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CHARACTERIZATIONS OF THE UNIFORM DISTRIBUTIONS BASED ON UPPER RECORD VALUES

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Abstract. We obtain two characterizations of uniform distribution based on ratios of upper record values by the properties of independence and identical distribution.

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function(cdf) $F(x)$ which is absolutely continuous and probability density function(pdf) $f(x)$. Suppose that $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

We say the random variable F follows a uniform distribution over the interval $[a, b]$ being denoted by $F \sim U(a, b)$, if the corresponding probability cumulative distribution function(cdf) $F(x)$ of X is of the form

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x < b, \\ 1, & x \geq b. \end{cases}$$

The distribution of record values is given in terms of hazard function and hazard rate (see [2]). The function $R(x)$ defined as $R(x) = -\ln(\bar{F}(x))$ for

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$\bar{F}(x) = 1 - F(x)$ is called hazard function for the upper record values. The function $r(x) = \frac{dR(x)}{dx} = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate for the upper records.

We say the random variable X belongs to the class C_1, if either $r(1-v) \leq r(1-vw)w$ or $r(1-v) \geq r(1-vw)w$ for all $0 < v < \infty$, $0 < w < \infty$. Several distributions including uniform, exponential and Pareto distributions are members of the class C_1.

Many characterization results involving spacings of record statistics can be found in the literature. In [3], Arslan et al. characterized that if $X_{U(m)} - X_{U(m-1)}$ and $X_{L(m)}$, $2 \leq m < n$, are identically distributed, then $F \sim U(0, \beta)$ where the random variables are symmetric about $\beta/2$. In [4], Arslan et al. proved that X_i characterizes the uniform distribution if and only if $X_{L(n)}$ and $X_{L(n-1)} \cdot V_1$, $n \geq 2$, are identically distributed where V_1 is independent of random variables X_i's. Recently, in [5], Nadarajah et al. derived that X is distributed uniformly over the interval $(0, 1)$ if and only if $W = -\ln(X_{L(n)}/X_{L(m)})$ has the Gamma distribution with shape parameter $n - m$.

The current investigation was induced by characterizations of uniform distribution by [3] and [4]. Namely, if we write $U = (1 - X_{U(n)})/(1 - X_{U(n-1)})$ and $V = 1 - X_{U(n-1)}$, one can ask whether the independence of U and V characterizes uniform distribution. Also we can ask whether the identity distribution of $(1 - X_{U(n)})/(1 - X_{U(n-1)})$ and $(1 - X_{U(n+1)})/(1 - X_{U(n)})$ characterizes uniformality.

In this paper, we investigate characterizations of the uniform distribution based on upper record values by the independence property and the assumption of identical distribution.

2. Main Results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables that have absolutely continuous (with respect to Lebesgue measure) cdf $F(x)$ with $F(0) = 0$ and the corresponding pdf $f(x)$ with $f(0) = 1$. Then X_n is distributed uniformly over the interval $(0, 1)$ if and only if $(1 - X_{U(n)})/(1 - X_{U(n-1)})$ and $1 - X_{U(n-1)}$ are independent for $n \geq 2$.*

Proof. If $F(x) = x$ for all $0 \leq x < 1$, then the joint pdf $f_{n-1,n}(x, y)$ of $X_{U(n-1)}$ and $X_{U(n)}$ is given by (see [2])

$$f_{n-1,n}(x, y) = \frac{[-\ln(1-x)]^{n-2}}{\Gamma(n-1)} \frac{1}{1-x},$$

for all $0 \leq x < y \leq 1$ and $n \geq 2$.

Consider the function $W = (1 - X_{U(n)})/(1 - X_{U(n-1)})$ and $V = 1 - X_{U(n-1)}$. It follows that $x_{U(n-1)} = 1 - v$, $x_{U(n)} = 1 - vw$. The Jacobian of the transformation is $J = v$. Thus we can obtain the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = [-\ln(v)]^{n-2}/\Gamma(n-1) \tag{1}$$

for all v, w, $0 < v \leq 1$, $0 \leq w < 1$.

The marginal pdf $f_W(w)$ of W is found by

$$f_W(w) = \int_0^1 \frac{[-\ln(v)]^{n-2}}{\Gamma(n-1)} dv = 1 \tag{2}$$

for all $0 \le w < 1$. Also, the pdf $f_V(v)$ of V is given by

$$f_V(v) = [-\ln(v)]^{n-2}/\Gamma(n-1) \tag{3}$$

for all $0 < v \le 1$.

From (1), (2) and (3), we obtain $f_V(v)f_W(w) = f_{V,W}(v,w)$. Hence V and W are independent.

Now we prove the sufficient condition. The joint pdf $f_{n-1,n}(x,y)$ of $X_{U(n-1)}$ and $X_{U(n)}$ is given by (see [2])

$$f_{n-1,n}(x,y) = [R(x)]^{n-2} r(x) f(y)/\Gamma(n-1)$$

for all $0 \le x < y \le 1$ and $n \ge 2$, where $R(x) = -\ln(\bar{F}(x))$ and $r(x) = \frac{d}{dx}R(x) = \frac{f(x)}{\bar{F}(x)}$.

Let us use the transformation $W = (1 - X_{U(n)})/(1 - X_{U(n-1)})$ and $V = 1 - X_{U(n-1)}$. The Jacobian of the transformation is $J = v$. Thus we obtain the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = [R(1-v)]^{n-2} r(1-v) f(1-vw) v/\Gamma(n-1) \tag{4}$$

for all v, w, $0 < v \le 1$, $0 \le w < 1$.

The pdf $f_V(v)$ of V is given by

$$f_V(v) = [R(1-v)]^{n-2} f(1-v)/\Gamma(n-1) \tag{5}$$

for all v, $0 < v \le 1$.

From (4) and (5), we get the pdf $f_W(w)$ of W

$$f_W(w) = f(1-vw)v/\bar{F}(1-v)$$

for all w, $0 \le w < 1$.

Since V and W are independent, the pdf $f_W(w)$ of W is a function of w only. Thus we must have $\frac{\partial}{\partial v}(f_W(w)) = 0$. That is,

$$-f'(1-vw)vw\bar{F}(1-v) + f(1-vw)\bar{F}(1-v) - f(1-v)f(1-vw)v = 0 \tag{6}$$

for all v, w, $0 < v \le 1$, $0 \le w < 1$.

Therefore, by the existence and uniqueness theorem, there exists a unique solution of the differential equation (6) that satisfies the initial conditions $F(0) = 0$ and $f(0) = 1$. Thus we get $F(x) = x$ for $0 \le x < 1$, from (6). This completes the proof. □

Theorem 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. random variables that have absolutely continuous (with respect to Lebesgue measure) cdf $F(x)$ with $F(0) = 0$ and $F(1) = 1$ and the corresponding pdf $f(x)$. Assume that F belongs to the class C_1. Then X_n is distributed uniformly over the interval $(0,1)$ if and* *only if the probability distributions of* $W_{n+1,n} = (1 - X_{U(n+1)})/(1 - X_{U(n)})$ *and* $W_{n,n-1} = (1 - X_{U(n)})/(1 - X_{U(n-1)})$ *are identically distributed for* $n \geq 2$.

Proof. If X_n is distributed uniformly over the interval $(0, 1)$, then it can be easily seen that

$$W_{n+1,n} = (1 - X_{U(n+1)})/(1 - X_{U(n)})$$

and

$$W_{n,n-1} = (1 - X_{U(n)})/(1 - X_{U(n-1)})$$

are identically distributed. We have to prove the converse.

From (4), the pdf g_n of $W_{n+1,n}$ can be written as

$$g_n(w) = \begin{cases} \int_0^1 \frac{[R(1-v)]^{n-1}}{\Gamma(n)} r(1-v) f(1-vw) v dv, & 0 \leq w < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $R(x) = -\ln(\bar{F}(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{\bar{F}(x)}$.

Thus it follows that

$$P(W_{n+1,n} < w) = \int_0^1 \frac{[R(1-v)]^{n-1}}{\Gamma(n)} r(1-v) \bar{F}(1-vw) dv$$

for all $0 \leq w < 1$.

Since $W_{n+1,n}$ and $W_{n,n-1}$ are identically distributed, we get

$$\begin{aligned} &\int_0^1 [R(1-v)]^{n-1} r(1-v) \bar{F}(1-vw) dv \\ &= (n-1) \int_0^1 [R(1-v)]^{n-2} r(1-v) \bar{F}(1-vw) dv \end{aligned} \tag{7}$$

for all $0 \leq w < 1$.

Substituting the identity

$$(n-1) \int_0^1 [R(1-v)]^{n-2} r(1-v) \bar{F}(1-vw) dv = \int_0^1 [R(1-v)]^{n-1} f(1-vw) w dv$$

in (7), we get on simplification

$$\int_0^1 [R(1-v)]^{n-1} \bar{F}(1-vw)[r(1-v) - r(1-vw)w] dv = 0 \tag{8}$$

for all $0 \leq w < 1$.

Thus if $F \in C_1$, then (8) is true if for almost all v and any fixed w, $0 \leq w < 1$,

$$r(1-v) = r(1-vw)w. \tag{9}$$

Integrating (9) with respect to v from v_1 to 1 and simplifying, we get

$$\bar{F}(1-v_1)\bar{F}(1-w) = \bar{F}(1-v_1 w) \tag{10}$$

for any fixed v_1 with $0 \leq v_1 \leq 1$.

By the theory of functional equations (see [1]), the only continuous solution of (10) with the boundary conditions $\bar{F}(0) = 1$ and $\bar{F}(1) = 0$ is

$$\bar{F}(x) = 1 - x$$

for all x, $0 \le x < 1$. This completes the proof. □

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