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FRACTIONAL CALCULUS AND INTEGRAL TRANSFORMS OF THE M-WRIGHT FUNCTION

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Abstract. This paper is concerned to investigate M-Wright function, which was earlier known as transcendental function of the Wright type. M-Wright function is a special case of the Wright function given by British mathematician (E.Maitland Wright) in 1933. We have explored the cosequences of Riemann-Liouville Integral and Differential operators on M-Wright function. We have also evaluated integral transforms of the M-Wright function.

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1. Introduction and Preliminaries

Some authors including Podlubny [8], Gorenflo et al. [10], [11] Germano et al. [6] and Kiryakova [12], [13] refer to the M-Wright function as Mainradi function. In 1994 a renowned mathematician suggested to Mainardi, that the function which was named after him was already discovered by E.Maitland Wright in 1940 [1]. Mainardi, in his first analysis of the time-fractional diffusion equation [2]-[5], in the Caputo sense, introduced the two Wright-type functions $F_\alpha(z)$ and $M_\alpha(z)$, as

$$F_\alpha(z) = W_{\text{-}\alpha,0}(\text{-z}),\ 0 < \alpha < 1 \tag{1}$$

$$M_\alpha(z) = W_{\text{-}\alpha,1\text{-}\alpha}(\text{-z}),\ 0 < \alpha < 1 \tag{2}$$

where z is a complex variable and α is a real parameter such that $0 < \alpha < 1$. Both functions turn out to be analytic in the whole complex plane, i.e. they are the entire functions interrelated through

$$F_\alpha(z) = \alpha z M_\alpha(z) \tag{3}$$

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As a matter of fact, these functions $F_\alpha(z)$ and $M_\alpha(z)$ are particular cases of the Wright function of the second kind $W_{\lambda,\mu}(z)$ [1],i.e.

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n+\mu)},\ \lambda > -1,\ \mu \in \mathbb{C},\ z \in \mathbb{C} \tag{4}$$

by setting λ = -α and $\mu = 0$ or μ = 1-α, respectively. Their series representations are:

$$F_\alpha(z) = \sum_{n=1}^{\infty} \frac{(\text{-}z)^n}{n!\Gamma(\text{-}\alpha n)} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(\text{-}z)^{n-1}}{(n)!}\Gamma(\alpha n+1)\sin(\pi\alpha n) \tag{5}$$

$$M_\alpha(z) = \sum_{n=0}^{\infty} \frac{(\text{-}z)^n}{n!\Gamma(\text{-}\alpha(n+1)+1)} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(\text{-}z)^{n-1}}{(n-1)!}\Gamma(\alpha n)\sin(\pi\alpha n) \tag{6}$$

The noteworthy particular case is

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}}\exp\left(\frac{-z^2}{4}\right) \tag{7}$$

$$M_{1/3}(z) = 3^{2/3} Ai\left(\frac{z}{3^{1/3}}\right) \tag{8}$$

For $\alpha = \frac{1}{2}, \frac{1}{3}$ $M_\alpha(z)$ reduce respectively to the well known Gaussian and Airy function [1]. In view of its properties our function can be considered as a generalized hyper-airy function.

Further properties of the M-Wright function can be found in the book of Mainardi [2], especially in Chapter 6 and Appendix F, and in the review paper [5]. In the literature, the M-Wright function $M_\nu(z)$ is also referred to as the Mainardi function. This name was was originally introduced in the community of fractional analysis by Podlubny [8].

The Euler transform and Laplace transform [7] of the function $f(z)$ is defined as

$$B[f(z):a,b] = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz \tag{9}$$

$$\mathfrak{L}[f(z);s] = \int_0^\infty e^{-sz}f(z)dz \tag{10}$$

2. Fractional Calculus Operators

Fractional calculus is the branch of mathematical analysis. Which deals with pseudodifferential operators that extend the standard notions of integrals and derivatives to any noninteger order. In this section, we introduce fractional integrals and differentials of the M-Wright function (6). The operator of Riemann-Liouville fractional integrals and derivatives given by Samko, Kilbas and Marichev [9, section,5.1], for $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$), are defined by

$$(I^\alpha_{o+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,\ (x>0) \tag{11}$$

$$(I_{o-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}\frac{f(t)}{(t-x)^{1-\alpha}}dt,(x>0) \tag{12}$$

$$(D_{0+}^{\alpha}f)(x)=\left(\frac{d}{dx}\right)^{[\Re(\alpha)]+1}(I_{0+}^{1-\alpha+[\Re(\alpha)]}f)(x)$$

$$=\left(\frac{d}{dx}\right)^{[\Re(\alpha)]+1}\frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])}\int_0^x\frac{f(t)}{(x-t)^{\alpha-[\Re(\alpha)]}}dt,\ (x>0) \tag{13}$$

and

$$(D_{0-}^{\alpha}f)(x)=\left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1}(I_{0-}^{1-\alpha+[\Re(\alpha)]}f)(x)$$

$$=\left(-\frac{d}{dx}\right)^{[\Re(\alpha)]+1}\frac{1}{\Gamma(1-\alpha+[\Re(\alpha)])}\int_x^{\infty}\frac{f(t)}{(x-t)^{\alpha-[\Re(\alpha)]}}dt,(x>0) \tag{14}$$

respectively, where $[\Re(\alpha)]$ is the integral part of $\Re(\alpha)$.

3. Fractional integration of the M-Wright function

In this section we establish a formula for the fractional integration of the the M-Wright function (6) and Wright function of the second kind (4).

Theorem 3.1. *Let $\alpha,\beta\in\mathbb{C},\Re(\alpha)>0$ and $0<\beta<1$, then the fractional integration (I_{0+}^{α}) of the M-Wright function* (6) *holds true:*

$$\left(I_{0+}^{\alpha}[t^{-\beta}M_{\beta}(at^{-\beta})]\right)(x)=x^{-\beta+\alpha}W_{-\beta,\alpha-\beta+1}(-ax^{-\beta})\ ,\ (x>0) \tag{15}$$

Proof. By virtue of (6) and (11), we have

$$\left(I_{0+}^{\alpha}[t^{-\beta}M_{\beta}(at^{-\beta})]\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k a^k t^{-\beta-\beta k}}{k!\Gamma(-\beta(k+1)+1)}dt$$

$$=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k a^k}{k!\Gamma(-\beta(k+1)+1)}\int_0^x(x-t)^{\alpha-1}t^{-\beta-\beta k}dt \tag{16}$$

Interchanging the the order of integration and summation and then evaluating the integral by modified beta function defined as:

$$\int_a^b(b-t)^{\beta-1}(t-a)^{\alpha-1}dt=(b-a)^{\alpha+\beta-1}B(\alpha,\beta),\ \text{for }\Re(\alpha)>0,\Re(\beta)>0 \tag{17}$$

thus equation (16) reduces to

$$=x^{-\beta+\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k(ax^{-\beta})^k}{k!\Gamma(-\beta(k+1)+1)}\frac{\Gamma(-\beta(k+1)+1)}{\Gamma(-\beta(k+1)+(1+\alpha))}$$

$$=x^{-\beta+\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k(ax^-\beta)^k}{k!\Gamma(-\beta(k+1)+(1+\alpha))}$$

Using the equation (4), we deduce

$$=x^{-\beta+\alpha}W_{-\beta,\alpha-\beta+1}(-ax^{-\beta}) \tag{18}$$

This completes the proof of the Theorem 3.1

□

Theorem 3.2. *Let $\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0$ and $0 < \beta < 1$, then the fractional integration (I^{α}_{o-}) of the Wright function of the second kind* (4) *holds true:*

$$\left(I^{\alpha}_{0-}[t^{\beta-1}W_{-\beta,-\alpha-\beta+1}(at^{\beta})]\right)(x) = x^{\alpha+\beta-1}M_{\beta}(ax^{\beta}),\ (x > 0) \tag{19}$$

Proof. By virtue of (4) and (12), we have

$$\left(I^{\alpha}_{-}[t^{\beta-1}W_{-\beta,-\alpha-\beta+1}(at^{\beta})]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}\sum_{k=0}^{\infty}\frac{a^{k}t^{\beta+\beta k-1}}{k!\Gamma(-\beta(k+1)+(1-\alpha))}dt$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{k}}{k!\Gamma(-\beta(k+1)+(1-\alpha))}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{\beta+\beta k-1}dt \tag{20}$$

Let $u = (t-x)/t$, then

$$\left(I^{\alpha}_{-}[t^{\beta-1}W_{-\beta,-\alpha-\beta+1}(at^{\beta})]\right)(x) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{k}x^{\alpha+\beta+\beta k-1}}{k!\Gamma(-\beta(k+1)+(1-\alpha))}$$

$$\times\int_{0}^{1}u^{\alpha-1}(1-u)^{(-\alpha-\beta-\beta k+1)-1}du$$

Evaluating the inner integral and using beta function formula, we get

$$= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{a^{k}x^{\alpha+\beta+\beta k-1}}{k!\Gamma(-\beta(k+1)+(1-\alpha))}B(\alpha, -\alpha-\beta-\beta k+1)$$

$$= x^{\alpha+\beta-1}\sum_{k=0}^{\infty}\frac{a^{k}x^{\beta k}}{k!\Gamma(-\beta(k+1)+(1-\alpha))}\frac{\Gamma(-\beta(k+1)+(1-\alpha))}{\Gamma(-\beta(k+1)+1)}$$

$$= x^{\alpha+\beta-1}\sum_{k=0}^{\infty}\frac{(ax^{\beta})^{k}}{k!\Gamma(-\beta(k+1)+1)}$$

Upon using (6), we deduce

$$= x^{\alpha+\beta-1}M_{\beta}(-ax^{\beta}) \tag{21}$$

This completes the proof of Theorem 3.2 □

4. Fractional differentiation of the M-Wright function

We are presenting the following formulas for the fractional differentiation of the M-Wright function (6) and Wright function of the second kind (4).

Theorem 4.1. *Let $\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0$ and $0 < \beta < 1$, then the fractional differentiation (D_{0+}^{α}) of the M-Wright function* (6) *holds true:*

$$(D_{0+}^{\alpha}[t^{-\beta}M_{\beta}(at^{-\beta})])(x) = x^{-\beta-\alpha}W_{-\beta,-\alpha-\beta+1}(-ax^{-\beta}),\ x > 0 \tag{22}$$

Proof. By using the definition of the M-Wright function (6) and fractional derivative formula (13), we have

$$(D_{0+}^{\alpha}[t^{-\beta}M_{\beta}(at^{-\beta})])(x) = \left(\frac{d}{dx}\right)^{n}(I_{0+}^{n-\alpha}(M_{\beta}(at^{-\beta}))(x),\ n = 1 + \{\Re(\alpha)\}$$

$$= \left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-t)^{n-\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^{k}a^{k}t^{-\beta-\beta k}}{k!\Gamma(-\beta(k+1)+1)} \tag{23}$$

Interchanging the order of integration and summations and evaluating the inner integral by the use of the modified beta function formula, we obtain

$$= \frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^{k}a^{k}}{k!\Gamma(-\beta(k+1)+1)}$$

$$\times\left(\frac{d}{dx}\right)^{n}x^{-\beta-\beta k+n-\alpha}B(-\beta-\beta k+1, n-\alpha)$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^{k}a^{k}}{k!\Gamma(-\beta-\beta k+1+n-\alpha)}\left(\frac{d}{dx}\right)^{n}x^{-\beta-\beta k+n-\alpha}$$

$$= x^{-\beta-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^{k}(ax^{-\beta})^{k}}{k!\Gamma(-\beta(k+1)+(1-\alpha))}$$

Upon using (6) deduce (22).

$$= x^{-\beta-\alpha}W_{-\beta,-\alpha-\beta+1}(-ax^{-\beta}) \tag{24}$$

This completes the proof of the Theorem 4.1

□

Theorem 4.2. *Let $\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0$ and $0 < \beta < 1$, then the fractional differentiation (D_{0-}^{α}) of the Wright function of the second kind* (4) *holds true :*

$$(D_{0-}^{\alpha}[t^{\beta-1}W_{-\beta,\alpha-\beta+1}(at^{\beta})])(x) = x^{\beta-\alpha-1}M_{\beta}(-ax^{\beta}) \tag{25}$$

Proof. By virtue of (4) and (14), we obtain

$$(D_{0-}^{\alpha}[t^{\beta-1}W_{-\beta,\alpha-\beta+1}(at^{\beta})])(x) = \left(-\frac{d}{dx}\right)^{n}\left(I_{0-}^{n-\alpha}[t^{\beta-1}W_{-\beta,\alpha-\beta+1}(at^{\beta})]\right)(x)$$

$$= (-1)^{n}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}$$

$$\times \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\beta(k+1)+(1+\alpha))} \int_x^{\infty} t^{\beta+\beta k-1}(t-x)^{n-\alpha-1} dt \tag{26}$$

if we set $u = (t-x)/t$, then the above expression transform into the form

$$I = (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)}$$

$$\times \sum_{k=0}^{\infty} \frac{a^k x^{n+\beta+\beta k-\alpha-1}}{k!\Gamma(-\beta(k+1)+(1+\alpha))} \int_0^1 u^{n-\alpha-1}(1-u)^{\alpha-\beta-\beta k-n} du$$

$$= (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)}$$

$$\times \sum_{k=0}^{\infty} \frac{a^k x^{n+\beta+\beta k-\alpha-1}}{k!\Gamma(-\beta(k+1)+(1+\alpha))} B(n-\alpha, \alpha-\beta-\beta k-n+1)$$

$$= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\beta(k+1)+(1+\alpha))} \frac{\Gamma(-\beta-\beta k+1+\alpha-n)}{\Gamma(-\beta-\beta k+1)}$$

$$\times (-1)^n \left(\frac{d}{dx}\right)^n x^{n+\beta+\beta k-\alpha-1}$$

$$= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\beta(k+1)+(1+\alpha))} \frac{\Gamma(-\beta-\beta k+1+\alpha-n))}{\Gamma(-\beta-\beta k+1)}$$

$$\times (n+\beta+\beta k-\alpha-1).....(n+\beta+\beta k-\alpha-1-n+1)x^{\beta+\beta k-\alpha-1}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\beta(k+1)+(1+\alpha))} \frac{\Gamma(-\beta-\beta k+1+\alpha-n)}{\Gamma(-\beta-\beta k+1)}$$

$$\times (-1)^n (1+\alpha-\beta-\beta k-n)_n \; x^{\beta+\beta k-\alpha-1}$$

$$= x^{\beta-\alpha-1} \sum_{k=0}^{\infty} \frac{(ax^{\beta})^k}{k!\Gamma(-\beta(k+1)+1)}$$

By using (6), we get

$$= x^{\beta-\alpha-1} M_{\beta}(-ax^{\beta}) \tag{27}$$

This completes the proof of the Theorem 4.2

5. Integral transforms of the M-Wright function

In this section we investigate Euler transform and Laplace transform of the M-Wright function.

Theorem 5.1. *Let $0 < \alpha < 1$ and $z \in \mathbb{C}$, the Euler transform of the M-Wright function is*

$$B[M_\alpha(z^\alpha) : -\alpha+1, 1] = W_{-\alpha,-\alpha+2}(1) \tag{28}$$

Proof. Applying (9) and (6), we have

$$B[M_\alpha(z^{-\alpha}) : -\alpha+1, 1] = \int_0^1 z^{-\alpha+1-1}(1-z)^{1-1}\sum_{n=0}^{\infty}\frac{(-z^{-\alpha})^n}{\Gamma(-\alpha(n+1)+1)n!}dz$$

Reciprocating the integration and summation which is verified under given conditions

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(-\alpha(n+1)+1)n!}\int_0^1 z^{-\alpha(n+1)}dz$$

Applying (17) we achieve our result in view of (4). □

Corollary 5.2. *Put $\alpha = \frac{1}{2}$ in (28), we get*

$$\int_0^1 z^{-\frac{1}{2}}e^{-\frac{1}{4z}} = \sqrt{\pi}\,{}_1\Psi_1\left[\begin{array}{cc} (1,0); & \\ & -1 \\ (\frac{3}{2},-\frac{1}{2}); & \end{array}\right]$$

Corollary 5.3. *Put $\alpha = \frac{1}{3}$ in (28), we get*

$$\int_0^1 z^{-\frac{1}{3}}Ai\left(\frac{z^{-\frac{1}{3}}}{3^{\frac{1}{3}}}\right) = 3^{\frac{2}{3}}\,{}_1\Psi_1\left[\begin{array}{cc} (1,0); & \\ & -1 \\ (\frac{1}{2},-\frac{3}{2}); & \end{array}\right]$$

Where ${}_1\Psi_1$ is the Wright-hypergeometric function.

Theorem 5.4. *Let $\alpha < 1$ and $\mid s^\alpha \mid < \infty$, the Laplace transform of M-Wright function is*

$$\mathfrak{L}[M_\alpha(z^{-\alpha}); s] = s^\alpha\,{}_0F_1(-;1;s^\alpha) \tag{29}$$

where ${}_0F_1$ is transformed form of confluent hypergeometric function.

Proof. Applying (10) and (6), we have

$$\mathfrak{L}[M_\alpha(z^{-\alpha}); s] = \int_0^\infty e^{-sz}\sum_{n=0}^{\infty}\frac{(-z^{-\alpha})^n}{\Gamma(-\alpha(n+1)+1)n!}dz$$

Reciprocating the integration and summation which is verified under given conditions

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(-\alpha(n+1)+1)n!}\int_0^\infty e^{-sz}z^{-\alpha(n+1)}dz$$

$$= s^\alpha\sum_{n=0}^{\infty}\frac{(-s^\alpha)^n}{(1)_n}$$

$$= s^{\alpha}{}_0F_1(-; 1; s^{\alpha})$$

□

Corollary 5.5. *Put* $\alpha = \frac{1}{2}$ *in* (29), *we get*

$$\int_0^\infty e^{-sz-\frac{1}{4z}} = \sqrt{\pi} s^{\frac{1}{2}} e^{-s^{1/2}}$$

Corollary 5.6. *Put* $\alpha = \frac{1}{3}$ *in* (29), *we get*

$$\int_0^\infty e^{-sz} Ai\left(\frac{z^{-\frac{1}{3}}}{3^{\frac{1}{3}}}\right) = 3^{\frac{2}{3}} s^{\frac{1}{3}} e^{-s^{1/3}}$$

Conclusion

In the present paper we have built up the Riemann-Liouville fractional integral and derivatives of M-Wright function (6). Also we have evaluated the Laplace and Euler transforms of the M-Wright function. The fractional calculus also finds applications in different fields of science, including theory of fractals, numerical analysis, physics, engineering, biology, economics and finance. The results in this paper may as well detect certain utility in above fields.

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