4. (see [13]) We define $\mathcal{Z}_G$ as the family of all C_G-simulation functions $\zeta : [0,+\infty)^2 \to \mathbb{R}$ satisfying the following:

$(\mathcal{Z}_G1)$ $\zeta(t,s) < G(s,t)$ for all $t,s > 0$, where $G : [0,+\infty)^2 \to \mathbb{R}$ is a C-class function;

$(\mathcal{Z}_G2)$ if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < C_G$.

Some examples of simulation functions and C_G-simulation functions are:

d) $\zeta(t,s) = \frac{s}{s+1} - t$ for all $t, s \geq 0$.

e) $\zeta(t,s) = s - \varphi(s) - t$ for all $t, s \geq 0$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function and $\varphi(t) = 0$ if and only if $t = 0$.

For more examples of simulation functions and C_G-simulation functions see [3, 6, 11, 13, 14, 18].

Each simulation function as in manuscript [11] is also a C_G-simulation function as in Definition 1.4, but the converse is not true. For this claim see Example 3.3 of [6] using the C-class function $G(s,t) = s - t$. For examples of simulation functions and C_G-simulation functions see [3, 6, 9, 11, 12, 13, 14, 17, 18].

2. MAIN RESULTS

In this section, we establish some results on the existence and uniqueness of Picard-Jungck operator for a pair of mappings by using C_G-simulation functions in the framework of generalized metric spaces. We begin with the following definition.

Let (f, g) be pair of self mappings of a set X. Recall that if $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair (f, g) is weakly compatible if f and g commute at their coincidence points. A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq X$ is a Picard-Jungck sequence of the pair (f, g) (based on x_0) if $y_n = fx_n = f^n x_0 = gx_{n+1} = g^{n+1}x_0$ for all $n \in \mathbb{N}\cup\{0\}$ (see also [6, Definition 4.4]).

A pair (f, g) is said to be a *weakly Picard-Jungck operator* (WPJO) if it has a unique point of coincidence point $z \in X$ and $z = \lim_{n\to\infty} f^n u = g^{n+1}u$ for all $u \in X$.

A pair (f, g) is said to be a *Picard-Jungck operator* (PJO) if it has a unique common fixed point $u \in X$ and $u = \lim_{n\to\infty} f^n u = g^{n+1}u$ for all $u \in X$.

A self-mapping f is said to be a *Picard operator* if it has a unique fixed point $z \in M$ and $z = \lim_{n\to\infty} f^n u$ for all $u \in X$.

A self-mapping f is said to be a *weakly Picard operator* if it has a fixed point $z \in M$ and $z = \lim_{n\to\infty} f^n u$ for all $u \in X$.

Definition 2.1. Let Let (f, g) be a pair of self mappings on a generalized metric space (X, d). An operator f is called a (Z_G, g)-contraction if there exists $\zeta \in Z_G$ such that for all $x, y \in X$ with $d(fx, fy) > 0$, we have

$$\zeta(d(fx,fy), d(gx,gy)) \geq C_G. \tag{1}$$

Now to state our first new result for the notion of (Z_G, g)-contraction, we need the following result.

Lemma 2.2. Let (f, g) be a pair of self mappings on a generalized metric space (X, d) and f be a (Z_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Then the sequence $\{d(y_n, y_{n+1})\}$ is decreasing and $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$.

Proof. Suppose that there is a Picard-Jungck sequence $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$ where $n \in \mathbb{N} \cup \{0\}$. Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Substituting $x = x_{n+1}$, $y = x_{n+2}$ in (1) we obtain that

$$\begin{aligned} C_G &\leq \zeta\left(d\left(fx_{n+1}, fx_{n+2}\right), d\left(gx_{n+1}, gx_{n+2}\right)\right) = \zeta\left(d\left(y_{n+1}, y_{n+2}\right), d\left(y_n, y_{n+1}\right)\right) \\ &< G\left(d\left(y_n, y_{n+1}\right), d\left(y_{n+1}, y_{n+2}\right)\right). \end{aligned}$$

Using $(C_G 1)$ of Definition 1.3, we have $d\left(y_n, y_{n+1}\right) > d\left(y_{n+1}, y_{n+2}\right)$. Hence, for all $n \in \mathbb{N} \cup \{0\}$ we get that $d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right)$.

Further we have to prove that $y_n \neq y_m$ for $n \neq m$. Indeed, suppose that $y_n = y_m$ for some $n > m$. Then we choose $x_{n+1} = x_{m+1}$ (which is obviously possible by the definition of Picard-Jungck sequence $\{y_n\}$) and hence also $y_{n+1} = y_{m+1}$. Then following the previous arguments, we have

$$d\left(y_n, y_{n+1}\right) < d\left(y_{n-1}, y_n\right) < \cdots < d\left(y_m, y_{m+1}\right) = d\left(y_n, y_{n+1}\right),$$

which is a contradiction.

Therefore there exists $D \geq 0$ such that $\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = D \geq 0$. Suppose that $D > 0$. Since $d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right)$ and both $d\left(y_{n+1}, y_{n+2}\right)$ and $d\left(y_n, y_{n+1}\right)$ tend to D, using $\mathcal{Z}_G 2$ of Definition 1.4, we get

$$C_G \leq \limsup_{n\to\infty} \zeta\left(d\left(y_{n+1}, y_{n+2}\right), d\left(y_n, y_{n+1}\right)\right) < C_G,$$

which is a contradiction. Hence $\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = D = 0$. □

Lemma 2.3. Let (f, g) be a pair of self mappings on a generalized metric space (X, d) and f be a (Z_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Then the Picard-Jungck sequence $\{y_n\}$ is a Cauchy sequence.

Proof. Suppose that there is a Picard-Jungck sequence $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$ where $n \in \mathbb{N} \cup \{0\}$.

If $y_k = y_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 2.2, we have $d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = 0$.

Now, we have to show that $\{y_n\}$ is a Cauchy sequence. Consider a real sequence $S_n = \sup\{d(y_i, y_j) : i, j \geq n\}$. It is clear that $0 \leq S_{n+1} \leq S_n < \infty$ for all $n \in \mathbb{N}$. Hence there exists $s \geq 0$ such that $\lim\limits_{n\to\infty} S_n = s$. Assume that $s > 0$ then by the assumption of S_n, for each $k \in \mathbb{N}$ there exists $n(k), m(k)$ such that $m(k) > n(k) \geq k$ with $S_k - \frac{1}{k} < d(y_{m(k)}, y_{n(k)}) \leq S_k$. Hence

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = s. \tag{2}$$

Putting $x = x_{m(k)+1}$, $y = x_{n(k)+1}$ in (1), we obtain

$$\begin{aligned} C_G &\leq \zeta\left(d\left(fx_{m(k)+1}, fx_{n(k)+1}\right), d\left(gx_{m(k)+1}, gx_{n(k)+1}\right)\right) \\ &< G\left(d\left(gx_{m(k)+1}, gx_{n(k)+1}\right), d\left(fx_{m(k)+1}, fx_{n(k)+1}\right)\right) \end{aligned}$$

$$= G\left(d\left(y_{m(k)}, y_{n(k)}\right), d\left(y_{m(k)+1}, y_{n(k)+1}\right)\right) \tag{3}$$

Using (C_G1) of Definition 1.3, it follows that $d\left(y_{m(k)}, y_{n(k)}\right) > d(y_{m(k)+1}$, $y_{n(k)+1})$. Since $s > 0$, $\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = \lim_{k\to\infty} d(y_{m(k)+1}, y_{n(k)+1}) > 0$. Therefore, using (3), we have

$$C_G \leq \limsup_{n\to\infty} \zeta\left(d\left(y_{m(k)+1}, y_{n(k)+1}\right), d\left(y_{m(k)}, y_{n(k)}\right)\right) < C_G,$$

which is a contradiction. Therefore, $s = 0$ and the Picard-Jungck sequence $\{y_n\}$ is a Cauchy sequence. □

Now, we recall the following result of Abbas and Jungck [1] to be used in the sequel.

Proposition 2.4. Let (f, g) be a pair of weakly compatible self mappings on a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is a unique common fixed point of f and g.

Theorem 2.5. Let (f, g) be a pair of self mappings on a generalized metric space (X, d) and f be a (Z_G, g)-contraction. Suppose that there exists a Picard-Jungck sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f, g). Also assume that at least one of the following conditions hold:

(**i**) $(f(X), d)$ or $(g(X), d)$ is complete;

(**ii**) (X, d) is complete, g is continuous and $g(X)$ is closed subspace of X.

Then pair (f, g) is WPJO. Moreover, if f and g are weakly compatible, then pair (f, g) is PJO.

Proof. First of all we shall prove that the point of coincidence of f and g is unique (if it exists). Suppose that z_1 and z_2 are distinct points of coincidence of f and g. From this it follows that there exist two points v_1 and v_2 ($v_1 \neq v_2$) such that $fv_1 = gv_1 = z_1$ and $fv_2 = gv_2 = z_2$. Then (1) implies that

$$\begin{aligned} C_G &\leq \zeta\left(d\left(fv_1, fv_2\right), d(gv_1, gv_2)\right) \\ &= \zeta\left(d\left(z_1, z_2\right), d\left(z_1, z_2\right)\right) \\ &< G\left(d\left(z_1, z_2\right), d\left(z_1, z_2\right)\right) \leq C_G, \end{aligned}$$

which is a contradiction.

In order to prove that a pair (f, g) is WPJO, suppose that there is a Picard-Jungck sequence $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$ where $n \in \mathbb{N} \cup \{0\}$.

If $y_k = y_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using Lemma 2.2, we have $d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = 0$. Now, using Lemma 2.3, we obtain that Picard-Jungck sequence $\{y_n\}$ is a Cauchy sequence.

Suppose that (i) holds, i.e., $(g(X), d)$ is complete. Then there exists $v \in X$ such that $y_{n-1} = fx_{n-1} = gx_n \to gv$ as $n \to \infty$. We shall prove that $fv = gv$.

It is clear that we can suppose $y_n \neq fv, gv$ for all $n \in \mathbb{N} \cup \{0\}$. Suppose that $d(fv, gv) > 0$ and using (1) and ($\mathcal{Z}_G 1$), we have

$$C_G \leq \zeta\left(d\left(fx_n, fv\right), d\left(gx_n, gv\right)\right) < G\left(d\left(gx_n, gv\right), d\left(fx_n, fv\right)\right).$$

Taking $n \to \infty$, we have $C_G \leq \zeta\left(d\left(gv, fv\right), d\left(gv, gv\right)\right) < G\left(d\left(gv, gv\right), d\left(gv, fv\right)\right)$. Using ($C_G 1$) of Definition 1.3, we get $d\left(gv, fv\right) < d\left(gv, gv\right)$, which is a contradiction. Hence, $fv = gv$ is a (unique) point of coincidence of f and g.

Similarly, we can prove that $fv = gv$ is a (unique) point of coincidence of f and g, when $\left(f\left(X\right), d\right)$ is complete.

Finally, suppose that (ii) holds. Since (X, d) is complete, then there exists $v \in X$ such that $y_n = fx_n \to v$, when $n \to \infty$. As g is continuous, and $g(X)$ is a closed subspace of X. Then, we choose $u \in X$ such that $y_n = g(x_{n+1}) \to g(u) = v$ when $n \to \infty$. Suppose that $d(fu, v) > 0$. Consider

$$C_G \leq \zeta\left(d\left(f\left(x_n\right), fu\right), d\left(gx_n, gu\right)\right) < G\left(d\left(gx_n, gu\right), d\left(f\left(x_n\right), fu\right)\right).$$

Taking $n \to \infty$, we have $C_G \leq \zeta\left(d\left(v, fu\right), d\left(gu, gu\right)\right) < G\left(d\left(gu, gu\right), d\left(v, fu\right)\right)$. Using ($C_G 1$) of Definition 1.3, we get $d\left(v, fu\right) < d\left(gu, gu\right)$. Hence, $fu = gu = v$ is a (unique) point of coincidence of f and g. Hence, the result is proved in both the cases.

Further, since f and g are weakly compatible, then according to Proposition 2.4, they have a unique common fixed point. □

Remark 2.1. Theorem 2.5 holds true if, in particular, (X, d) is complete, g is continuous and f and g are commuting.

Corollary 2.6. Let (X, d) be a complete generalized metric space, $f : X \to X$ be self-mappings and f satisfies

$$\zeta\left(d\left(fx, fy\right), d\left(x, y\right)\right) \geq C_G,$$

for all $x, y \in X$ with $d(fx, fy) > 0$ and $\zeta \in Z_G$. Suppose that there exists a Picard sequence $\{u_n\} \subseteq M$ by $u_{n+1} = f^n u_0 = fu_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then f is a Picard operator.

Corollary 2.7. Let (X, d) be a complete generalized metric space and let $f : X \to X$ be a self mapping. If there exist $n \in \mathbb{N}$ such that f^n satisfies

$$\zeta\left(d\left(f^n x, f^n y\right), d\left(x, y\right)\right) \geq C_G,$$

for all $x, y \in X$ with $d(f^n x, f^n y) > 0$ and $\zeta \in Z_G$. Then f is a Picard operator.

Proof. From Corollary 2.6, it is obvious that f^n is a Picard operator, thus there exists a unique $z \in X$ such that $f^n z = z$ and $\lim_{m \to \infty} \left(f^n u\right)^m = z$, for all $u \in X$. Also, we observe that $f^{n+1} z = f^n z$, that is $f^n(fz) = fz$, thus fz is also a fixed point of f^n. Thus $fz = z$.
Further, if z^* is another fixed point of f, then it must be a fixed point of f^n. Hence $z = z^*$. Therefore f has a unique fixed point.

Now, let m be a positive integer greater than n. Then there exist $l \geq 1$ and $s \in \{0, 1, 2, ..., n-1\}$ such that $m = nl + s$. Here, we notice that

$$\lim_{m\to\infty} f^m u = \lim_{l\to\infty} f^{nl}(f^s u) = z.$$

Hence the result. □

Example 2.8. Let $X = \{1, 2, 3, 4\}$. Define the metric $d : X \times X \to [0, \infty)$ by $d(1,2) = d(2,1) = 3, d(1,3) = d(3,1) = d(2,3) = d(3,2) = 1, d(1,4) = d(4,1) = d(2,4) = d(4,3) = d(3,4) = 4$ and $d(x,x) = 0$, for all $x \in X$. Then (X, d) is a complete generalized metric space but not a metric space.

Define a function $f : X \to X$ as $f(1) = 3, f(2) = 3, f(3) = 3, f(4) = 1$.

Here, $d(fx, fy) = \begin{cases} d(1,3) = 1, (x = 4, y \neq 4), (x \neq 4, y = 4) \\ d(1,1) = 0, (x = 4, y = 4) \\ d(3,3) = 0, (x \neq 4, y \neq 4) \end{cases}$

So $d(fx, fy) > 0$ if and only if $x = 4$, $y \neq 4$.

However, putting $\zeta(t,s) = G(s,t) - t$, $G(s,t) = m\ s, m < 1$, $C_G = 0$, we have that f is a $\mathcal{Z}$-contraction with respect to ζ.

Consider $x = 4, y \neq 4$ (or $x \neq 4, y = 4$). We have

$$\zeta(d(fx, fy), d(x,y)) \geq C_G = 0 \Leftrightarrow md(x,y) - d(fx, fy) \geq 0.$$

Hence $4m - 1 > 0$, for all $m \in (\frac{1}{4}, 1)$.

Thus all the conditions of Corollary 2.6 are satisfied and 3 is a unique fixed point of f.

Here it is to note that Banach's contraction is not satisfied with usual metric $d(x,y) = |x - y|$. If we choose $x = 2$, $y = 4$, $d(fx, fy) = d(f2, f4) = d(3,1) = 2$, then $d(fx, fy) \leq k\ d(x,y)$, $k \in (0,1)$, implies that $k \geq 1$.

Let (X, d) be a metric space and $T : X \to X$ be a self mapping. It is said that T has Property P if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, where $F(T)$ denotes the set of all fixed points of T.

We will now present results regarding the property P for some well-known types of self-mappings.

Theorem 2.9. If $T : X \to X$ satisfies

$$\zeta(d(Tx, Ty), d(x,y)) \geq C_G,$$

for all $x, y \in X$, and some $\zeta \in \mathcal{Z}_{\mathcal{G}}$, then T has Property P.

Proof. Let $u \in F(T^n)$, $n > 1$. If $u \neq Tu$, then we have

$$\begin{aligned} C_G &\leq \zeta\left(d\left(TT^{n-1}u, TT^n u\right), d\left(T^{n-1}u, T^n u\right)\right) \\ &< G(d\left(T^{n-1}u, T^n u\right), d\left(TT^{n-1}u, TT^n u\right)) \\ &= G(d\left(T^{n-1}u, u\right), d\left(u, T^{n+1}u\right)). \end{aligned}$$

It follows from (C_G1) that $d\left(T^{n-1}u,u\right) > d\left(u,T^{n+1}u\right) = d(u,Tu)$. Therefore, we get

$$\begin{aligned} d\left(u,Tu\right) &= d\left(T^{n}u,T^{n+1}u\right) < d\left(T^{n-1}u,T^{n}u\right) \\ &< d\left(T^{n-2}u,T^{n-1}u\right) < \cdots < d\left(u,Tu\right), \end{aligned}$$

which is impossible.

We use the fact that $d\left(u,T^{n+1}u\right) = d\left(u,Tu\right)$ and $d\left(T^{n-1}u,u\right)$ are both positive numbers. Hence the result. □

Finally, we have the following open question: Does the following claim hold using (Z_G, g)-quasi-contraction of Ćirić-Das-Naik type?

Definition 2.10. Let (f,g) be a pair of self mappings on a generalized metric space (X,d). A mapping f is called a (Z_G, g)-quasi-contraction of Ćirić-Das-Naik type if there exist $\zeta \in Z_G$, $\lambda \in (0,1)$ such that

$$\zeta\left(d\left(fx,fy\right), \lambda \max\left\{d\left(gx,gy\right), d\left(gx,fx\right), d\left(gy,fy\right), d\left(gx.fy\right), d\left(gy,fx\right)\right\}\right) \geq C_G$$

for all $x,y \in X$ with $d(fx,fy) > 0$.

In the case that $g = i_X$ (identity mapping on X) and $C_G = 0$ we get a $\mathcal{Z}$-quasi-contraction of Ćirić type.

Claim. *Let f be a (Z_G, g)-quasi-contraction of Ćirić-Das-Naik type in a generalized metric space (X,d) and suppose that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f,g). Also assume that at least one of the following conditions hold:*

(i) *$(f(X), d)$ or $(g(X), d)$ is complete;*

(ii) *(X,d) is complete and f and g are continuous and compatible.*

Then pair (f,g) is WPJO.

Acknowledgements. The author is thankful to the learned referee for suggesting some improvements in the manuscript.

References

1. M. Abbas and J. Jungck, *Common fixed point results for non-commuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008) 416-420.
2. S. Alizadeh, F. Moradlou and P. Salimi, *Some fixed point results for $(\alpha,\beta)-(\psi,\phi)$-contractive mappings*, Filomat **28** (2014), 635-647.
3. A.H. Ansari, H. Isik and S. Radenović, *Coupled fixed point theorems for contractive mappings involving new function classes and applications*, Filomat **31** (2017), 1893–1907.
4. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922) 133–181.
5. A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Pub. Math. **57** (2000), 31-37.
6. A.-F.R.-L.-de-Hierro, E. Karapınar, C.R.-L.-de-Hierro and J.M.-Morenoa, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345-355.

7. A.-F.R.-L.-de-Hierro and N. Shahzad, *New fixed point theorem under R-contractions*, Fixed Point Theory Appl. **2015**, 2015:98.
8. A.-F.R.-L.-de-Hierro and N. Shahzad, *Common fixed point theorems under (R,S)-contractivity conditions*, Fixed Point Theory Appl. **2016**, 2016:55.
9. H. Isik, N.B. Gungor, C. Park, J.S. Young, *Fixed point theorems for almost Z-contractions with an application*, Mathematics **37** (2018), 6 doi: 10.3390/math6030037.
10. G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly **83** (1976), 261-263.
11. F. Khojasteh, S. Shukla and S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat **29** (2015), 1189-1194 .
12. A. Latif, H. Isik, A.H. Ansari, *Fixed points and functional equation problems via cyclic admissible generalized contractive type mappings*, J. Nonlinear Sci. Appl. **9** (2016), 1129-1142.
13. X.L. Liu, A.H. Ansari, S. Chandok and S. Radenovic, *On some results in metric spaces using auxiliary simulation functions via new functions*, J. Comput. Anal. Appl. **24** (2018), 1103-1114.
14. A. Nastasi and P. Vetro, *Fixed point results on metric and partial metric spaces via simulation functions*, J. Nonlinear Sci. Appl. **8** (2015), 1059-1069 .
15. M. Olgun, O. Bicer and T. Alyildiz, *A new aspect to Picard operators with simulation functions*, Turk. J. Math. **40** (2016), 832-837.
16. S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, *Some results on weakly contractive maps*, Bull. Iranian Math. Soc. **38** (2012), 625-645.
17. S. Radenović and S. Chandok, *Simulation type functions and coincidence points*, Filomat **32** (2018), 141-147.
18. S. Wang, A.H. Ansari and S. Chandok, *Some fixed point results for non-decreasing and mixed monotone mappings with auxiliary functions*, Fixed Point Theory Appl. **2015**, 2015:209.

Sumit Chandok received his M.Sc. and Ph.D. from Guru Nanak Dev University, Amritsar, India. His research interests include Fixed Point Theory, Functional Equations and Nonlinear Analysis. He has published good number of publications in the journals of international repute.

School of Mathematics, Thapar Institute of Engineering & Technology, Patiala-147004, Punjab, India.
e-mail: sumit.chandok@thapar.edu