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ESSENTIAL NORMS OF SUMS OF TOEPLITZ PRODUCTS ON THE PLURIHARMONIC DIRICHLET SPACE

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Abstract. On the setting of the pluriharmonic Dirichlet space, we describe the essential norm of an operator which is a finite sum of products of several Toeplitz operators.

1. Introduction

Let B be the unit ball in the complex n-space $\mathbb{C}^n$ and V be the Lebesgue volume measure on $\mathbb{C}^n$ normalized so that $V(B) = 1$. The Sobolev space $\mathscr{S}$ is the completion of the space of all smooth functions f on B for which

$$\|f\| := \left\{ \left| \int_B f\, dV \right|^2 + \int_B \left(|\mathcal{R}f|^2 + |\widetilde{\mathcal{R}}f|^2 \right) dV \right\}^{1/2} < \infty$$

where

$$\mathcal{R}f(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}(z), \quad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^n \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \cdots, z_n) \in B$. The Sobolev space $\mathscr{S}$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_B f\, dV \int_B \bar{g}\, dV + \int_B \left(\mathcal{R}f\overline{\mathcal{R}g} + \widetilde{\mathcal{R}}f\overline{\widetilde{\mathcal{R}}g} \right) dV.$$

A function $u \in C^2(B)$ is said to be pluriharmonic if the one-variable function $\lambda \mapsto u(a + \lambda b)$, defined for $\lambda \in \mathbb{C}$ such that $a + \lambda b \in B$, is harmonic for each $a \in B$ and $b \in \mathbb{C}^n$. The pluriharmonic Dirichlet space $\mathscr{D}_{ph}$ is then a subspace of $\mathscr{S}$ consisting of all pluriharmonic functions

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on B. Then one can check $\mathscr{D}_{ph}$ is closed in $\mathscr{S}$. We let Q be the Hilbert space orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}_{ph}$ and put

$$\mathscr{L}^{1,\infty} = \left\{ \varphi \in \mathscr{S} \ : \varphi, \frac{\partial \varphi}{\partial z_j}, \frac{\partial \varphi}{\partial \bar{z}_j} \in L^\infty, j = 1, \cdots, n \right\}.$$

It is known that each function in $\mathscr{L}^{1,\infty}$ can be extended to a continuous function on $\bar{B}$, the closed unit ball; see Theorem 5.4 of [1] for example. Thus we will use the same notation between a function in $\mathscr{L}^{1,\infty}$ and its continuous extension to $\bar{B}$. For $\varphi \in \mathscr{L}^{1,\infty}$, we note that $\mathcal{R}\varphi, \widetilde{\mathcal{R}}\varphi \in L^\infty$. Given $u \in \mathscr{L}^{1,\infty}$, the *Toeplitz operator* T_u with symbol u is the linear operator on $\mathscr{D}_{ph}$ defined by

$$T_u f = Q(uf)$$

for functions $f \in \mathscr{D}_{ph}$. Then one can see that T_u is bounded on $\mathscr{D}_{ph}$.

In this paper we consider operators which are finite sums of Toeplitz products of several Toeplitz operators. More explicitly, we consider operators of the form

$$\sum_{i=1}^{N} \prod_{j=1}^{M_i} T_{u_{ij}} \tag{1}$$

where $u_{ij} \in \mathscr{L}^{1,\infty}$. We then study the characterizing problem of when an operator of the form (1) is compact. Moreover, we will describe the essential norm for such an operator. Recall that the essential norm $\|L\|_e$ of a bounded linear operator L on $\mathscr{D}_{ph}$ is defined as

$$\|L\|_e = \inf_K \|L + K\|$$

where the infimum is taken over all compact operators K on $\mathscr{D}_{ph}$. Thus we note L is compact if and only if $\|L\|_e = 0$.

On the setting of the Bergman spaces or holomorphic Dirichlet spaces, the corresponding problem has been well studied. Axler and Zheng [2] considered the problem on the Bergman space of the unit disk and proved that such an operator is compact if and only if the Berezin transform of the operator vanishes on the boundary of the disk. Later their result has been extended to bounded symmetric domains in [5]. Recently, on the holomorphic Dirichlet space of the ball or polydisk, the same problem has been studied as in [7] and [9].

We in this paper continue to study the same problem on the pluriharmonic Dirichlet space under consideration and describe the essential norm of an operator of the form (1) in terms of the boundary value of the corresponding sum of products of symbols. The following is the main result of our paper.

Theorem 1. *Given $u_{ij} \in \mathscr{L}^{1,\infty}$, we have*

$$\left\| \sum_{i=1}^{N} \prod_{j=1}^{M_i} T_{u_{ij}} \right\|_e = \max_{\eta \in \partial B} \left| \sum_{i=1}^{N} \prod_{j=1}^{M_i} u_{ij}(\eta) \right|.$$

In Section 2, we collect some basic facts. In Section 3, we will prove Theorem 1. As a consequence, we characterize the compactness for such an operator to be compact; see Corollary 8. Also we study the compact product problem with pluriharmonic symbols; see Corollary 9.

2. Preliminaries

The Dirichlet space $\mathscr{D}$ is a closed subspace of $\mathscr{S}$ consisting of all holomorphic functions in $\mathscr{S}$. We let P be the Hilbert space orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}$. Each point evaluation is easily verified to be bounded linear functionals on both $\mathscr{D}$ and $\mathscr{D}_{ph}$. Hence, for each $z \in B$, there exist functions $K_z \in \mathscr{D}$ and $R_z \in \mathscr{D}_{ph}$ which have the following reproducing properties:

$$f(z) = \langle f, K_z \rangle, \qquad u(z) = \langle u, R_z \rangle$$

for functions $f \in \mathscr{D}$ and $u \in \mathscr{D}_{ph}$. As is well known, a real-valued function on B is pluriharmonic if and only if it is the real part of a holomorphic function on B. Hence every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function; see Chapter 4 of [10]. Using this fact, we can see that $\mathscr{D}_{ph} = \mathscr{D} + \overline{\mathscr{D}}$. Hence there is a useful relation between R_z and K_z:

$$R_z = K_z + \overline{K_z} - 1.$$

Since $P\varphi = \langle \varphi, K_z \rangle$ for $z \in B$, the formula above leads us to the following useful connection between P and Q:

$$Q(\varphi) = P(\varphi) + \overline{P(\overline{\varphi})} - P(\varphi)(0) \tag{2}$$

for functions $\varphi \in \mathscr{S}$. We let $L^2 = L^2(B, V)$ be the usual Lebesgue space equipped with the usual norm $\|\ \|_2$ and A^2 be the well known Bergman space consisting of all holomorphic functions in L^2. Let β be the Bergman projection which is the orthogonal projection from L^2 onto A^2. It is known that there is a useful connection between P and β:

$$\mathcal{R}(P\psi) = \beta\left(\mathcal{R}\psi\right) - \beta\left(\mathcal{R}\psi\right)(0)$$

for functions $\psi \in \mathscr{S}$; see [7] for detail. It follows from (2) that

$$\begin{aligned} \mathcal{R}(Q\psi) &= \beta\left(\mathcal{R}\psi\right) - \beta\left(\mathcal{R}\psi\right)(0) \\ \widetilde{\mathcal{R}}(Q\psi) &= \overline{\beta\left(\mathcal{R}\bar{\psi}\right)} - \overline{\beta\left(\mathcal{R}\bar{\psi}\right)(0)} \end{aligned} \tag{3}$$

for functions $\psi \in \mathscr{S}$.

For a function $\varphi \in L^\infty$, we let S_φ denote the Bergman space Toeplitz operator on A^2 defined by

$$S_\varphi f = \beta(\varphi f)$$

for functions $f \in A^2$. Clearly S_φ is bounded on A^2. Given a bounded linear operator L on A^2, the Berezin transform $\widehat{L}$ of L is the function on B defined by

$$\widehat{L}(a) = \int_B (Lb_a)\overline{b_a}\, dV, \qquad a \in B$$

where b_a denotes the well known normalized Bergman kernel of A^2. It is known that $\widehat{L}$ is a continuous function on B. Moreover, it turns out that the Berezin transform of an operator which is a product of Bergman space Toeplitz operators preserves the boundary continuity of symbols. More explicitly, it is known that for given symbols $\varphi, \psi \in L^\infty$ which are continuous on $\bar{B}$, the Berezin transform $\widehat{S_\varphi S_\psi}$ is continuous up to $\bar{B}$ and

$$\widehat{S_\varphi S_\psi} = \varphi\psi \quad \text{on} \quad \partial B \tag{4}$$

holds; see Proposition 2.1 of [4] for example. Also, the Berezin transform turns out to provide a compactness criterion for operators which are sums of products of Bergman space Toeplitz operators. Specially, for symbols $\varphi, \psi, u \in L^\infty$, it is known that $S_\varphi S_\psi - S_u$ is compact on A^2 if and only if

$$\lim_{|a|\to 1} [S_\varphi \widehat{S_\psi - S_u}](a) = 0;$$

see Theorem A of [5] for more general results. Recall that each function in $\mathscr{L}^{1,\infty}$ can be extended to a continuous function on $\bar{B}$. Now, combining these observations with (4), we have the following characterization.

Lemma 2. *Let $\varphi, \psi, u \in \mathscr{L}^{1,\infty}$. Then $S_\varphi S_\psi - S_u$ is compact on A^2 and only if $\varphi\psi = u$ on ∂B.*

3. The proof of Theorem 1

We let $\mathscr{D}_0$ be the space of all $f \in \mathscr{D}$ such that $f(0) = 0$. Note that $\mathscr{D}_{ph} = \mathscr{D}_0 \oplus \overline{\mathscr{D}}$. The following is taken from Proposition 2 of [8].

Proposition 3. *If a sequence $u_j = f_j + \overline{g_j} \in \mathscr{D}_0 + \overline{\mathscr{D}}$ converges to 0 weakly in $\mathscr{D}_{ph}$, then f_j and g_j converge to 0 weakly in $\mathscr{D}$. Also, if h_j converges to 0 weakly in $\mathscr{D}$, then h_j converges to 0 weakly in $\mathscr{D}_{ph}$.*

It is easy to see that the identity operator from $\mathscr{D}_{ph}$ into b^2 is bounded. Moreover, it turns out that it is in fact compact; see Proposition 3 of [8]. In particular, for a sequence φ_k converging weakly to 0 in $\mathscr{D}_{ph}$, we have $||\varphi_k||_2 \to 0$ as $k \to \infty$.

The following lemma will be useful in our proofs. Recall that β is the Bergman projection and put

$$\langle f, g\rangle_2 = \int_B f\overline{g}\, dV, \qquad f, g \in L^2.$$

Lemma 4. *Let $u \in \mathscr{L}^{1,\infty}$. If φ_j converges to 0 weakly in $\mathscr{D}_{ph}$, then we have*

$$\lim_{j\to\infty} \beta[\mathcal{R}(u\varphi_j)](0) = 0.$$

Proof. For each j, we write $\varphi_j = f_j + \overline{g_j} \in \mathscr{D}_0 + \overline{\mathscr{D}}$. By Proposition 3, f_j converges to 0 weakly in $\mathscr{D}$. We first claim that $\mathcal{R}\varphi_j$ converges to 0 weakly in A^2. To prove this, let $\epsilon \in A^2$ be an arbitrary function and choose $\psi \in \mathscr{D}$ such that $\mathcal{R}\psi = \epsilon - \epsilon(0)$. Since $\mathcal{R}f_j(0) = 0$ for each j, we see

$$\begin{aligned}
\langle \mathcal{R}\varphi_j, \epsilon\rangle_2 &= \langle \mathcal{R}f_j, \mathcal{R}\psi + \epsilon(0)\rangle_2 \\
&= \langle \mathcal{R}f_j, \mathcal{R}\psi\rangle_2 + \overline{\epsilon(0)}\mathcal{R}f_j(0) \\
&= \langle \mathcal{R}f_j, \mathcal{R}\psi\rangle_2 \\
&= \langle f_j, \psi\rangle - f_j(0)\overline{\psi(0)} \\
&= \langle f_j, \psi\rangle
\end{aligned}$$

for each j, which implies $\langle \mathcal{R}\varphi_j, \epsilon\rangle_2 \to 0$ as $j \to \infty$ and $\mathcal{R}\varphi_j$ converges to 0 weakly in A^2. Now, to complete the proof, we note that

$$\left|\int_B (\mathcal{R}u)\varphi_j\, dV\right| \le ||\mathcal{R}u||_\infty \left(\int_B |\varphi_j|^2\, dV\right)^{\frac{1}{2}} \to 0$$

as $j \to \infty$ by the remark just after Proposition 3 again. It follows that

$$\begin{aligned}\lim_{j\to\infty} \beta[\mathcal{R}(u\varphi_j)](0) &= \lim_{j\to\infty} \int_B [(\mathcal{R}u)\varphi_j + u(\mathcal{R}\varphi_j)]\, dV \\ &= \lim_{j\to\infty} \langle \mathcal{R}\varphi_j, \bar{u}\rangle_2 \\ &= \lim_{j\to\infty} \langle \mathcal{R}\varphi_j, \beta(\bar{u})\rangle_2 \\ &= 0\end{aligned}$$

because $\mathcal{R}f_j$ converges weakly to 0 in A^2. The proof is complete. □

For each $a \in B$, we let

$$E_a(z) = \sum_{|\alpha|>0} \frac{(n+|\alpha|)!}{n!|\alpha|\alpha!} \overline{a^\alpha} z^\alpha, \qquad z \in B$$

and put $e_a := E_a||E_a||^{-1}$ for notational simplicity. Then, it is known that e_a converges weakly to 0 in $\mathscr{D}$ as $a \to \partial B$ and $\lim_{a\to\zeta}\langle ue_a, e_a\rangle = u(\zeta)$ for every $\zeta \in \partial B$; see Lemma 6 of [7]. It follows that

$$\lim_{a\to\zeta}\langle T_u e_a, e_a\rangle = \lim_{a\to\zeta}\langle ue_a, e_a\rangle = u(\zeta) \tag{5}$$

holds for every $u \in \mathscr{L}^{1,\infty}$ and $\zeta \in \partial B$.

The following shows that a semi-commutator of two Toeplitz operators is always compact on $\mathscr{D}_{ph}$.

Proposition 5. *For any $u, v \in \mathscr{L}^{1,\infty}$, $T_uT_v - T_{uv}$ is compact on $\mathscr{D}_{ph}$.*

Proof. Put $T = T_uT_v - T_{uv}$ for notational simplicity and let f_k be a sequence converging weakly to 0 in $\mathscr{D}_{ph}$. To prove the compactness of T on $\mathscr{D}_{ph}$, we need to show $||Tf_k|| \to 0$ as $k \to \infty$. First note that each T_vf_k converges weakly to 0 in $\mathscr{D}_{ph}$ and then

$$\lim_{k\to\infty} \|f_k\|_2 = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T_vf_k\|_2 = 0 \tag{6}$$

by the remark just after Proposition 3. Since $Q\varphi(0) = \int_B \varphi\, dV$ for every $\varphi \in \mathscr{S}$, we have

$$|Tf_k(0)| \le \int_B |uT_vf_k + uvf_k|\, dV \le \|u\|_\infty\|T_vf_k\|_2 + \|uv\|_\infty\|f_k\|_2$$

for each k. It follows from (6) that

$$\lim_{k\to\infty} |Tf_k(0)| = 0. \tag{7}$$

Since $Tf_k = Q[uQ(vf_k) - uvf_k]$, we have by (3)

(8)

$$\begin{aligned}
\mathcal{R}(Tf_k) &= \beta[\mathcal{R}(uQ(vf_k)) - \mathcal{R}(uvf_k)] - \beta[\mathcal{R}(uQ(vf_k)) - \mathcal{R}(uvf_k)](0)\\
&= \beta[(\mathcal{R}u)T_vf_k] + \beta[u\mathcal{R}(Q(vf_k))] - \beta[\mathcal{R}(uT_vf_k)](0)\\
&\quad - \beta[(\mathcal{R}(uv)f_k] - \beta[uv\mathcal{R}f_k] + \beta[\mathcal{R}(uvf_k)](0)\\
&= \beta[(\mathcal{R}u)T_vf_k] + \beta[u\{\beta(\mathcal{R}(vf_k)) - \beta(\mathcal{R}(vf_k))(0)\}]\\
&\quad - \beta[\mathcal{R}(uT_vf_k)](0) - S_{\mathcal{R}(uv)}f_k - S_{uv}\mathcal{R}f_k + \beta[\mathcal{R}(uvf_k)](0)\\
&= \beta[(\mathcal{R}u)T_vf_k] + \beta[u\{\beta((\mathcal{R}v)f_k + v\mathcal{R}f_k) - \beta(\mathcal{R}(vf_k))(0)\}]\\
&\quad - \beta[\mathcal{R}(uT_vf_k)](0) - S_{\mathcal{R}(uv)}f_k - S_{uv}\mathcal{R}f_k + \beta(\mathcal{R}(uvf_k))(0)\\
&= S_{\mathcal{R}u}T_vf_k + S_uS_{\mathcal{R}v}f_k + S_uS_v\mathcal{R}f_k - \beta(u)\beta[\mathcal{R}(vf_k)](0)\\
&\quad - \beta[\mathcal{R}(uT_vf_k)](0) - S_{\mathcal{R}(uv)}f_k - S_{uv}\mathcal{R}f_k + \beta[\mathcal{R}(uvf_k)](0)\\
&= S_{\mathcal{R}u}T_vf_k + S_uS_{\mathcal{R}v}f_k + [S_uS_v - S_{uv}]\mathcal{R}f_k - \beta(u)\beta[\mathcal{R}(vf_k)](0)\\
&\quad - \beta[\mathcal{R}(uT_vf_k)](0) - S_{\mathcal{R}(uv)}f_k + \beta[\mathcal{R}(uvf_k)](0)
\end{aligned}$$

for all k. Since f_k and T_vf_k converge weakly to 0, we have by Lemma 4

$$\lim_{k\to\infty}\beta[\mathcal{R}(vf_k)](0) = \lim_{k\to\infty}\beta[\mathcal{R}(uT_vf_k)](0) = \lim_{k\to\infty}\beta[\mathcal{R}(uvf_k)](0) = 0.$$

Also, since $\mathcal{R}f_k$ converges weakly to 0 in A^2 and $S_uS_v - S_{uv}$ is compact by Lemma 2, we see $\|[S_uS_v - S_{uv}](\mathcal{R}f_k)\|_2 \to 0$ as $k \to \infty$. Combining these with (8) and (6), we have

$$\begin{aligned}
\|\mathcal{R}(Tf_k)\|_2 &\le ||S_{\mathcal{R}u}||\,||T_vf_k||_2 + ||S_uS_{\mathcal{R}v}||\,||f_k||_2 + ||[S_uS_v - S_{uv}]\mathcal{R}f_k||_2\\
&\quad + ||\beta u||_2|\beta[\mathcal{R}(vf_k)](0)| + |\beta[\mathcal{R}(uT_vf_k)](0)|\\
&\quad + ||S_{\mathcal{R}(uv)}||\,||f_k||_2 + |\beta[\mathcal{R}(uvf_k)](0)|\\
&\to 0
\end{aligned}$$

as $k \to \infty$. Also, one can see by (3) again

$$\begin{aligned}
\overline{\widetilde{\mathcal{R}}(Tf_k)} &= S_{\mathcal{R}\bar{u}}\overline{T_vf_k} + S_{\bar{u}}S_{\mathcal{R}\bar{v}}\overline{f_k} + [S_{\bar{u}}S_{\bar{v}} - S_{\overline{uv}}]\mathcal{R}\overline{f_k} - \beta(\bar{u})\beta[\mathcal{R}(\overline{vf_k})](0)\\
&\quad - \beta[\mathcal{R}(\overline{uT_vf_k})](0) - S_{\mathcal{R}(\overline{uv})}\overline{f_k} + \beta[\mathcal{R}(\overline{uvf_k})](0)
\end{aligned}$$

for all k. Note that the complex conjugate of a sequence converging weakly to 0 in $\mathscr{D}_{ph}$ also converges weakly to 0. Now, by the similar argument above, one can see that $\|\widetilde{\mathcal{R}}(Tf_k)\|_2 \to 0$ as $k \to \infty$. Combining the above together with (7), we see

$$\lim_{k\to\infty}\|Tf_k\|^2 = \lim_{k\to\infty}\Big[|Tf_k(0)|^2 + \|\mathcal{R}(Tf_k)\|_2^2 + \|\widetilde{\mathcal{R}}(Tf_k)\|_2\Big] = 0,$$

which implies the compactness of T as desired. The proof is complete. $\square$

Given $u, v \in \mathscr{L}^{1,\infty}$, observing

$$T_u T_v = T_{uv} + [T_u T_v - T_{uv}],$$

we see that $T_u T_v$ is a compact perturbation of T_{uv} by Proposition 5. By using the exactly same argument as in Proposition 9 of [9] together with Proposition 5, we can see that the same is true for operators which are finite sums of products of several Toeplitz operators as shown in the following.

Proposition 6. *Given $u_{ij} \in \mathscr{L}^{1,\infty}$, there exists a compact operator K on $\mathscr{D}_{ph}$ such that*

$$\sum_{i=1}^{N} \prod_{j=1}^{M_i} T_{u_{ij}} = T_{\sum_{i=1}^{N} \prod_{j=1}^{M_i} u_{ij}} + K.$$

Before we prove the main result, we describe the essential norm of a single Toeplitz operator as a preliminary result.

Lemma 7. *For $u \in \mathscr{L}^{1,\infty}$, we have $||T_u||_e = \max_{\eta \in \partial B} |u(\eta)|$.*

Proof. Put $\rho = \max_{\eta \in \partial B} |u(\eta)|$ for simplicity. Choose a point $\zeta \in \partial B$ such that $|u(\zeta)| = \rho$. For any compact operator K on $\mathscr{D}_{ph}$, since e_a converges weakly to 0 in $\mathscr{D}_{ph}$ as $a \to \partial B$ by Proposition 3, we note that

$$||T_u + K|| \geq \lim_{a \to \zeta} |\langle (T_u + K)e_a, e_a \rangle| = \lim_{a \to \zeta} |\langle T_u e_a, e_a \rangle| = |u(\zeta)|$$

by (5), thus $\rho \leq ||T_u||_e$ holds.

Now, we prove the reverse inequality. By Lemma 1.2 of [6], there is an orthonormal sequence ψ_j in $\mathscr{D}_{ph}$ for which $||T_u \psi_j|| \to ||T_u||_e$ as $j \to \infty$. Write $\psi_j = f_j + \overline{g_j} \in \mathscr{D}_0 \oplus \bar{\mathscr{D}}$ for each j. Since ψ_j converges weakly to 0 in $\mathscr{D}_{ph}$, f_j and g_j converge weakly to 0 in $\mathscr{D}$ by Proposition 3. In particular, we note f_j and g_j converge uniformly to 0 on every compact subsets of B. Thus, by the remark after Proposition 3, we see

$$\lim_{j \to \infty} \int_B |\xi \psi_j|^2 \, dV \leq ||\xi||_\infty^2 \lim_{j \to \infty} \int_B |\psi_j|^2 \, dV = 0 \tag{9}$$

where $\xi = u, \mathcal{R}u$, or $\widetilde{\mathcal{R}}u$. On the other hand, since u is continuous on $\bar{B}$, for any $\epsilon > 0$, there exists $r \in (0,1)$ such that $|u(z)| \leq \rho + \epsilon$ for every $r < |z| < 1$. Fix $t \in (r, 1)$. Then note

$$\lim_{j \to \infty} \int_{|z| \leq t} |f_j(z)|^2 \, dV(z) = 0.$$

Writing

$$f_j(z) = \sum_\alpha a^j_\alpha z^\alpha, \qquad z \in B$$

for the Taylor series expansions of f_j, we note

$$\int_{|z|\le t} |f_j|^2\, dV = \sum_{|\alpha|\ge 0} |a^j_\alpha|^2 \int_{|z|\le t} |z^\alpha|^2\, dV = \sum_{|\alpha|\ge 0} |a^j_\alpha|^2 \frac{t^{2|\alpha|+2n}}{|\alpha|+n} \int_{\partial B} |\zeta^\alpha|^2\, d\sigma$$

and

$$\begin{aligned}\int_{|z|\le r} |\mathcal{R} f_j|^2\, dV &= \sum_{|\alpha|>0} |a^j_\alpha|^2 |\alpha|^2 \frac{r^{2|\alpha|+2n}}{|\alpha|+n} \int_{\partial B} |\zeta^\alpha|^2\, d\sigma \\ &= \sum_{|\alpha|>0} |a^j_\alpha|^2 \frac{t^{2|\alpha|+2n}}{|\alpha|+n} |\alpha|^2 \left(\frac{r}{t}\right)^{2|\alpha|+2n} \int_{\partial B} |\zeta^\alpha|^2\, d\sigma\end{aligned}$$

for each j. Since $|\alpha|^2 \left(\frac{r}{t}\right)^{2|\alpha|+2n} \to 0$ as $|\alpha| \to \infty$, we see

$$\lim_{j\to\infty} \int_{|z|\le r} |u \mathcal{R} f_j|^2\, dV \le ||u||^2_\infty \lim_{j\to\infty} \int_{|z|\le r} |\mathcal{R} f_j|^2\, dV = 0$$

and hence

$$\begin{aligned}\lim_{j\to\infty} \int_B |u\mathcal{R} f_j|^2\, dV &= \lim_{j\to\infty} \int_{|z|>r} |u \mathcal{R} f_j|^2\, dV \\ &\le (\rho+\epsilon)^2 \lim_{j\to\infty} \int_B |\mathcal{R}\psi_j|^2\, dV.\end{aligned} \tag{10}$$

Also, using the similar argument above for g, we can see

$$\lim_{j\to\infty} \int_B |u\overline{\mathcal{R} g_j}|^2\, dV \le (\rho+\epsilon)^2 \lim_{j\to\infty} \int_B |\widetilde{\mathcal{R}}\psi_j|^2\, dV. \tag{11}$$

Note that by the boundedness of Q,

$$||T_u\psi_j||^2 \le \left|\int_B u\psi_j\, dV\right|^2 + \int_B \left(|(\mathcal{R}u)\psi_j + u\mathcal{R} f_j|^2 + |(\widetilde{\mathcal{R}}u)\psi_j + u\overline{\mathcal{R} g_j}|^2 \right) dV$$

for each j. It follows from (9), (10) and (11) that

$$\begin{aligned}\lim_{j\to\infty} ||T_u\psi_j||^2 &\le (\rho+\epsilon)^2 \lim_{j\to\infty} \int_B \left(|\mathcal{R}\psi_j|^2 + \widetilde{\mathcal{R}}\psi_j|^2 \right) dV \\ &\le (\rho+\epsilon)^2 \lim_{j\to\infty} ||\psi_j||^2\end{aligned}$$

for any $\epsilon > 0$. Since $||\psi_j|| = 1$ for each j, the above shows

$$\lim_{j\to\infty} ||T_u\psi_j|| \le \rho.$$

Now, recalling $\lim_{j\to\infty}||T_u\psi_j|| = ||T_u||_e$, we have $||T_u||_e \le \rho$, as desired. The proof is complete. □

Now we are ready to prove our main theorem.

Proof of Theorem 1. By Proposition 6 and Lemma 7, we see

$$\left\|\sum_{i=1}^{N}\prod_{j=1}^{M_i} T_{u_{ij}}\right\|_e = \left\|T_{\sum_{i=1}^{N}\prod_{j=1}^{M_i} u_{ij}}\right\|_e = \max_{\eta\in\partial B}\left|\sum_{i=1}^{N}\prod_{j=1}^{M_i} u_{ij}(\eta)\right|,$$

as desired. The proof is complete. □

As an immediate consequence of Theorem 1, we obtain the following characterization.

Corollary 8. *For $u_{ij} \in \mathscr{L}^{1,\infty}$, the following statements are equivalent.*

(a) $\displaystyle\sum_{i=1}^{N}\prod_{j=1}^{M_i} T_{u_{ij}}$ *is compact on $\mathscr{D}_{ph}$.*

(b) $\displaystyle\sum_{i=1}^{N}\prod_{j=1}^{M_i} u_{ij} = 0$ *on ∂B.*

As an application of Corollary 8, we consider the compact product problem of when the compactness of a product of several Toeplitz operators with pluriharmonic symbols implies the triviality of one of symbols. For $n \ge 2$ and pluriharmonic functions $u_1, \cdots, u_N$ which are continuous on ∂B, it is known that $u_1 \cdots u_N = 0$ on ∂B if and only if $u_j = 0$ on B for some j; see Corollary 3.5 of [3] for details. Thus, the following is a simple consequence of Corollary 8.

Corollary 9 ($n \ge 2$)**.** *Let $u_1, \cdots, u_N \in \mathscr{L}^{1,\infty}$ be pluriharmonic functions. Then $T_{u_1} \cdots T_{u_N}$ is compact on $\mathscr{D}_{ph}$ if and only if $u_j = 0$ for some j.*

Consider two harmonic symbols u, v which are nonzero on the unit disk and $uv = 0$ on the unit circle. Then T_uT_v is compact by Corollary 8, but neither u nor v is identically zero. This observation tells us that Corollary 9 can not be extended to the one dimensional case in general. But, for a single Toeplitz operator with pluriharmonic symbol, we have the following characterization for full range of dimensions.

Corollary 10. *Let $u \in \mathscr{L}^{1,\infty}$ be a pluriharmonic function. Then the following are equivalent.*

(a) *T_u is compact on $\mathscr{D}_{ph}$.*
(b) *$u = 0$ on B.*
(c) *$T_u = 0$ on $\mathscr{D}_{ph}$.*

Proof. Since a pluriharmonic symbol which vanishes on ∂B vanishes on B, implication (a) $\Rightarrow$ (b) follows from Corollary 8. Also, since (b) $\Rightarrow$ (c) $\Rightarrow$ (a) is clear, we complete the proof. $\square$

In view of Corollary 10, one might ask whether the same is true for general symbols. But the answer is *no*. For example, the Toeplitz operator $T_{1-|z|^2}$ is compact on $\mathscr{D}_{ph}$ by Corollary 8, but $T_{1-|z|^2}$ is not equal to 0. Indeed,

$$T_{1-|z|^2}1(0) = \int_B (1 - |w|^2)\, dV(w) \neq 0.$$

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